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On Λ-Fractional Differential Geometry

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Abstract: Applying a new fractional derivative, the Λ- fractional derivative, with the corresponding Λ-fractional space, fractional differential geometry is discussed. The Λ-fractional derivative satisfies the conditions for the existence of a differential, demanded by the differential topology, in the Λ-fractional space, where the Λ-derivatives behave like the conventional ones. Thus, fractional differential geometry is established in that Λ-space in the conventional way. The results are pulled back to the initial space. The present work concerns the geometry of fractional curves and surfaces.

Keywords: Λ- fractional derivative, Λ-fractional space, Λ-fractional differential geometry, Λ- fractional tangent, Λ-fractional curvature, Λ-fractional focal curve, Λ-fractional tangent space of surfaces.

1 Introduction

Fractional analysis has recently been considered as an indispensable tool in describing real life models. The origins of fractional calculus go back to Leibnitz [1], Liouville [2] and Riemann [3]. Fractional calculus has been employed to describe more intricate real world models ever since. While conventional mathematical analysis is almost restricted to a local description of a function and fractional analysis is inherently a global one, the latter is considered as more suitable for describing the real world. In fact various viscoelastic responses have been described by fractional differential analysis [4], as well as other physical problems, dependent upon time derivatives [5]. Moreover problems described by fractals are better expressed through fractional analysis [6]. Also various control problems have been analyzed through Fractional calculus. Extensive information about fractional analysis and fractional differential equations are explicated in [7,8,9,10]. Lazopoulos [11] has introduced an elastic uniaxial model based upon fractional derivatives. This model succeeded in lifting Noll's axiom of local-action. Hence fractional analysis from the solely time dependent problems, extended to space dependent ones, just for considering inhomogeneous space fields. Nevertheless, Carpinteri et al. [12] have also introduced a fractional approach considering non-local mechanics. Let us point out that many researchers suggested Fractional Calculus for solving problems in mechanics, [13,14], Jumarie [15,16]. Drapaca and Sivaloganathan [17], Sumelka [18] have adopted fractional analysis in problems of continuum mechanics with microstructure,where non-local elasticity is necessary. Another favorite field of fractional continuum mechanics is hydrodynamics [13,19]. Moreover, another application of fractional calculus was the description of peridynamic theory [20,21,22]. Nevertheless in mechanics, viscoelasticity is the main area for fractional calculus applications [4,5]. In addition fractional differential geometry describes successfully rigid body dynamics, in holonomic and non-holonomic systems [23,24,25,26]. Differential geometry which is revisited by fractional calculus might be found in quantum mechanics, physics and relativity [27,28]. Theory, along with applications in various physical areas, may also be found in various books [8,9,10, 29]. Different aspects concerning fractional geometry of manifolds [13,15,16,23,30] have been presented. Further, researchers attempted to apply fractional differential geometry to various fields of mechanics, quantum mechanics, physics, relativity, finance, probabilities etc. Moreover, fractal functions exhibiting self-similarity are non-differentiable functions but they exhibit fractional differentiability of order $0 < \gamma < 1$. See Ref.[6,37,38,39]. However, mathematicians are doubting about the basis of fractional geometry, since the various and many fractional derivatives do not satisfy the

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requirements of differential topology for forming differentials and able to formulate geometry. Adda [7,31] has proposed for the fractional differential instead of the classical one, where g(x) is the fractional derivative of a function f(x). In the first attempt to establish fractional differential, Lazopoulos [32,33] introduced the L-fractional derivative. Nevertheless, even that fractional derivative failed to satisfy the differential topology conditions for existence of a fractional differential. Lazopoulos then [34] introduced the Λ-fractional derivative that conforms with the conditions required by differential topology in the Λ-fractional space conjugate to the original one. Then the various results are pulled back to the original space. No differential geometry is valid in the original space. Hence development of fractional differential geometry is possible only in the Λ-space and the various results may be transfer may be transferred to the original space. Lazopoulos [40] has already presented the Λ-fractional beam theory, using the proposed Λ-fractional curvature of the elastic curves of the beams. Further, the fractional deformation of a bar based upon the Λ- fractional derivative has also been presented [41]. Recently, the fractional plane elasticity theory with biharmonic functions has been presented [42], along with the discussion of Fractional Taylor's Series and the Variational Euler-Lagrange equations[43]. In the present work, fractional differential geometry is formulated in the Λ-fractional space for curves and surfaces. Furthermore the various results may be transferred back to the original space. Specifically, the fractional differential geometry of curves with their tangent spaces, their normals, the curvature vectors and the curves of curvature centers is established. In addition the fractional Serret-Frenet equations will be discussed in the Λ-fractional space. Further, the present work studies the tangent space on a surface, helping in the accurate description of the fractional differentials of surfaces, defining the fractional first and second fundamental fractional differential forms and the fractional normal plane. Moreover, the fractional normal curvature of the curves on a manifold is also discussed. In addition, the covariant Λ-fractional derivative, the fractional Christofell's coefficients, the Λ-fractional covariant derivative, the extremals of fractional functions and functionals and the geodesics of fractional manifolds are formulated. In fact those topics enclose the basic principles of the fractional differential geometry. Lazopoulos [40] has already presented the Λ-fractional beam theory, using the proposed Λ-fractional curvature of the elastic curves of the beams.

2 Basic properties of fractional calculus

With many applications in engineering and physics, fractional calculus has been considered as one of the most active fields in applied mathematics. Fractional calculus has lately become a branch of pure mathematics with many applications in physics and engineering. There are many definitions of fractional derivatives. In fact, Fractional Calculus was stemmed by Leibniz, looking for the possibility of defining the derivative $\frac{d^\gamma g}{dx^\gamma}$ when $n = \frac{1}{2}$.The various types of the fractional derivatives exhibit some advantages over the others. Nevertheless they all are non local. On the other hand, the conventional derivatives express strictly locality. Information about fractional analysis and its applications may be found in the classical books of Kilbas et al. [29], Podlubny [9], Samko et al. [8]. Recalling the n-fold integral of a function f(x):

$$ {}_aI_x^n f(x) = \frac{1}{\Gamma(n)} \int_a^x \frac{f(s)}{(x-s)^{1-n}} ds \tag{1} $$

Leibniz defined the γ-multiple integral with $0 < \gamma \leq 1$ by,

$$ {}_aI_x^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_a^x \frac{f(s)}{(x-s)^{1-\gamma}} ds \tag{2} $$

with $\Gamma(\gamma)$ Euler's Gamma function. Further, the left Riemann-Liouville (R-L) derivatives are defined by:

$$ {}_a^{RL}D_x^\gamma f(x) = \frac{d}{dx}\left({}_aI_x^{1-\gamma}(f(x))\right) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_a^x \frac{f(s)}{(x-s)^\gamma} ds \tag{3} $$

with corresponding definitions for the right fractional integrals and derivatives Podlubny [9].

3 The Λ-fractional derivative

The L-fractional derivative was introduced by the authors, in an attempt to devise a fractional derivative satisfying the properties of a derivative demanded by the differential topology, for the existence of the corresponding differential. Indeed, the differential topology requirements for the existence of a differential are, see [31,35,36]:

Linearity: D(af(x)+bg(x))=aDf(x)+bDg(x).

Leibniz rule: $D(f(x)\cdot g(x)) = Df(x)\cdot g(x) + f(x)\cdot Dg(x)$.

Chain rule: D(g(f))(x)=Dg(f(x))· Df(x).

