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Application of Lerch Polynomials to Approximate Solution of Singular Fredholm Integral Equations with Cauchy Kernel

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Abstract: In this paper, we develop an effective method based on Lerch polynomials to provide an approximate solution of various cases of Cauchy type singular integral equations. The method reduces the solution of given singular integral equation to the solution of a matrix equation corresponding to linear system of algebraic equations with unknown Lerch coefficients. Error estimation of the presented method is mentioned. Finally, several examples show the reliability and efficiently of the proposed method.

Keywords: Singular integral equations, Lerch polynomials, Cauchy kernel, Error estimation.

1 Introduction

Integral equations play an important role in many branches of modern mathematics and appear in various applications, including fluid mechanics, engineering, contact problems in the theory of elasticity, biology, etc. Several researchers are interested in discussing different types of integral equations [1-10].

Singular integral equations with Cauchy kernels have many applications in a wide variety of physics and enigneering fields like airfoils, fracture mechanics elastodynamics, aerodynamics see [11-15]. Since it is difficult to find analytical solutions of singular integral equations with Cauchy kernels, so many researchers have been developed several numerical methods for solving these equations such as Homotopy perturbation method [16,17], Bernisten method [18,19], Bessel polynomials method [20], Reproducing kernel Hilbert space [21], Differential transform method [22], iteration method [23], Jacobi polynomial method [24], Cubic spline method [25], rational function method [26], Chebyshev polynomials method [27], Nyström method [28] and others. A general form of Cauchy singular integral equation is given as [29]:

$$\int_{-1}^{1}\frac{\varphi(t)}{t-x}dt+\int_{-1}^{1}k(x,t)\varphi(t)dt=f(x), \qquad -1<x<1, \tag{1}$$

where $f(x)$ and $k(x,t)$ are given real-valued continuous functions, $\varphi(x)$ is the unknown function. When $k(x,t)=0$ equation (1) reduced to the following equation:

$$\int_{-1}^{1}\frac{\varphi(t)}{t-x}dt=f(x), \qquad -1<x<1. \tag{2}$$

The complete analytical solutions of equation (2) are given by the following cases see [30,31]:

Case (I) The solution $\varphi(x)$ is bounded at both the end points $x=\pm1$:

$$\varphi(x)=-\frac{\sqrt{1-x^2}}{\pi^2}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}(t-x)}\,dt\ , \tag{3}$$

the solution exist if and only if the following condition satisfied:

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\,dt=0\ . \tag{4}$$

Case (II) The solution $\varphi(x)$ is unbounded at both the end points $x=\pm1$

$$\varphi(x)=-\frac{1}{\pi^2\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-t^2}\,f(t)}{t-x}\,dt+\frac{A}{\sqrt{1-x^2}}, \tag{5}$$

where A is an arbitrary constant.

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Case (III) The solution $\varphi(x)$ is bounded at the end point $x=1$ but unbounded at the end point $x=-1$

$$\varphi(x) = -\frac{1}{\pi^2}\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{f(t)}{t-x}\,dt. \qquad (6)$$

Case (IV) The solution $\varphi(x)$ is bounded at the end point $x=-1$ but unbounded at the end point $x=1$

$$\varphi(x) = -\frac{1}{\pi^2}\sqrt{\frac{1+x}{1-x}}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{f(t)}{t-x}\,dt. \qquad (7)$$

In recent years, Sezer et al. [32] have introduced Lerch matrix collocation method for solving two-dimensional and three- dimensional Volterra type integral and partial integro-differential equations. Also, in [33] Sezer et al. introduced A new approximation based on residual error estimation for the solution of a class of unsteady convection-diffusion problem. In this paper, Lerch matrix collocation method is developed to solve Cauchy singular integral equations of the first kind of the form (2).

2 Lerch polynomials

The explicit formula of the Lerch polynomials are given as [34-36]:

$$L_n(x,\lambda) = \sum_{k=1}^{n}\frac{k!}{n!}\,s(n,k)\binom{k+\lambda-1}{k}x^k, \qquad (8)$$

where $L_0(x,\lambda)=1$ is the initial value, λ is a parameter and $s(n,k)$ is Stirling numbers of the first kind.

Some properties about the Stirling numbers of the first kind can be written as [37,38]:

$$s(n+1,0)=0,\ s(n,n)=1,\ s(n,1)=(-1)^{n-1}\,(n-1)!,$$
$$s(n,n-1)=-\binom{n}{2},\quad n\geq 0.$$

The Lerch polynomials are defined by the generating functions as follows:

$$(1-x\log(1+t))^{-\lambda} = \sum_{n\geq 0} L_n(x,\lambda)t^n.$$

From (8), the first six Lerch polynomials are computed as follows:

$$L_0(x,\lambda)=1,$$
$$L_1(x,\lambda)=\lambda x,$$
$$L_2(x,\lambda) = -\frac{\lambda}{2}x+\frac{\lambda(\lambda+1)}{2}x^2,$$
$$L_3(x,\lambda) = \frac{\lambda}{3}x-\frac{\lambda(\lambda+1)}{2}x^2+\frac{\lambda(\lambda+1)(\lambda+2)}{6}x^3,$$
$$L_4(x,\lambda) = -\frac{\lambda}{4}x+\frac{11\lambda(\lambda+1)}{24}x^2-\frac{\lambda(\lambda+1)(\lambda+2)}{4}x^3$$
$$+\frac{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}{24}x^4,$$
$$L_5(x,\lambda) = \frac{\lambda}{5}x-\frac{5\lambda(\lambda+1)}{12}x^2+\frac{7\lambda(\lambda+1)(\lambda+2)}{24}x^3$$
$$-\frac{\lambda(\lambda+1)(\lambda+2)(\lambda+3)}{12}x^4$$
$$+\frac{\lambda(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)}{120}x^5.$$

Lerch polynomials can be converted into matrix form by using the relation (2.1):

$$\begin{pmatrix} L_0(x,\lambda) \\ L_1(x,\lambda) \\ L_2(x,\lambda) \\ \vdots \\ L_N(x,\lambda) \end{pmatrix} =$$
$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \frac{1!}{1!}s(1,1)\binom{\lambda}{1} & \dots & 0 \\ 0 & \frac{1!}{2!}s(2,1)\binom{\lambda}{1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1!}{N!}s(N,1)\binom{\lambda}{1} & \dots & \frac{N!}{N!}s(N,N)\binom{\lambda+N-1}{N} \end{pmatrix}\begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^N \end{pmatrix}.$$

3 Transformation of equation (2)

The unknown function $\varphi(x)$ in equation (2) can be represented by the following form:

$$\varphi^{(i)}(x) = \frac{\psi^{(i)}(x)\,\Psi^{(i)}(x)}{\sqrt{1-x^2}},\ i=1,2,3,4, \qquad (9)$$

where $\psi^{(i)}(x)$ is a well-behaved function of x in the interval $[-1,1]$ and

$$\begin{cases} \Psi^{(1)}(x) = 1-x^2, & for\ Case\ (I)\ , \\ \Psi^{(2)}(x) = 1, & for\ Case\ (II), \\ \Psi^{(3)}(x) = 1-x, & for\ Case\ (III), \\ \Psi^{(4)}(x) = 1+x, & for\ Case\ (IV). \end{cases} \tag{10}$$

Now, in order to remove the singular term of equation (2) at $t=x$, we have convert equation (2) to the equivalent transformation for cases I, II, III, IV respectively as follows:

Case (I): By using (9) and (10) the unknown function $\varphi^{(1)}(x)$ can be represented in the form:

$$\varphi^{(1)}(x) = \sqrt{1-x^2}\,\psi^{(1)}(x), \qquad -1 \le x \le 1. \tag{11}$$

Substituting from (11) into (2), we have

$$\int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x}\,\psi^{(1)}(t)dt = f(x), \qquad -1 \le x \le 1. \tag{12}$$

Therefore,

$$\int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x}\,\psi^{(1)}(t)dt = \int_{-1}^{1} \sqrt{1-t^2}\,\frac{\psi^{(1)}(t)-\psi^{(1)}(x)}{t-x}\,dt + \int_{-1}^{1} \sqrt{1-t^2}\,\frac{\psi^{(1)}(x)}{t-x}\,dt. \tag{13}$$

In the sense of Cauchy principle value,

$$\int_{-1}^{1} \sqrt{1-t^2}\,\frac{1}{t-x}\,dt = -\pi x, \qquad -1 \le x \le 1. \tag{14}$$

Thus equation (12) can be converted into

$$-\pi x\,\psi^{(1)}(x) + \int_{-1}^{1} \sqrt{1-t^2}\,\frac{\psi^{(1)}(t)-\psi^{(1)}(x)}{t-x}\,dt = f(x), \quad -1 \le x \le 1. \tag{15}$$

Case (II): By using (9) and (10) the unknown function $\varphi^{(2)}(x)$ of equation (2) can be represented in the form:

$$\varphi^{(2)}(x) = \frac{1}{\sqrt{1-x^2}}\,\psi^{(2)}(x), \qquad -1 < x < 1. \tag{16}$$

Substituting from (12) into (2), we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,\frac{\psi^{(2)}(t)}{t-x}\,dt = f(x), \qquad -1 < x < 1. \tag{17}$$

Therefore,

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,\frac{\psi^{(2)}(t)}{t-x}\,dt = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,\frac{\psi^{(2)}(t)-\psi^{(2)}(x)}{t-x}\,dt + \psi^{(2)}(x)\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}(t-x)}\,dt. \tag{18}$$

In the sense of Cauchy principle value,

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}(t-x)}\,dt = 0. \tag{19}$$

Thus equation (17) can be converted into

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,\frac{\psi^{(2)}(t)-\psi^{(2)}(x)}{t-x}\,dt = f(x), -1 < x < 1. \tag{20}$$