Although the various fractional derivatives satisfy the linearity property, they fail to satisfy the composition and Leibniz's rules. Lazopoulos [32] introduced the L-fractional derivative in an attempt for the fractional derivative to satisfy the differential topology requirements for the existence of differential and hence the existence of fractional differential geometry. Nevertheless the initial definition of the L-fractional derivative failed to satisfy all the requirements for the existence of differential. The revision of the L-fractional derivative is targeting to that purpose. The $\Lambda-$fractional derivative has been introduced as:

$$ {}^{\Lambda}_{a}D^{\gamma}_{x}f(x) = \frac{{}^{RL}_{a}D^{\gamma}_{x}f(x)}{{}^{RL}_{a}D^{\gamma}_{x}x} \tag{4}$$

Considering the definition of the fractional derivative, Eq.(3), the Λ-FD is expressed by:

$$ {}^{\Lambda}_{a}D^{\gamma}_{x}f(x) = \frac{\frac{d_aI^{1-\gamma}_{x}f(x)}{dx}}{\frac{d_aI^{1-\gamma}_{x}x}{dx}} = \frac{d_aI^{1-\gamma}_{x}f(x)}{d_aI^{1-\gamma}_{x}x} \tag{5}$$

Defining as X=${}_aI_x^{1-\gamma}x$ and F(X)=${}_aI_x^{1-\gamma}f(x)$, the Λ-FD is defined as a conventional derivative in the fractional space (X, F(X)). In fact, the fractional differential geometry is defined as a conventional differential geometry in the Λ-fractional space, (X, F(X)), where the derivative,

$$ {}^{\Lambda}_{a}D^{\gamma}_{x}f(x) = \frac{dF(X)}{dX} \tag{6}$$

Further the relation,

$$ {}^{RL}_{a}D^{\gamma}_{x}({}_aI^{\gamma}_{x}f(x)) = f(x) \tag{7}$$

is quite important for the pulling back of the various functions from the fractional Λ-space to the original one. It is evident that, in the just presented Λ-fractional derivatives, only left fractional integrals and RL fractional derivatives were considered. If we were to involve the right fractional integrals and RL derivatives, then the Λ-fractional derivatives should be defined by

$$ {}^{\Lambda}_{a}D^{\gamma}_{x}f(x) = \frac{d_aI^{1-\gamma}_{x}f(x) - d_xI^{1-\gamma}_{b}f(x)}{2d_aI^{1-\gamma}_{x}x} = \frac{dF(X)}{dX} \tag{8}$$

with

$$ F(x) = {}_aI^{1-\gamma}_{x}f(x) = \frac{{}_aI^{1-\gamma}_{x}f(x) - {}_xI^{1-\gamma}_{b}f(x)}{2} = \frac{1}{\Gamma(1-\gamma)}\left(\int_a^x \frac{f(s)}{(x-s)^{\gamma}}ds - \int_x^b \frac{f(s)}{(x-s)^{\gamma}}ds\right) = F(x(X)) \tag{9}$$

It will be clarified in the application, how from the initial space (x, f(x)) the fractional Λ-space (X, F(X)) is defined. Further the pull back of the results in the initial space will also be demonstrated. For simplicity reasons only the left fractional integrals and derivatives will be taken into consideration. Nevertheless, applications with symmetric space may be found in [20]. Just to clarify the ideas, let us work as an example on the function,

$$ f(x) = x^2 \tag{10}$$

Then the Λ-fractional plane (X, F(X)) is defined (with a=0) by

$$ X = \frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} \tag{11}$$

$$ F(X) = {}_aI^{1-\gamma}_{x}f(x(X)) = {}_aI^{\gamma}_{x}f(x) = \frac{1}{\Gamma(1-\gamma)}\int_a^x \frac{s^2}{(x-s)^{\gamma}}ds = \frac{2}{(-6+11\gamma-6\gamma^2+\gamma^3)\Gamma(1-\gamma)}x^{(3-\gamma)} \tag{12}$$

Further considering from Eq.(7),

$$ x = ((2-3\gamma+\gamma^2)\Gamma(1-\gamma)X)^{\frac{1}{2-\gamma}} \tag{13}$$

Eq.(8) yields

$$ F(X) = -\frac{2(((2-3\gamma+\gamma^2)\Gamma(1-\gamma)X)^{\frac{1}{2-\gamma}})^{3-\gamma}}{\Gamma(1-\gamma)(-6+11\gamma-6\gamma^2+\gamma^3)} \tag{14}$$

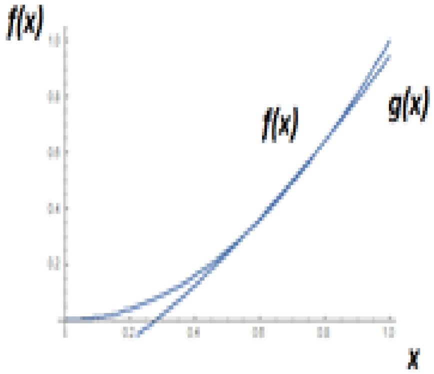

Fig. 1: The function $f(x) = x^5$ in the initial space.

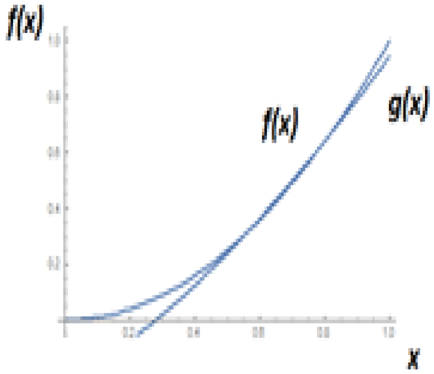

Fig. 2: The function $f(x) = x^5$ in the initial space.

Therefore, the curve in the original plane (x, f(x)) is shown in Fig.1
corresponds to the Λ-fractional plane (space) shown in Fig.2, for $\gamma = 0.6$. Thus, the derivative

$$\frac{dF(X)}{dX} = \frac{(24(5-\gamma)(2-3\gamma+\gamma^2)\Gamma(1-\gamma)(2-3\gamma+\gamma^2)X\Gamma(1-\gamma))^{\frac{3}{2-\gamma}}}{(2-\gamma)\Gamma(6-\gamma)} \tag{15}$$

For X_0=0.6 and γ=0.6, the derivative in the Λ-fractional plane is equal to $D(F(X_0)) = 1.1580$. Since the Λ-derivative behaves in the conventional way in the Λ-fractional space, the tangent Y(X) of the curve at a point X0 is defined by the line,

$$Y(X) = F(X_0) + \frac{d}{dX}(F(X_0)(X - X_0) \tag{16}$$

In the original plane (x, f(x)) the corresponding tangent space is defined by the curve that will be built as follows: Recalling Eq.(9), the x_0=0.81 in the initial plane, corresponding to X_0=0.60 is defined. Then substituting in the derivative $\frac{dF(X)}{dX}$ in the fractional plane the X as a function of x, the ${}_a^{\Lambda}D_x^{\gamma}f(x)$ is defined. Hence the corresponding function in the real space (x, f(x)) may be pulled back by the relation ${}_a^{RL}D_x^{1-\gamma}({}_a^{\Lambda}D_x^{\gamma}f(x))$.Indeed

$${}_a^{RL}D_x^{1-\gamma}({}_a^{\Lambda}D_x^{\gamma}f(x)) = \frac{1}{\Gamma(\gamma)}\frac{d}{dx}\int_a^x \frac{{}_a^{\Lambda}D_x^{\gamma}f(x)}{(x-s)^{1-\gamma}}ds \tag{17}$$

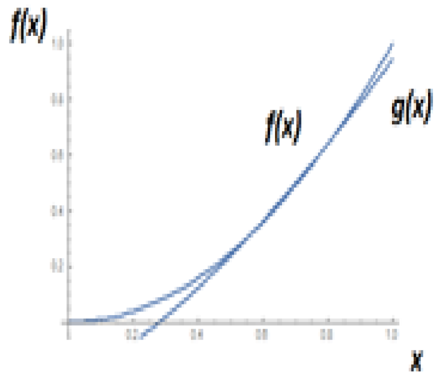

Fig. 3: The function $f(x) = x^5$ in the initial space.