Case (III): From (9) and (10) the unknown function $\varphi^{(3)}(x)$ of equation (2) can be represented in the form:

$$\varphi^{(3)}(x) = \sqrt{\frac{1-x}{1+x}}\,\psi^{(3)}(x), \quad -1 < x \le 1. \tag{21}$$

Substituting from (21) into (2), we have

$$\int_{-1}^{1} \sqrt{\frac{1-t}{1+t}}\,\frac{\psi^{(3)}(t)}{t-x}\,dt = f(x), \quad -1 < x \le 1. \tag{22}$$

Therefore,

$$\begin{aligned}\int_{-1}^{1} \sqrt{\frac{1-t}{1+t}}\,\frac{\psi^{(3)}(t)}{t-x}dt &= \int_{-1}^{1} \frac{\sqrt{1-t^2}}{1+t}\,\frac{\psi^{(3)}(t)}{t-x}\,dt \\ &= \int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x}\,\Big(\frac{\psi^{(3)}(t)}{1+t} - \frac{\psi^{(3)}(x)}{1+x} + \frac{\psi^{(3)}(x)}{1+x}\Big)\,dt \\ &= \int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x}\,\Big(\frac{\psi^{(3)}(t)}{1+t} - \frac{\psi^{(3)}(x)}{1+x}\Big)\,dt + \frac{\psi^{(3)}(x)}{1+x}\int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x}\,dt \\ &= \frac{1}{1+x}\Big(\int_{-1}^{1} \sqrt{1-t^2}\,\frac{\psi^{(3)}(t)-\psi^{(3)}(x)}{t-x}\,dt - \int_{-1}^{1} \sqrt{1-t^2}\,\frac{\psi^{(3)}(t)}{1+t}\,dt\Big) - \frac{\pi x\psi^{(3)}(x)}{1+x},\end{aligned}$$

and equation (22) can be converted into

$$\frac{1}{1+x}\Big(\int_{-1}^{1} \sqrt{1-t^2}\,\frac{\psi^{(3)}(t)-\psi_3(x)}{t-x}\,dt - \int_{-1}^{1} \sqrt{1-t^2}\,\frac{\psi^{(3)}(t)}{1+t}\,dt\Big) - \frac{1}{1+x}\pi x\psi^{(3)}(x) = f(x)\,, -1 < x \le 1. \tag{23}$$

Case (IV): From (9) and (10) the unknown function $\varphi^{(4)}(x)$ of equation (2) can be represented in the form:

$$\varphi^{(4)}(x)=\sqrt{\frac{1+x}{1-x}}\,\psi^{(4)}(x),\quad -1\leq x<1. \tag{24}$$

Substituting from (24) into (2), we have

$$\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\,\frac{\psi^{(4)}(t)}{t-x}\,dt=f(x),\quad -1\leq x<1. \tag{25}$$

Similarly, as in case (III) equation (25) can be converted into

$$\frac{1}{1-x}\Big(\int_{-1}^{1}\sqrt{1-t^2}\,\frac{\psi^{(4)}(t)-\psi^{(4)}(x)}{t-x}\,dt+\int_{-1}^{1}\sqrt{1-t^2}\,\frac{\psi^{(4)}(t)}{1-t}\,dt\Big)-\frac{1}{1-x}\pi x\psi^{(4)}(x)=f(x), \tag{26}$$
$$-1\leq x<1.$$

In equations (15), (20), (23) and (26), $\frac{\psi(t)-\psi(x)}{t-x}=\psi'(x)$ if $t=x$, then $\frac{\psi(t)-\psi(x)}{t-x}\in C([-1,1]\times[-1,1])$ for any case, the singularity of equation (2) has been removed.

4 The method of solution

The approximate solution of equations (15), (20), (23) and (26) is assumed to be in the truncated Lerch series form as:

$$\psi^{(i)}(x)\cong\psi_N^{(i)}(x,\lambda)=\sum_{n=0}^{N}a_n^{(i)}\,L_n(x,\lambda),i=1,2,3,4, \tag{27}$$

where $i=1,2,3,4$ for case I, II, III and IV respectively, $a_n^{(i)}$ are the unknown Lerch coefficients for $n=0,1,\ldots,N$, and $L_n(x,\lambda)$ is Lerch polynomials which is defined by (8).

We rewrite equations (15), (20), (23) and (26) in the following form:

$$F^{(i)}(x)+G^{(i)}(x)=f(x),\qquad i=1,2,3,4, \tag{28}$$

where

$$F^{(1)}(x)=\int_{-1}^{1}\sqrt{1-t^2}\,\frac{\psi^{(1)}(t)-\psi^{(1)}(x)}{t-x}\,dt,$$
$$G^{(1)}(x)=-\pi x\psi^{(1)}(x),\qquad for\ case\ (I), \tag{29}$$

$$F^{(2)}(x)=\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\,\frac{\psi^{(2)}(t)-\psi^{(2)}(x)}{t-x}dt,$$
$$G^{(2)}(x)=0,\ for\ case\ (II), \tag{30}$$