In the present case for the function $f(x) = x^2$

$$ {}_{a}^{RL}D_x^{1-\gamma}({}_{a}^{\Lambda}D_x^{\gamma}x^2) = 1.41 \tag{18}$$

Thus, the fractional tangent space g(x) in the original space (x,f(x)) is defined by

$$ g(x) = f(x)_{x_0} + {}_{a}^{RL}D_x^{1-\gamma}({}_{a}^{RL}D_x^{\gamma}f(x)(\frac{dF(X_0)}{dX})_{x=x_0})(\frac{x^{2-\gamma}}{\Gamma(1-\gamma)(2-3\gamma+\gamma^2)} - X_0) \tag{19}$$

In the present case at $X_0 = 0.6$ for γ=0.6, $x_0 = 0.81$ the tangent space is defined by

$$ g(x) = (x^2)_{x=0.81} + 1.41(\frac{1.79x^{1.4}}{\Gamma(0.4)} - 0.6) \tag{20}$$

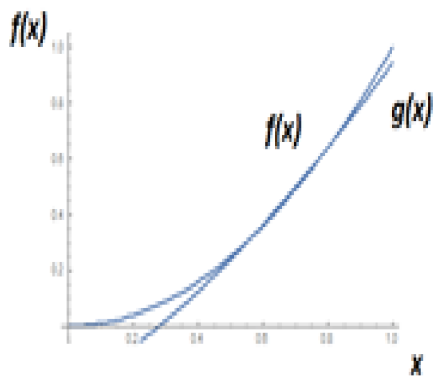

Fig. 4: The function $f(x) = x^5$ in the initial space.

4 The fractional arc-length

Let y=f(x) be a function, with fractional derivative of order $0<\gamma<1$. The fractional differential in the Λ-fractional plane (X, Y(X)) is defined by:

$$dY(X)=\frac{dY(X)}{dX}dX \tag{21}$$

where X and Y(X) are defined by $X={}_aI_x^{1-\gamma}x$ and $F(X)={}_aI_x^{1-\gamma}f(x)$. Then the arc-length in the Λ-Fractional Plane is defined by:

$$S(X)=\left(\frac{dF(X)^2}{dX^2}+1\right)^{1/2}dX \tag{22}$$

Furthermore, the arc-length s(x) in the original plane is defined by,

$$s(x)={}_a^{RL}D_x^{1-\gamma}(S(X))={}_a^{RL}D_x^{1-\gamma}\left(S\left(\frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}\right)\right) \tag{23}$$

Nevertheless, for the parametric curves of the type:

$$x=g(t), \qquad y=f(t) \tag{24}$$

The fractional differential of the arc-length in the Λ- fractional plane is expressed by:

$$dS(T)=\sqrt{\frac{dY(T)}{dT}^2+\frac{dX(T)}{dT}^2} \tag{25}$$

and the arc-length

$$S(T)=\int_0^T dS(T) \tag{26}$$

The arc-length s(t) in the original plane is defined by the integral equation,

$$s(t)={}_a^{RL}D_t^{1-\gamma}(S(T))={}_a^{RL}D_t^{1-\gamma}\left(S\left(\frac{t^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}\right)\right) \tag{27}$$

5 The fractional tangent space of a space curve

Let a representation of a space curve C be r=r(s) in the initial space, where s is the fractional length of the curve. Then, the fractional tangent space of the curve in the Λ-space is defined by the first order derivative:

$$\mathbf{R_1}=\frac{d^\gamma\mathbf{r}}{d^\gamma s}=\frac{dI^{1-\gamma}\mathbf{r}}{dI^{1-\gamma}s}=\frac{d\mathbf{R}(S)}{dS} \tag{28}$$

Since,

$$d|\mathbf{R}(S)|=|dS| \tag{29}$$

The length $|\mathbf{R}_1|$ of the tangent vector in the Λ-Fractional Space is unity. Further, the corresponding tangent vector expressed in variables of the original plane is defined through the equation,

$$R_1(s)=\mathbf{R_1}(S)=\mathbf{R_1}\left(\frac{s^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}\right) \tag{30}$$

In addition, that tangent vector may be pulled back to the original space through the equation,

$$r^t(s_0)={}_a^{RL}D_t^{1-\gamma}\mathbf{R_1}\left(\frac{s_0^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}\right) \tag{31}$$

The tangent space of the curve $\mathbf{r^1}=\mathbf{r}(s)$ at the point $\mathbf{r_0}=\mathbf{r}(s_0)$ is defined through the Fractional Space with,

$$\mathbf{R^t}=\mathbf{R}(S_0)+k\mathbf{R_1}(S_0) \qquad 0<k \tag{32}$$

and the corresponding tangent space in the original space may be defined with,

$$\mathbf{r^t}(s) = \mathbf{r}(s_0) + ({}_a^{RL}D_t^{1-\gamma}\mathbf{R_1}(\frac{s_0^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}))(\frac{s^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} - S_0) \quad (33)$$

The study may go on for the definition of the Fractional curvature and fractional radius of curvature in the fractional space, following conventional approaches in the Λ-fractional space and then, the results may be pulled back to the original space. The plane through $\mathbf{R_0} = \mathbf{R}(S_0)$, orthogonal to the tangent line at $\mathbf{R_0}$ defines the normal plane to the curve at S_0. That normal plane in the Λ-fractional space is defined by:

$$(\mathbf{Y} - \mathbf{R}(S_0)) \cdot \mathbf{T}(S_0) = (\mathbf{Y} - \mathbf{R}(S_0)) \cdot \mathbf{R_1}(S_0) = 0 \quad (34)$$

The corresponding normal space γ in the original space is defined by,

$$\mathbf{y}(s) = {}_a^{RL}D_t^{1-\gamma}\mathbf{Y}(S(s)) \quad (35)$$

6 Fractional curvature of curves

Considering the fractional tangent vector in the Λ-fractional space:

$$\mathbf{T} = \mathbf{R_1}(S) = \frac{dR(S)}{dS} = {}_a^{\Lambda}D_t^{\gamma}(\mathbf{r}(s)) \quad (36)$$

The fractional derivative:

$$\mathbf{R_2}(S) = \frac{dT}{dS} = {}_a^{\Lambda}D_s^{\gamma}({}_a^{\Lambda}D_s^{\gamma}\mathbf{T}) = \mathbf{T_1}(S) \quad (37)$$

The fractional curvature vector $\mathbf{K}$ on the curve C at the point $\mathbf{R}(S)$ is defined by

$$\mathbf{K} = \mathbf{K}(S) = \mathbf{T_1} \quad (38)$$

In fact, the fractional curvature vector $\mathbf{T_1}$ on the curve C in the Λ-fractional space is orthogonal to $\mathbf{T}$ and parallel to the fractional normal plane. The fractional curvature of C at $\mathbf{R}$(S) in the Λ-fractional space is the magnitude of the Fractional curvature vector:

$$K = |\mathbf{K}(S)| \quad (39)$$

Likewise, the fractional radius of curvature in the fractional space is defined as the reciprocal of the curvature K at $\mathbf{R}(S)$:

$$P = \frac{1}{K} = |\frac{1}{\mathbf{K}(S)}| \quad (40)$$

7 The fractional Serret-Frenet equations

Let $\mathbf{r}(s)$ be a curve with its conjugate in the Λ-fractional space $\mathbf{R}(S)$ with unit speed, where the fractional velocity vector,

$$\mathbf{T}(S) = \mathbf{R_1}(S) = \frac{dT}{dS} = {}_a^{\Lambda}D_s^{\gamma}\mathbf{r}(S) = \frac{d\mathbf{R}(S)}{dS} \quad (41)$$

is of unit length. Then, the vector

$$\mathbf{T_1}(S) = \mathbf{R_2}(S) = \frac{dT}{dS} = {}_a^{\Lambda}D_s^{\gamma}({}_a^{\Lambda}D_s^{\gamma}\mathbf{r}(S)) = \frac{d\mathbf{R}^2(S)}{dS^2} \quad (42)$$

is normal to the curve **R=R**(S) since $\mathbf{T}(S)\mathbf{T}(S) = 1$ and

$$\mathbf{T_1}(S) \cdot \mathbf{T}(S) = 0 \quad (43)$$

since for Liouville-Riemann's derivative ${}_{a}^{RL}D_{s}^{\gamma}\beta = 0$ for any constant β. Consider $\mathbf{T_1}(S) = K(S)\mathbf{N}(S)$, where $\mathbf{N}(S)$ is the unit principal normal to $\mathbf{R}$ at S. Consequently, there is a possibility for the definition of the equations for the fractional focal curve $\mathbf{C}(S)$ by:

$$(\mathbf{C}(S) - \mathbf{R}(S)) \cdot \mathbf{R_1}(S) = 0 \tag{44}$$

$$(\mathbf{C}(S) - \mathbf{R}(S)) \cdot K(S)\mathbf{N}(S) = 1 \tag{45}$$

Hence, the centre of curvature $\mathbf{C}(S)$ is defined by the point $\mathbf{R}(S) + P(S)\mathbf{N}(S)$ with $P(S) = \frac{1}{K(S)}$.In addition the principal normal vector $\mathbf{N}(S)$, that is orthogonal to the tangent line, is pointing towards the locus of the curvature centers that is called the focal line in the Λ-fractional space. In that fractional space, the binormal unit vector $\mathbf{B}(S) = \mathbf{T}(S) \times \mathbf{N}(S)$ forms a right oriented orthonormal basis $\mathbf{T}(S), \mathbf{N}(S), \mathbf{B}(S)$ for the tangent vector space of the fractional mapping $\mathbf{R}(S)$ of the initial curve $\mathbf{r}(s)$. Also, the derivatives of the aforementioned orthonormal basis with respect to S, i.e $\mathbf{T_1}(S), \mathbf{N_1}(S), \mathbf{B_1}(S)$, depend linearly upon the vectors, $\mathbf{T}(S), \mathbf{N}(S), \mathbf{B}(S)$. Yet, from the evident equations:

$$\mathbf{T_1} \cdot \mathbf{T} = 0 \; with \; \mathbf{T_1} \cdot \mathbf{N} = 0 \; and \; \mathbf{T_1} \cdot \mathbf{N} + \mathbf{N_1} \cdot \mathbf{T} = 0 \tag{46}$$

the Serret-Frenet equations are formulated for the conjugate Λ-fractional space:

$$\mathbf{T_1} = k\mathbf{N} \tag{47}$$

$$\mathbf{N_1} = -k\mathbf{T} + \tau\mathbf{B} \tag{48}$$

$$\mathbf{B_1} = -\tau\mathbf{N} \tag{49}$$

The coefficient τ is the torsion of the curve $\mathbf{R}(S)$ in the fractional space conjugate of the curve $\mathbf{r}(s)$. These equations are the Fractional Equations for the fractional Serret-Frenet system in the fractional Λ-space.

8 The fractional radius of curvature of a plane curve

For a plane curve $\mathbf{r}(s)$, we study the fractional curvature or the conjugate $\mathbf{R}(s)$ in the Λ-fractional space. In fact, according to Porteous [18], we study at each point of the fractional curve $\mathbf{R}(S)$, how closely the neighbourhood of the curve approximates to a parameterized circle. In the Λ-fractional tangent space at a point $\mathbf{R}(T0)$, the circle with centre $\mathbf{C}$ and radius $\mathbf{P}$ is described by all $\mathbf{R}(T)$ in the differential space such that:

$$(\mathbf{R}(T) - \mathbf{C}(T)) \cdot (\mathbf{R}(T) - \mathbf{C}(T)) = P^2 \tag{50}$$

Further Eq.(46) yields:

$$\mathbf{C} \cdot \mathbf{R} - \frac{1}{2}\mathbf{R} \cdot \mathbf{R} = \frac{1}{2}(\mathbf{C} \cdot \mathbf{C} - P^2) \tag{51}$$

with the right-hand side between constant. Therefore, the differentiation of the function:

$$V(\mathbf{C}) : T \to \mathbf{C} \cdot \mathbf{R}(T) - \frac{1}{2}\mathbf{R}(T) \cdot \mathbf{R}(T) \tag{52}$$

Yields,

$$\frac{dV(\mathbf{C})}{dT} = (\mathbf{C} - \mathbf{R}(T)) \cdot \mathbf{R_1}(T) = 0 \tag{53}$$

$$\frac{d^2V(\mathbf{C})}{dT^2} = (\mathbf{C} - \mathbf{R}(T)) \cdot \mathbf{R_2}(T) - \mathbf{R_1}(T) \cdot \mathbf{R_1}(\mathbf{T}) = 0 \tag{54}$$

Suppose that$\mathbf{R}$ is a parametric curve with R(t) in the virtual tangent space of the Λ-fractional space. Then Eq.53 indicates that:

$$\frac{dV(\mathbf{T})}{dT} = 0 \quad or \quad \frac{dV(T)}{dT} = 0 \tag{55}$$

when the tangent vector $\mathbf{R_1}(T)$ in the Λ-fractional space vector is orthogonal to the vector - $\mathbf{R}(T)$, that is the normal line. When $\mathbf{R_2}(T)$ is not linearly dependent upon $\mathbf{R_1}(T)$, there will be a unique point $\mathbf{C}-\mathbf{R}(T)$ on the normal line such that also $\frac{d^2V}{dT^2}=0$.Specifically, for plane curves

$$\mathbf{r}(x)=x\mathbf{e_1}+y(x)\mathbf{e_2} \tag{56}$$

In addition, the corresponding Λ-fractional space with a=0 is defined by

$$X=\frac{1}{\Gamma(1-\gamma)}\int_0^x \frac{s}{(x-s)^\gamma}ds \quad , \quad Y(X)=\frac{1}{\Gamma(1-\gamma)}\int_0^x \frac{y(s)-y(0)}{(x-s)^\gamma}ds \tag{57}$$

Then, the equation defining in the Λ-fractional space the fractional centres of curvature $\mathbf{C}=C1\mathbf{e_1}+C_2\mathbf{e_2}$ become,

$$(C_1-X)+(C_2-Y(X)))\frac{dY(X)}{dX}=0 \tag{58}$$

$$(C_2-Y(X))\frac{d^2Y(X)}{dX^2}-(1+(\frac{dY(X)}{dX})^2)=0 \tag{59}$$

Further, in the Λ-fractional space, the fractional radius of curvature is defined by,

$$\mathbf{P}^\gamma=P_1^\gamma\mathbf{e_1}+P_2^\gamma\mathbf{e_2}=(C_1-X)\mathbf{e_1}+(C_2-Y(X))\mathbf{e_2} \tag{60}$$

Hence, the fractional curvature in the Λ-fractional space is defined by its components,

$$P_1^\gamma=-\frac{1+(\frac{dY(X)}{dX})^2}{\frac{dY^2(X)}{dX^2}}\frac{dY(X)}{dX} \tag{61}$$

$$P_2^\gamma=\frac{1+(\frac{dY(X)}{dX})^2}{\frac{dY^2(X)}{dX^2}} \tag{62}$$

9 Applications

a.The fractional geometry of a parabola.