$$F^{(3)}(x)=\frac{1}{1+x}\Big(\int_{-1}^{1}\sqrt{1-t^2}\frac{\psi^{(3)}(t)-\psi^{(3)}(x)}{t-x}dt-\int_{-1}^{1}\sqrt{1-t^2}\frac{\psi^{(3)}(t)}{1+t}dt\Big),$$
$$G^{(3)}(x)=-\frac{\pi x\psi^{(3)}(x)}{1+x},\ for\ case\ (III), \tag{31}$$

$$F^{(4)}(x)=\frac{1}{1-x}\Big(\int_{-1}^{1}\sqrt{1-t^2}\,\frac{\psi^{(4)}(t)-\psi^{(4)}(x)}{t-x}\,dt+\int_{-1}^{1}\sqrt{1-t^2}\,\frac{\psi^{(4)}(t)}{1-t}\,dt\Big),$$
$$G^{(4)}(x)=-\frac{\pi x\psi^{(4)}(x)}{1-x},for\ case\ (IV). \tag{32}$$

By using the collocation points

$$x_l=-1+\frac{2}{N}l,l=0,1,\ldots,N, \tag{33}$$

into equation (28), we get

$$F^{(i)}(x_l)+G^{(i)}(x_l)=f(x_l), \tag{34}$$
$$i=1,2,3,4,\ \ l=0,1,\ldots,N.$$

Substituting from (27) into (29), (30), (31) and (32) we obtain

$$F^{(1)}(x_l,\lambda)\simeq\sum_{n=0}^{N}a_n^{(1)}\int_{-1}^{1}\sqrt{1-t^2}\,\frac{L_n(t,\lambda)-L_n(x_l,\lambda)}{t-x_l}\,dt,\quad for\ case\ (I),$$
$$F^{(2)}(x_l,\lambda)\simeq\sum_{n=0}^{N}a_n^{(2)}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\,\frac{L_n(t,\lambda)-L_n(x_l,\lambda)}{t-x_l}\,dt,\quad for\ case\ (II), \tag{35}$$
$$F^{(3)}(x_l,\lambda)\simeq\frac{1}{1+x_l}\Big[\sum_{n=0}^{N}a_n^{(3)}\int_{-1}^{1}\sqrt{1-t^2}\frac{L_n(t,\lambda)-L_n(x_l,\lambda)}{t-x_l}dt-\sum_{n=0}^{N}a_n^{(3)}\int_{-1}^{1}\sqrt{1-t^2}\,\frac{L_n(t,\lambda)}{1+t}\,dt\Big],\ for\ case\ (III),$$
$$F^{(4)}(x_l,\lambda)\simeq\frac{1}{1-x_l}\Big[\sum_{n=0}^{N}a_n^{(4)}\int_{-1}^{1}\sqrt{1-t^2}\frac{L_n(t,\lambda)-L_n(x_l,\lambda)}{t-x_l}dt+\sum_{n=0}^{N}a_n^{(4)}\int_{-1}^{1}\sqrt{1-t^2}\frac{L_n(t,\lambda)}{1-t}dt\Big],\ for\ case\ (IV),$$

and

$$G^{(1)}(x_l,\lambda) \simeq -\pi x_l \sum_{n=0}^{N} a_n^{(1)} L_n(x_l,\lambda), \ \ for\ case\ (I),$$

$$G^{(2)}(x_l,\lambda) = 0, \ \ for\ case\ (II), \tag{36}$$

$$G^{(3)}(x_l,\lambda) \simeq -\frac{\pi x_l}{1+x_l} \sum_{n=0}^{N} a_n^{(3)} L_n(x_l,\lambda), \ \ for\ case\ (III),$$

$$G^{(4)}(x_l,\lambda) \simeq -\frac{\pi x_l}{1-x_l} \sum_{n=0}^{N} a_n^{(4)} L_n(x_l,\lambda), \ \ for\ case\ (IV).$$

Substituting from (35) and (36) into (34) we obtain

$$\sum_{n=0}^{N} a_n^{(1)} \left[\int_{-1}^{1} \sqrt{1-t^2}\, \frac{L_n(t,\lambda)-L_n(x_l,\lambda)}{t-x_l}\, dt - \pi x_l\, L_n(x_l,\lambda)\right] = f(x_l),\ for\ case\ (I),$$

$$\sum_{n=0}^{N} a_n^{(2)} \left[\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} \frac{L_n(t,\lambda)-L_n(x_l,\lambda)}{t-x_l} dt\right] = f(x_l),\ for\ case\ (II), \tag{37}$$

$$\sum_{n=0}^{N} a_n^{(3)} \frac{1}{1+x_l}\left[\int_{-1}^{1} \sqrt{1-t^2} \frac{L_n(t,\lambda)-L_n(x_l,\lambda)}{t-x_l} dt - \int_{-1}^{1} \sqrt{1-t^2}\, \frac{L_n(t,\lambda)}{1+t} dt - \pi x_l\, L_n(x_l,\lambda)\right] = f(x_l),\ for\ case\ (III),$$

$$\sum_{n=0}^{N} a_n^{(4)} \frac{1}{1-x_l}\left[\int_{-1}^{1} \sqrt{1-t^2}\, \frac{L_n(t,\lambda)-L_n(x_l,\lambda)}{t-x_l}\, dt - \int_{-1}^{1} \sqrt{1-t^2}\, \frac{L_n(t,\lambda)}{1-t}\, dt - \pi x_l\, L_n(x_l,\lambda)\right] = f(x_l),\ for\ case\ (IV).$$

We can write equation (37) in the following form

$$\sum_{n=0}^{N} a_n^{(i)} [\mathbf{F}^{(i)}(x_l,\lambda) + \mathbf{G}^{(i)}(x_l,\lambda)] = f(x_l), \quad i = 1,2,3,4. \tag{38}$$

Hence, the matrix form (38) corresponding to all cases of equation (2) can be written in the form

$$A_i X_i = F, i = 1,2,3,4, \tag{39}$$

where

$$[A_i] = \mathbf{F}^{(i)}(x_l,\lambda) + \mathbf{G}^{(i)}(x_l,\lambda), \ \ i = 1,2,3,4,$$

$$X_i = [a_0^{(i)}, \ \ a_1^{(i)}, \ \ a_2^{(i)}, \ \ \ldots, \ \ a_N^{(i)}]^T, \ \ i = 1,2,3,4$$

and

$$F = [f(x_0), \ \ f(x_1), \ \ \ldots, \ \ f(x_N)]^T.$$

After solving equations (39) for Cases (i), (ii), (iii) and (iv), the unknown coefficients $a_n^{(i)}$ are determined and the approximate solutions of (11), (16), (21) and (24) by using (27) are given by

$$\varphi_N^{(1)}(x,\lambda) = \sqrt{1-x^2} \sum_{n=0}^{N} a_n^{(1)}\, L_n(x,\lambda), \ for\ case\ I,$$

$$\varphi_N^{(2)}(x,\lambda) = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{N} a_n^{(2)}\, L_n(x,\lambda), \ for\ case\ II, \tag{40}$$

$$\varphi_N^{(3)}(x,\lambda) = \sqrt{\frac{1-x}{1+x}} \sum_{n=0}^{N} a_n^{(3)}\, L_n(x,\lambda), \ for\ case\ III,$$

$$\varphi_N^{(4)}(x,\lambda) = \sqrt{\frac{1+x}{1-x}} \sum_{n=0}^{N} a_n^{(4)}\, L_n(x,\lambda), \ for\ case\ IV.$$

5 Error estimation

In this section, we give an error estimation for the approximate solutions of equation (2). Let $\varphi_n(x)$ be approximate solution for all cases I, II, III, IV and $e_n(x) = \varphi(x) - \varphi_n(x)$ be the error function associated with $\varphi_n(x)$ where $\varphi(x)$ is the exact solution of (2) for all cases I, II, III, IV. Since $\varphi_n(x)$ is the approximate solution, it satisfies

$$\int_{-1}^{1} \frac{\varphi_n(t)}{t-x} dt = f(x) + H_n(x) \qquad -1 < x < 1, \tag{41}$$

where $H_n(x)$ is a perturbation term and it is obtained from:

$$H_n(x) = \int_{-1}^{1} \frac{\varphi_n(t)}{t-x} dt - f(x). \tag{42}$$

Subtracting (2) from (41), we obtain

$$\int_{-1}^{1} \frac{e_n(t)}{t-x} dt = H_n(x), \tag{43}$$

for the error function $e_n(x)$. To find an approximation $\hat{e}(x)$ to $e_n(x)$ we can solve (43) by the same ways as we did for (2). In this case, only the function $f(x)$ will be replaced by the perturbation term $H_n(x)$.

6 Numerical examples

In this section several numerical examples are given to illustrate the effectiveness and reliability of Lerch polynomial method. These examples have been solved by our method with $N = 5$ and $\lambda = 1$, all numerical computation were performed by using Maple 18.

Example 1. Consider the following Cauchy integral equation of the first kind [18,21,27].

$$\int_{-1}^{1}\frac{\varphi(t)}{t-x}dt = x^4+5x^3+2x^2+x-\frac{11}{8}, \quad -1<x<1. \quad (44)$$

The exact solution of equation (44) for all cases is given by:

Case (I):

$$\varphi(x) = -\frac{1}{\pi}\sqrt{1-x^2}(x^3+5x^2+\frac{5}{2}x+\frac{7}{2}). \quad (45)$$

Case (II):

$$\varphi(x) = \frac{1}{\pi\sqrt{1-x^2}}(x^5+5x^4+\frac{3}{2}x^3-\frac{3}{2}x^2-\frac{5}{2}x-\frac{7}{2}). \quad (46)$$

Case (III):

$$\varphi(x) = -\frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}(x^4+6x^3+\frac{15}{2}x^2+6x+\frac{7}{2}). \quad (47)$$

Case (IV):

$$\varphi(x) = \frac{1}{\pi}\sqrt{\frac{1+x}{1-x}}(x^4+4x^3-\frac{5}{2}x^2+x-\frac{7}{2}). \quad (48)$$

Now applying Lerch polynomial method for equation (44) and using the collocation points (33) when $N=5,\ \lambda=1$.