Let $\mathbf{r}$ be a parabola $t\rightarrow(t,t^2)$. Then its vector equation in the original space is defined by,

$$\mathbf{r}(t)=t\mathbf{e_1}+t^2\mathbf{e_2} \tag{63}$$

Hence the fractional Λ-space is defined by,

$$\mathbf{R}(T)={}_aI_x^{1-\gamma}t\mathbf{e_1}+{}_aI_x^{1-\gamma}t^2\mathbf{e_2}=\frac{t^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}\mathbf{e_1}-\frac{2t^{3-\gamma}}{(-6+11\gamma-6\gamma^2+\gamma^3)\Gamma(1-\gamma)}\mathbf{e_2} \tag{64}$$

Consequently,

$$T=\frac{t^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} \tag{65}$$

and

$$Y(T)=-\frac{2t^{3-\gamma}}{(-6+11\gamma-6\gamma^2+\gamma^3)\Gamma(1-\gamma)}=-\frac{2((2-3\gamma+\gamma^2)T)^{\frac{3-\gamma}{2-\gamma}}\Gamma(1-\gamma)^{\frac{1}{2-\gamma}}}{(-6+11\gamma-6\gamma^2+\gamma^3)} \tag{66}$$

The functions $y(t)=t^2$ in the original space and (T,Y(T)) in the fractional Λ-space are represented in the Figs.(5,6) Therefore,

$$\mathbf{R}(T)=T\mathbf{e_1}+Y(T)\mathbf{e_2} \tag{67}$$

and

$$\mathbf{R_1}(T)=\mathbf{e_1}+\frac{\mathbf{dY(T)}}{\mathbf{dT}}\mathbf{e_2}=\mathbf{e_1}-\frac{2((2-3\gamma+\gamma^2)T\Gamma(1-\gamma))^{\frac{1}{2-\gamma}}}{-2+\gamma}\mathbf{e_2} \tag{68}$$

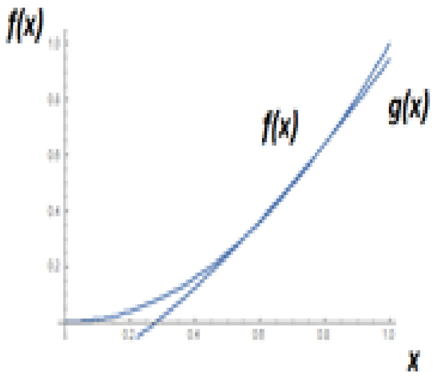

Fig. 5: The curve in the original space.

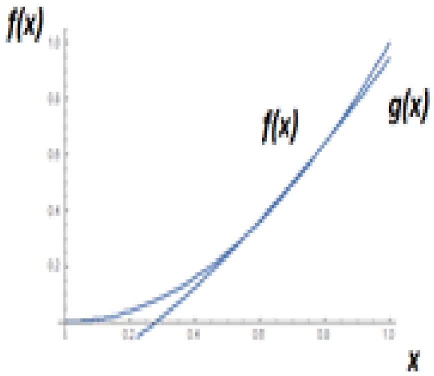

Fig. 6: The conjugate curve in the fractional Λ- space.

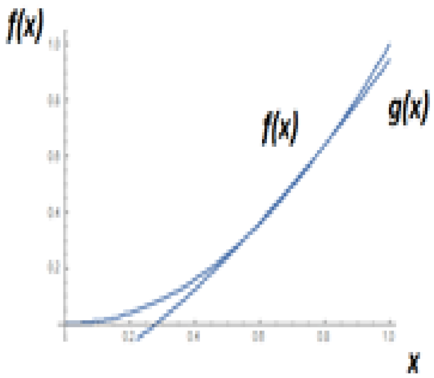

Fig. 7: The curve and its tangent in the fractional Λ- space for $\gamma = 0.6$ and Y=0.6.

Hence the tangent G(T) line at a point T_0 in the fractional Λ-space is defined by:

$$G(T) = Y(T_0) + \frac{dY(T)}{dT}|_{T_0}(T - T_0) \tag{69}$$

Let us remind that the relation between t and T is,

$$T = \frac{t^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} \tag{70}$$

The corresponding tangent space in the original space in the original space (t, t^2) is defined by the curve,

$$f(t) = t_0^2 + {}_0^C D_{t_0}^{1-\gamma}(\frac{dY(T)}{dT})(\frac{t^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} - T_0) \tag{71}$$

For the specific case of $\gamma = 0.6$ and $T_0 = 0.6$ in the Λ- fractional space, the corresponding point in the original space is t_0=0.81. Therefore the fractional tangent space g(t) in the original plane is defined by:

$$g(t) = 0.6561 + 1.4083(0.8051t^{1.4} - 0.6) \tag{72}$$

Fig 8. shows the original curve and the fractional tangent space of the curve. Furthermore, the curvature centers of the

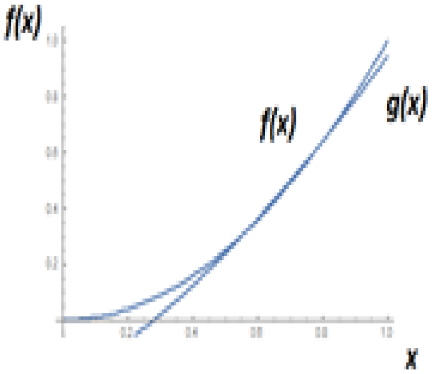

Fig. 8: The curve y(t) and the fractional tangent space in the initial space at the point t=0.81 and $\gamma = 0.6$

parabola in the Λ-space describe a curve,

$$\mathbf{r}(x) = C_1(T)\mathbf{e_1} + C_2(T)\mathbf{e_2} \tag{73}$$

become

$$(C_1 - T) + (C_2 - Y(T)))\frac{dY(T)}{dT} = 0 \tag{74}$$

$$(C_2 - Y(T))\frac{d^2Y(T)}{dT^2} - (1 + (\frac{dY(T)}{dT})^2) = 0 \tag{75}$$

Simce,

$$Y(T) = -\frac{2((2-3\gamma+\gamma^2)T)^{\frac{3-\gamma}{2-\gamma}}\Gamma(1-\gamma)^{\frac{1}{2-\gamma}}}{(-6+11\gamma-6\gamma^2+\gamma^3)} \tag{76}$$

and

$$\frac{dY(T)}{dT} = -\frac{2((2-3\gamma+\gamma^2)T\Gamma(1-\gamma))^{\frac{1}{2-\gamma}}}{(-2+\gamma)} \tag{77}$$

$$\frac{d^2Y(T)}{dT^2} = \frac{2((2-3\gamma+\gamma^2)T\Gamma(1-\gamma))^{\frac{1}{2-\gamma}}}{(-2+\gamma)^2 T} \tag{78}$$

Hence, solving the system, the curve of curvature centers is defined in the Λ-space with

$$C_1 = T + 1.671^{0.71}(-T + 0.84T^{0.29}(1 + 2.78T^{1.43}) \tag{79}$$

$$C_2 = 0.84T^{0.29}(1 + 2.78T^{1.43}) \tag{80}$$

Recalling the equation for the curve G(T) , the curve of the centers of curvature and the corresponding conjugate curve Y(T) in the Λ-space is shown in Fig.9 . Proceeding to the definition of the curve c(t) of the image of the curve $\mathbf{C}(T)$ in

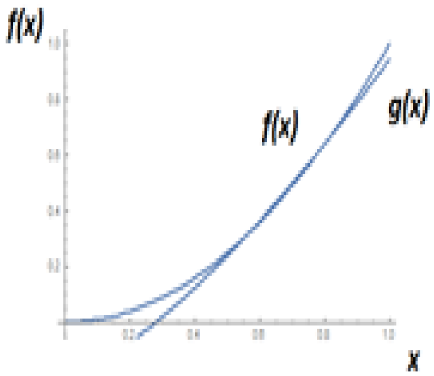

Fig. 9: The conjugate curve Y(T) and the fractional tangent space in the Λ-space for $\gamma = 0.6$

the initial plane (t,y(t)), where

$$\mathbf{c}(t) = c_1\mathbf{e_1} + c_2\mathbf{e_2} \tag{81}$$

Indeed,

$$\mathbf{c_i}(t) = {}_a^C D_{t_0}^{1-\gamma}(C_i(T)) \tag{82}$$

With

$$T = \frac{t^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} \tag{83}$$

Performing the algebra,

$$c_1 = 2.4t - 1.7^2 + 3.9^3 \tag{84}$$

$$c_2 = 0.7t - 3.2t^3 \tag{85}$$

The image of the curve of curvature centers in the original space is shown in Fig.10

10 On the Λ-fractional differential geometry of surfaces.

Let us consider a manifold with points M(u,v) defined by the vectors

$$M(u,v) = \mathbf{x}(u,v) \tag{86}$$

with,

$$x_i = x_i(u,v), \quad u_1 \leqslant u \leqslant u_2, \quad v_1 \leqslant v \leqslant v_2, \quad i = 1,2,3. \tag{87}$$

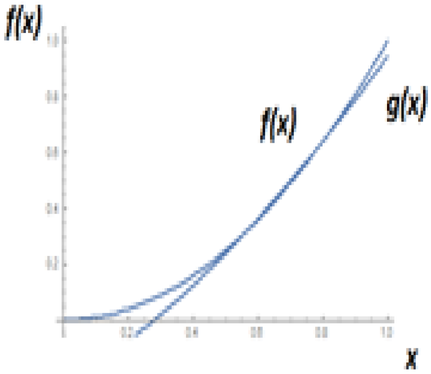

Fig. 10: The original space with the curve y(t) and the image of curve centers c(t) for $\gamma = 0.6$

in the initial space x_i. Transferring the surface in the fractional Λ-space (X, Y, Z) the manifold is defined by,

$$X = \frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}, \tag{88}$$

$$Y = \frac{y^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}, \tag{89}$$

$$Z = {}_aI_x{}^{1-\gamma}{}_aI_x^{1-\gamma} z(x,y) = \frac{1}{\Gamma(1-\gamma)^2}\int_b^y \left(\int_a^x \frac{z(s,t)}{(x-s)^\gamma} ds\right)\frac{dt}{(y-t)^\gamma}, \tag{90}$$

$$z = x^4 y^2 \qquad 0 < x < 1, \qquad 0 < y < 1 \tag{91}$$

in the initial space, the corresponding manifold in the Λ-fractional space (X,Y,Z) is defined

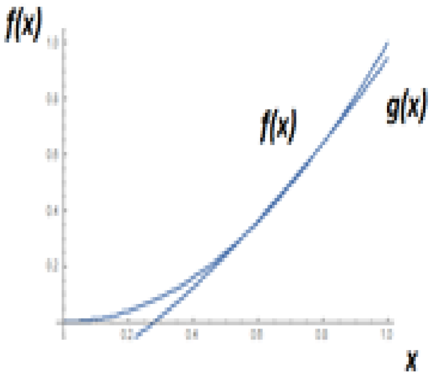

Fig. 11: The surface z in the initial space.

By Eqs.(88,89) and Eq.(90) with a=b=0,

$$Z = -\frac{48\cdot((X\cdot(2-3\gamma+\gamma^2)\cdot\Gamma(1-\gamma))^{\frac{1}{2-\gamma}})^{5-\gamma}\cdot((Y\cdot(2-3\gamma+\gamma^2)\cdot\Gamma(1-\gamma))^{\frac{1}{2-\gamma}})^{3-\gamma}}{(-6+11\gamma-6\gamma^2+\gamma^3)\cdot\Gamma(1-\gamma)\cdot\Gamma(6-\gamma)}. \tag{92}$$

For γ=0.8, the surface Z in the Λ-fractional space is defined by

$$Z = 1.039193^{3.5} Y^{1.83333} \tag{93}$$

and it is shown in Fig.12

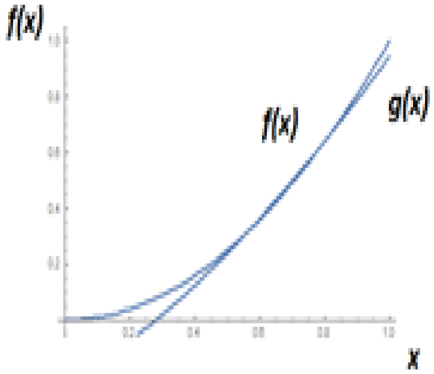

Fig. 12: The surface Z in the Λ-fractional space.

Further, the tangent space of the surface in the Λ-fractional space with γ=0.8, at the point X=Y=0.4 is defined by,

$$Z = (1.0392^{3.5} Y^{1.8333})_{(X=Y=0.4)} + \frac{dZ_{(X=Y=0.4)}}{dX}(X-0.4) + \frac{dZ_{(X=Y=0.4)}}{dY}(Y-0.4). \tag{94}$$

Simplifying Eq.(**??**, the surface Z becomes:

$$Z = 0.00769123 + 0.0673(X-0.4) + 0.035251(Y-0.4), \tag{95}$$

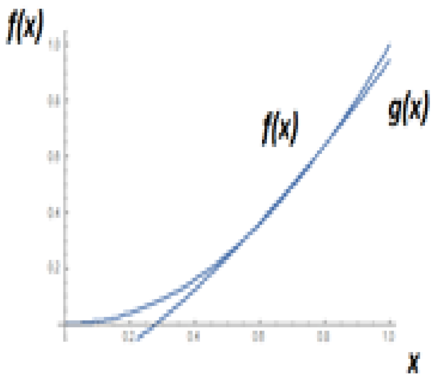

Fig. 13: The surface with the tangent space in the Λ-fractional space at the point X=Y=0.4.

The corresponding surface in the initial space to the tangent plane in the Λ-fractional space is defined by,

$$z = (x^4 y^2)_{(x=y=0.5052)} + ({}_a^{RL}D_{y=0.5052}^{1-\gamma}\ {}_a^{RL}D_{x=0.5052}^{\gamma}(\frac{dZ}{dX}))(X(x)-0.4)) + ({}_a^{RL}D_{y=0.5052}^{1-\gamma}\ {}_a^{RL}D_{x=0.5052}^{\gamma}(\frac{dZ}{dY}))(Y(y)-0.4). \tag{96}$$

The surface defined by Eq.(96) is shown in Fig.14 in the initial space.