For case (I):we obtain the following matrix equation:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{4} & -\frac{25}{24} & \frac{73}{48} & -\frac{1073}{480} \\ \frac{3}{5} & \frac{7}{50} & -\frac{77}{500} & \frac{4591}{15000} & -\frac{24017}{50000} & \frac{370537}{500000} \\ \frac{1}{5} & \frac{23}{50} & -\frac{161}{500} & \frac{5831}{15000} & -\frac{66467}{150000} & \frac{817819}{1500000} \\ -\frac{1}{5} & \frac{23}{50} & -\frac{69}{500} & \frac{3071}{15000} & -\frac{32563}{150000} & \frac{415739}{1500000} \\ -\frac{3}{5} & \frac{7}{50} & \frac{7}{500} & \frac{2071}{15000} & -\frac{5793}{50000} & \frac{90057}{500000} \\ -1 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{24} & -\frac{7}{48} & \frac{47}{480} \end{pmatrix} \begin{pmatrix} a_0^{(1)} \\ a_1^{(1)} \\ a_2^{(1)} \\ a_3^{(1)} \\ a_4^{(1)} \\ a_5^{(1)} \end{pmatrix}$$

$$= \frac{1}{\pi}\begin{pmatrix} -\frac{35}{8} \\ -\frac{11027}{5000} \\ -\frac{7667}{5000} \\ -\frac{5267}{5000} \\ \frac{5773}{5000} \\ \frac{61}{8} \end{pmatrix}.$$

Solving the above matrix equation we obtain the values of the constants as:

$$[a_0^{(1)} = -\frac{7}{2\pi},\ a_1^{(1)} = -\frac{31}{6\pi},\ a_2^{(1)} = -\frac{6}{\pi},\ a_3^{(1)} = -\frac{1}{\pi},$$

$$a_4^{(1)} = a_5^{(1)} = 0].$$

Substituting from these constants into (40) we obtain the approximate solution which is the same as the exact solution (45).

For case (II):we obtain the following matrix equation:

$$\begin{pmatrix} 0 & 1 & -\frac{3}{2}\pi & \frac{17}{6}\pi & -\frac{59}{12}\pi & \frac{128}{15}\pi \\ 0 & 1 & -\frac{11}{10}\pi & \frac{269}{150}\pi & -\frac{1303}{500}\pi & \frac{2451}{625}\pi \\ 0 & 1 & -\frac{7}{10}\pi & \frac{161}{150}\pi & -\frac{2027}{1500}\pi & \frac{3608}{1875}\pi \\ 0 & 1 & -\frac{3}{10}\pi & \frac{101}{150}\pi & -\frac{1153}{1500}\pi & \frac{2173}{1875}\pi \\ 0 & 1 & \frac{1}{10}\pi & \frac{89}{150}\pi & -\frac{237}{500}\pi & \frac{536}{625}\pi \\ 0 & 1 & \frac{1}{2}\pi & \frac{5}{6}\pi & -\frac{1}{12}\pi & \frac{13}{15}\pi \end{pmatrix} \begin{pmatrix} a_0^{(2)} \\ a_1^{(2)} \\ a_2^{(2)} \\ a_3^{(2)} \\ a_4^{(2)} \\ a_5^{(2)} \end{pmatrix} =$$

$$\begin{pmatrix} -\frac{35}{8} \\ -\frac{11027}{5000} \\ -\frac{7667}{5000} \\ -\frac{5267}{5000} \\ \frac{5773}{5000} \\ \frac{61}{8} \end{pmatrix}.$$

Solving the above matrix equation we obtain the values of the constants as:

$$[a_0^{(2)} arbitrary,\ a_1^{(2)} = -\frac{167}{60\pi},\ a_2^{(2)} = \frac{19}{6\pi},\ a_3^{(2)} = \frac{41}{4\pi},$$
$$a_4^{(2)} = \frac{7}{\pi},\ a_5^{(2)} = \frac{1}{\pi}].$$

Substituting from these constants into (40) we obtain the approximate solution in the following form:

$$\varphi_5^{(2)}(x,1) = \frac{1}{\pi\sqrt{1-x^2}}(x^5+5x^4+\frac{3}{2}x^3-\frac{3}{2}x^2-\frac{5}{2}x)+ \frac{a_0^{(2)}}{\sqrt{1-x^2}}. \quad (49)$$

Comparing the exact solution (46) with the approximate solution (49) it is clear that the approximate solution is the same as the exact solution if we take $a_0^{(2)} = -\frac{7}{2\pi}$.