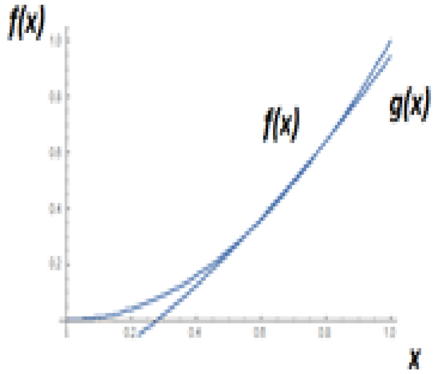

Fig. 14: The surface with its tangent surface at the point x=y=0.5052.

11 Fundamental differential forms on fractional differential manifolds

a.The first fractional fundamental form in the Λ-space.

Following formal procedure [17], the first fractional fundamental form in the fractional Λ-space is defined by the quantity,

$$I = d\mathbf{X} \cdot d\mathbf{X} = (\frac{d\mathbf{X}}{dU}dU + \frac{d\mathbf{X}}{dV}dV) \cdot (\frac{d\mathbf{X}}{d^{\gamma}U}dU + \frac{d\mathbf{X}}{d^{\gamma}V}dV) = EdU^2 + 2FdUdV + GdV^2 \tag{97}$$

with,

$$E = \frac{d\mathbf{X}}{dU} \cdot \frac{d\mathbf{X}}{dU}, \quad F = \frac{d\mathbf{X}}{dU} \cdot \frac{d\mathbf{X}}{dV}, \quad G = \frac{d\mathbf{X}}{dV} \cdot \frac{d\mathbf{X}}{dV}, \tag{98}$$

Since the first fractional fundamental form in the Λ space should be positive definite, it should apply that

$$EG - F^2 > 0. \tag{99}$$

a.The second fractional fundamental form in the Λ-space.

Consider the manifold $M(u,v) = \mathbf{x}(u,v)$. Then, the fractional manifold M^{Λ} in the fractional Λ-space is defined by,

$$M^{\Lambda}(U,V) = \mathbf{X}(U,V) > 0, \tag{100}$$

where, U,V are defined by similar Eqs.(24,25) Further, there is a fractional unit normal $\mathbf{N}$ at each point of the fractional manifold to the fractional tangent plane, in the Λ-space,

$$\mathbf{N} = \frac{\frac{d\mathbf{X}}{dU} \times \frac{d\mathbf{X}}{dV}}{|\frac{d\mathbf{X}}{dU} \times \frac{d\mathbf{X}}{dV}|} \tag{101}$$

that is a function of U and V with the fractional differential,

$$d\mathbf{N} = \frac{d\mathbf{N}}{dU}dU + \frac{d\mathbf{N}}{dV}dV. \tag{102}$$

Recalling that in the Λ-fractional space the derivatives are local and follow the rules of the common ones, they become zero for constants and taking into consideration that $\mathbf{N} \cdot \mathbf{N} = 1$, we get

$$d\mathbf{N} \cdot \mathbf{N} = 0. \tag{103}$$

Indicating that the vector $d\mathbf{N}$ is parallel to the fractional tangent space in the Λ-space.The second fractional fundamental form is defined by [17],

$$II = -d\mathbf{X}\cdot d\mathbf{N} = -(\frac{d\mathbf{X}}{dU}dU + \frac{d\mathbf{X}}{dV}dV)\cdot(\frac{d\mathbf{N}}{d^{\gamma}U}dU + \frac{d\mathbf{N}}{d^{\gamma}V}dV,) = LdU^2 + 2MdUdV + NdV^2 \tag{104}$$

where,

$$L = -\frac{d\mathbf{X}}{dU}\cdot\frac{d\mathbf{N}}{dU}, \quad F = -\frac{1}{2}(\frac{d\mathbf{X}}{dU}\cdot\frac{d\mathbf{N}}{dV} + \frac{d\mathbf{N}}{dU}\cdot\frac{d\mathbf{X}}{dV}), \quad N = -\frac{d\mathbf{X}}{dV}\cdot\frac{d\mathbf{N}}{dV}. \tag{105}$$

12 The fractional normal curvature

Let p be a point on a surface $\mathbf{x} = \mathbf{x}(u,v)$ and $\mathbf{x}(t) = \mathbf{x}(u(t),v(t))$ a regular curve c at p in the original space. Then, the corresponding surface in Λ-fractional space is defined by $\mathbf{X} = \mathbf{X}(U,V)$. Further the corresponding curve C in the Λ-space is defined by $\mathbf{X}(T) = \mathbf{X}(U(T),V(T))$, V(T) passing through the corresponding to p, P point in the Λ-space. The normal curvature $\mathbf{K_N}$ vector of C at P is the vector projection of the curvature vector $\mathbf{K}$ onto the normal vector $\mathbf{N}$ at P. The component of $\mathbf{K}$ in the direction of the normal $\mathbf{N}$ is called the normal fractional curvature of C at P and is denoted by $\mathbf{K_N}$.Therefore,

$$\mathbf{K_N} = (\mathbf{K}\cdot\mathbf{N})\mathbf{N}. \tag{106}$$

Since the unit tangent to C at P is the vector,

$$\mathbf{T} = \frac{d\mathbf{X}}{dS} = \frac{\frac{d\mathbf{X}}{dT}}{|\frac{d\mathbf{X}}{dT}|}, \tag{107}$$

where S denotes the fractional arc length of the curve in the fractional λ-space, and $\mathbf{T}$ is the unit perpendicular to the normal $\mathbf{N}$ along the curve, see Lazopoulos [16], we get,

$$0 = \frac{(\mathbf{T}\cdot\mathbf{N})}{dT} = (\frac{d\mathbf{T}}{dT}\cdot\mathbf{N} + \mathbf{T}\cdot\frac{d\mathbf{N}}{dT}). \tag{108}$$

Therefore, the normal curvature of a curve in the fractional Λ-space is equal to:

$$K_N = \mathbf{K}\cdot\mathbf{N} = \frac{\frac{d\mathbf{T}}{dT}\cdot\mathbf{N}}{|\frac{d\mathbf{X}}{dT}|} = -\mathbf{T}\cdot\frac{\frac{d\mathbf{N}}{dT}}{|\frac{d\mathbf{X}}{dT}|} = -\frac{d\mathbf{X}}{dT}\cdot\frac{\frac{d\mathbf{N}}{dT}}{|\frac{d\mathbf{X}}{dT}|^2} = -\frac{(\frac{d\mathbf{X}}{dU}\frac{dU}{dT} + \frac{d\mathbf{X}}{dV}\frac{dV}{dT})\cdot(\frac{d\mathbf{N}}{dU}\frac{dU}{dT} + \frac{d\mathbf{X}}{dV}\frac{dV}{dT})}{(\frac{d\mathbf{X}}{dU}\frac{dU}{dT} + \frac{d\mathbf{X}}{dV}\frac{dV}{dT})\cdot(\frac{d\mathbf{N}}{dU}\frac{dU}{dT} + \frac{d\mathbf{X}}{dV}\frac{dV}{dT})} = \frac{L(\frac{dU}{dT})^2 + 2M\frac{dU}{dT}\frac{dV}{dT} + N(\frac{dV}{dT})^2}{E(\frac{dU}{dT})^2 + 2F\frac{dU}{dT}\frac{dV}{dT} + G(\frac{dV}{dT})^2}. \tag{109}$$