For case (III): we obtain the following matrix equation:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{5}\pi & \frac{16}{25}\pi & -\frac{113}{125}\pi & \frac{2527}{1875}\pi & -\frac{18761}{9375}\pi & \frac{139523}{46875}\pi \\ -\frac{4}{5}\pi & \frac{24}{25}\pi & -\frac{134}{125}\pi & \frac{894}{625}\pi & -\frac{18412}{9375}\pi & \frac{130342}{46875}\pi \\ -\frac{6}{5}\pi & \frac{24}{25}\pi & -\frac{111}{125}\pi & \frac{779}{625}\pi & -\frac{5431}{3125}\pi & \frac{39259}{15625}\pi \\ -\frac{8}{5}\pi & \frac{16}{25}\pi & -\frac{92}{125}\pi & \frac{2212}{1875}\pi & -\frac{15344}{9375}\pi & \frac{113228}{46875}\pi \\ -2\pi & 0 & -\pi & \pi & -\frac{5}{3}\pi & \frac{7}{3}\pi \end{pmatrix} \begin{pmatrix} a_0^{(2)} \\ a_1^{(2)} \\ a_2^{(2)} \\ a_3^{(2)} \\ a_4^{(2)} \\ a_5^{(2)} \end{pmatrix} =$$

$$\begin{pmatrix} 0 \\ -\frac{11027}{12500} \\ -\frac{7667}{6250} \\ -\frac{15801}{12500} \\ \frac{5773}{3125} \\ \frac{61}{4} \end{pmatrix}.$$

Solving the above matrix equation we obtain the values of the constants as:

$$[a_0^{(3)} = -\frac{7}{2\pi},\ a_1^{(3)} = -\frac{98}{1875}a_5^{(3)} - \frac{259}{24\pi},$$

$$a_2^{(3)} = -\frac{13}{20}a_5^{(3)} - \frac{169}{12\pi}, a_3^{(3)} = \frac{7}{20}a_5^{(3)} - \frac{15}{2\pi},$$

$$a_4^{(3)} = 2a_5^{(3)} - \frac{1}{\pi},$$

$a_5^{(3)}$ is arbitrary constant].

If $a_5^{(3)}$ takes value close to zero, then the approximate solution is the same as the exact solution which is given by (47).

For case (IV): we obtain the matrix equation

$$\begin{pmatrix} 2\pi & 0 & \pi & -\pi & \frac{5}{3}\pi & -\frac{7}{3}\pi \\ \frac{8}{5}\pi & \frac{16}{25}\pi & \frac{12}{125}\pi & \frac{652}{1875}\pi & -\frac{3136}{9375}\pi & \frac{30148}{46875}\pi \\ \frac{6}{5}\pi & \frac{24}{25}\pi & -\frac{9}{125}\pi & \frac{269}{625}\pi & -\frac{929}{3125}\pi & \frac{6989}{15625}\pi \\ \frac{4}{5}\pi & \frac{24}{25}\pi & \frac{14}{125}\pi & \frac{164}{625}\pi & -\frac{668}{9375}\pi & \frac{8402}{46875}\pi \\ \frac{2}{5}\pi & \frac{16}{25}\pi & \frac{33}{125}\pi & \frac{337}{1875}\pi & \frac{281}{9375}\pi & \frac{3853}{46875}\pi \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0^{(2)} \\ a_1^{(2)} \\ a_2^{(2)} \\ a_3^{(2)} \\ a_4^{(2)} \\ a_5^{(2)} \end{pmatrix} =$$

$$\begin{pmatrix} -\frac{35}{4}\pi \\ -\frac{11027}{3125} \\ -\frac{23001}{12500} \\ -\frac{5267}{6250} \\ \frac{5773}{12500} \\ 0 \end{pmatrix}.$$

Solving the above matrix equation we obtain the values of the constants as:

$$[a_0^{(4)} = -\frac{7}{2\pi}, \quad a_1^{(4)} = -\frac{98}{1875}a_5^{(3)} + \frac{11}{24\pi},$$

$$a_2^{(4)} = -\frac{13}{20}a_5^{(3)} + \frac{25}{12\pi}, \; a_3^{(4)} = \frac{7}{20}a_5^{(3)} + \frac{11}{2\pi},$$

$$a_4^{(4)} = 2a_5^{(3)} + \frac{1}{\pi},$$

$a_5^{(4)}$ is arbitrary constant].

If $a_5^{(4)}$ takes value close to zero, then the approximate solution is the same as the exact solution which is given by (48).

Comparing the above results with the results of [21] for $N = 150,\ N = 200$ and results of [27] for $N = 20$ it is clear that our method gives the accurate results.

Example 2. Consider the following Cauchy integral equation of the first kind [18].

$$\int_{-1}^{1} \frac{\varphi(t)}{t-x}dt = -x^4 + \frac{3}{2}x^2 - \frac{3}{8}, \; -1 < x < 1. \qquad (50)$$

The exact solution of equation (6.7) for all cases is given by:

$$\textbf{Case (I)}: \varphi(x) = -\frac{1}{\pi}\sqrt{1-x^2}(x-x^3). \qquad (51)$$

$$\textbf{Case (II)}: \varphi(x) = \frac{1}{\pi\sqrt{1-x^2}}(-x+2x^3-x^5). \qquad (52)$$

$$\textbf{Case (III)}: \varphi(x) = -\frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}(x+x^2-x^3-x^4). \quad (53)$$

$$\textbf{Case (IV)}: \varphi(x) = -\frac{1}{\pi}\sqrt{\frac{1+x}{1-x}}(x-x^2-x^3+x^4). \quad (54)$$

Applying Lerch polynomial method for equation (50) and using the collocation points (33) when $N = 5,\ \lambda = 1$.
For case (I)we obtain the following matrix equation:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{4} & -\frac{25}{24} & \frac{73}{48} & -\frac{1073}{480} \\ \frac{3}{5} & \frac{7}{50} & -\frac{77}{500} & \frac{4591}{15000} & -\frac{24017}{50000} & \frac{370537}{500000} \\ \frac{1}{5} & \frac{23}{50} & -\frac{161}{500} & \frac{5831}{15000} & -\frac{66467}{150000} & \frac{817819}{1500000} \\ -\frac{1}{5} & \frac{23}{50} & -\frac{69}{500} & \frac{3071}{15000} & -\frac{32563}{150000} & \frac{415739}{1500000} \\ -\frac{3}{5} & \frac{7}{50} & \frac{7}{500} & \frac{2071}{15000} & -\frac{5793}{50000} & \frac{90057}{500000} \\ -1 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{24} & -\frac{7}{48} & \frac{47}{480} \end{pmatrix} \begin{pmatrix} a_0^{(1)} \\ a_1^{(1)} \\ a_2^{(1)} \\ a_3^{(1)} \\ a_4^{(1)} \\ a_5^{(1)} \end{pmatrix} =$$

$$\frac{1}{\pi}\begin{pmatrix} \frac{1}{8} \\ \frac{177}{5000} \\ -\frac{1583}{5000} \\ -\frac{1583}{5000} \\ \frac{177}{5000} \\ \frac{1}{8} \end{pmatrix}. \qquad (55)$$

Solving the matrix equation (55) we obtain the values of the constants as:

$$[a_0^{(1)} = 0, \quad a_1^{(1)} = -\frac{5}{6\pi}, \quad a_2^{(1)} = \frac{1}{\pi}, \quad a_3^{(1)} = \frac{1}{\pi},$$

$$a_4^{(1)} = a_5^{(1)} = 0].$$

Substituting from these constants into (40) we obtain the approximate solution which is the same as the exact solution (51).
For case (II): Similarly as in case (I) we obtain the values of the constants as:

$$[a_0^{(2)} arbitrary, \quad a_1^{(2)} = -\frac{27}{40\pi}, \quad a_2^{(2)} = \frac{7}{4\pi}, \quad a_3^{(2)} = \frac{3}{4\pi},$$

$$a_4^{(2)} = -\frac{2}{\pi}, \quad a_5^{(2)} = -\frac{1}{\pi}].$$

Substituting from these constants into (40) we obtain the approximate solution in the following form:

$$\varphi_5^{(2)}(x,1) = \frac{1}{\pi\sqrt{1-x^2}}(-x+2x^3-x^5) + \frac{a_0^{(2)}}{\sqrt{1-x^2}}. \quad (56)$$

Comparing the exact solution (52) with the approximate solution (56) it is clear that the approximate solution is the same as the exact solution if we take $a_0^{(2)} = 0$.

For case (III): Similarly as in cases (I) and (II) we obtain the values of the constants as:

$$[a_0^{(3)} = 0, \; a_1^{(3)} = -\frac{98}{1875}a_5^{(3)} - \frac{31}{24\pi}, \; a_2^{(3)} = -\frac{13}{20}a_5^{(3)} + \frac{7}{12\pi},$$

$$a_3^{(3)} = \frac{7}{20}a_5^{(3)} + \frac{5}{2\pi}, \; a_4^{(3)} = 2a_5^{(3)} + \frac{1}{\pi},$$

$$a_5^{(3)} \quad \text{is arbitrary constant} \,].$$

If $a_5^{(3)}$ takes value close to zero, then the approximate solution is the same as the exact solution which is given by (53).

For case (IV) Similarly as the above cases the values of the constants are given by

$$[a_0^{(4)} = 0, a_1^{(4)} = -\frac{98}{1875}a_5^{(4)} - \frac{3}{8\pi}, a_2^{(4)} = -\frac{13}{20}a_5^{(4)} + \frac{17}{12\pi},$$

$$a_3^{(4)} = \frac{7}{20}a_5^{(4)} - \frac{1}{2\pi}, \; a_4^{(4)} = 2a_5^{(4)} - \frac{1}{\pi},$$

$$a_5^{(4)} \quad \text{is arbitrary constant} \,].$$

If $a_5^{(4)}$takes value close to zero, then the approximate solution is the same as the exact solution which is given by (54).

7 Conclusion

In this paper, Lerch matrix collocation method for solving various cases of Cauchy type singular integral equations of the first kind is developed. The singularity of equation (2) was successfully removed by applying smooth transformations. Numerical examples show the reliability and efficiently of the proposed method.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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