Recalling Eqs.(97,104), the normal curvature is defined by,

$$K_N = \frac{II}{I} \tag{110}$$

13 Definition of Christoffel coefficients of fractional order

The idea of covariant derivative is a generalization of the directional derivative of a scalar function to the vector and general tensor fields. If $v_x \in R^n_x$ and $\mathbf{F}$ is a vector field, then

$$\nabla_{v_x}\mathbf{F} = \lim_{t\to 0}\frac{\mathbf{F}(\mathbf{x}+\mathbf{tv_x}) - \mathbf{F}(\mathbf{x})}{t} = \frac{d}{dt}\mathbf{F}(\mathbf{x}+\mathbf{tv_x}|_{t=0}. \tag{111}$$

If $\mathbf{F}$=(u^1,.................,u”(x)), then the covariant derivative of the vector F with respect to vector field $\mathbf{v}$ is given by,

$$\nabla_{v_x}\mathbf{F} = (\nabla_{v_x}u^1, ...,\nabla_{v_x}u''(x)) \tag{112}$$

Hence,

$$(\nabla_v\mathbf{F})_x = \nabla_{v_x}\mathbf{F} \tag{113}$$

A generic point $M \in R^n$ is defined by the coordinates,

$$x_j = f_j(u_i,u_2,......u_n) = f_j(\mathbf{u}) \tag{114}$$

Recalling Einstein's contraction convention, we have

$$dM = \partial_i M du_i = e_i du_i \tag{115}$$

with

$$e_i = \partial_i M du_i = D_{u_i} M \tag{116}$$

Furthermore,

$$de_i = \partial_k e_i du_k \tag{117}$$

Likewise, we may write,

$$\partial_k e_j = \Gamma^i_{jk} e_i \tag{118}$$

where, Γ^i_{jk} is the Christoffel coefficient. Hence,

$$de_i = \Gamma^i_{jk}(du_k)e_i \tag{119}$$

Further, differentiating the fundamental tensor, $g_{ij}= e_i e_j$

$$dg_{ij} = e_i de_j + de_i e_j = g_{ih}\Gamma^i_{jk}(du_k) + g_{jh}\Gamma^h_{jk}(du_k). \tag{120}$$

Consequently,

$$\partial_k g_{ij} = g_{ih}\Gamma^i_{jk} + g_{jh}\Gamma^h_{jk}. \tag{121}$$

The contravariant tensor g^{ij} of the tensor g_{ij} is defined by,

$$g_k g_{hj} = \delta^i_j. \tag{122}$$

14 The Λ-fractional covariant derivative

For the vector in the Λ-fractional space

$$\mathbf{V}(\mathbf{t}) = V_i(t)E_i(t). \tag{123}$$

Its covariant fractional derivative $D_{cv}\mathbf{V}$ is defined by

$$\frac{\mathbf{dV}}{dT} = \partial_{cv} V_i E_i. \tag{124}$$

Therefore,

$$\frac{\mathbf{dV}}{dT} = \partial_T V_i E_i + V_i \partial_T E_i = \partial_T V_i E_i + V_i \Gamma^i_{jk} \partial_T U_k E_j = \partial_T V_i E_i + V_j \Gamma^i_{jk} \partial_T U_k E_i. \tag{125}$$

Hence,

$$\partial^a_{cv} v_i = \partial^a_t v_i + \Gamma^i_{jk} \partial^a_{cv} v_j. \tag{126}$$

Discussing, further, the fractional velocity and acceleration on a manifold M $\in R^n$ in the Λ- fractional space,the tangent space is defined by

$$dM = \partial_i M dU_i = E_i d^a U_i. \tag{127}$$

Hence the Λ-velocity is defined by,

$$\overset{\Lambda}{\mathbf{V}} = \frac{d\mathbf{M}}{dT}. \tag{128}$$

Furthermore, if

$$\overset{\Lambda}{\mathbf{V}} = \overset{\Lambda}{V}_i E_i \tag{129}$$

then,

$$\overset{\Lambda}{V}_i = \frac{dU_i}{dT} = \partial_T U_i. \tag{130}$$

In addition, the Λ-fractional acceleration is defined as the covariant derivative of the Λ-fractional velocity

$$\overset{\Lambda}{\mathbf{G}} = \frac{d\mathbf{V}}{dT} = \overset{\Lambda}{G}_i E_i \tag{131}$$

with

$$\overset{\Lambda}{G}_i = \partial_{cv} V_i = \partial_T V_i E_i + \Gamma^i_{jk} V_k V_j. \tag{132}$$

If F(X) is a function in the Λ- fractional space, possibly non-differentiable, with finite Λ-fractional derivative of order $0< \gamma \leq 1$ at a point $\tilde{\mathrm{X}}$ in the Λ-fractional space, then is extremal if

$$dF(\tilde{\mathrm{X}}) = 0 \tag{133}$$

and specifically, it is a local maximum if $d^2F(\tilde{\mathrm{X}}) < 0$, whereas it is a local minimum if $d^2F(\tilde{\mathrm{X}}) > 0$. Recalling that the action integral in the Λ-fractional space,

$$A(T) = \int_0^{T_0} L(Q, \partial_T Q, T) dT. \tag{134}$$

The necessary condition for the extremum of the action integral is,

$$\frac{\partial L}{\partial Q} - \partial_T \left(\frac{\partial L}{\partial(\partial_T Q)}\right) = 0. \tag{135}$$

With the boundary conditions

$$\frac{\partial L}{\partial(\partial_T Q)} = 0, \;\; or \;\; \delta Q = 0 \;\; at \;\; T = 0 \;\; or \;\; T_0. \tag{136}$$

Proceeding to the discussion of the geodesics of a surface $M(U_i) = \mathbf{X}(\mathbf{U_i})$ in the Λ-fractional space, the arc-length S from a point T=0 to any T is defined by,

$$S = \int_0^{T_0} \sqrt{(\partial_T S)^2} dT = \int_0^{T_0} \sqrt{G_{ij} \partial_T U_i \partial_T^a U_j} dT. \tag{137}$$

The minimum length S defined between the points T=0 and T is expressed by,

$$\partial_T \left(\frac{1}{G} G_{ij} \partial_T U_j\right) - \frac{1}{2G} (\partial_T G_{jk}) \partial_T U_j \partial_T^a U_k = 0 \tag{138}$$

with

$$G = \sqrt{G_{ij} \partial_T U_i \partial_T^a U_j}. \tag{139}$$

Considering further,

$$\partial_T G_{jk} = \partial_T (E_j E_k) = (\partial_{cv} E j) E_k + E_j (\partial_{cv} E_k) = (\partial_T E_j + \Gamma^j_{im} (\partial_T U_m E_i) E_k + E_j (\partial_T E_k + \Gamma^j_{in} (\partial_t U_n) E_i). \tag{140}$$

The Eq.140 defines the geodesics on the manifold M in the Λ- fractional space. Then through the well-known transferring rule,

$$f(x) = {}_0^{RL}D_x^{1-\gamma}(F(X(x))) \tag{141}$$

the various results may be transferred to the initial space from the Λ-fractional space.

15 Conclusion

Since the well-known fractional derivatives fail to satisfy the necessary conditions for corresponding to a fractional differential, direct fractional differential geometry is not possible. Adopting the new definition of fractional derivative, the Λ-fractional derivative, along with a new fractional space, the Λ-fractional space, where the Λ-fractional derivative behaves as a conventional one, fractional differential geometry is formulated in the Λ-fractional space. Then the results are transferred into the initial space. The fractional geometry of curves is first discussed. Some further results concerning fractional manifolds, that are of major importance for various applications in mechanics and generally in physics, are summarized in the present work. The fractional Christoffel's coefficients, the Λ-covariant derivative, The fractional velocity and acceleration on a manifold, and the geodesics of a fractional manifold. All those results are necessary for applying fractional calculus in physics.

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