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Optimal Packing of Two Disks on Torus

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Abstract: The article is devoted to recently established connection between the packing problem of disks on torus and the effective conductivity of composites with circular inclusions. The packing problem is usually investigated by geometrical arguments, the conductivity problem by means of elliptic functions. An algorithm is developed in order to determine the optimal location of two disks on torus formed by the hexagonal lattice and square lattice. The corresponding minimization function is constructed in terms of expressions consisting of elliptic functions with unknown arguments. The numerically found roots coincide with the previously established optimal points by a pure geometrical study.

Keywords: Optimal packing, Torus, Composite, Effective conductivity, Hexagonal lattice, Square lattice.

1 Introduction

We consider an optimal packing problem for non-overlapping disks on the plane in the double periodic statement, i.e. in the torus topology [1,2,3,4,5,6,7,8,9]. In the present paper, we follow an example considered where the relation between the pure geometrical packing problem for non-overlapping disks and the optimal design problem of the theory of composites was established.

In general case, the series associated to periodic analytical functions, Eisenstein's series, are considered. The series are used under consideration of the Weierstrass invariants [10,11,12,13,14,15]. A supplementary Weierstrass' function is introduced. The function is used for simulation of the disks. Then the disks are summarized. The disks are embedded to the square with probability and this radius on Ox axis. The same on the axis Oy. Therefore, it is verified an imposition of the disks. In the case of the disks imposition are thrown out.

The process is repeated for all disks. We are going to use the Eisenstein structural sums for multivariable functions [16,17,18,19]. It should be noted, that they are applied to compete a property of the composite fibre materials. The random process is called isotropy [20,21,22,23,24]. It is possible, that there are a lot of disks.

The problem of packing circles, ellipses and other figures on a plane or inside any given areas has been studied by many authors and various methods have been developed to solve them [25,26,27]. When solving the problem of optimal packing of non-overlapping disks on a plane using Mathematica.

This will allow us to consider only connected graphs in our exhaustive search of all possible packing graphs and locally maximally dense packing without free circles in what follows. Finally, we observe that we can lower bound the number of edges (and their arrangement) incident to a vertex in the packing graph associated to a locally maximally dense packing with no free circles.

2 Statement of the problem

One of the topical problems both in geometry and physics is an optimal design problem. Majority of valuable works were devoted to the problem for elliptic functions, for example, the square lattice, the hexagonal lattice, problems with the fundamental translation vectors. Note, that the vectors are periods for the square lattice, and there are two fundamental translation vectors for the hexagonal lattice. Moreover all these combinations are linear. It is considered, that the periods can be continued and shifted.

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Consider the torus, the hexagonal lattice, with the fundamental translation vectors. Also we consider Weierstrass functions and their invariants.

Let us consider the hexagonal lattice with the fundamental translation vectors

$$\omega 1 = 1;$$

$$\omega 2 = e^{i\pi/3};$$

Also we consider Weierstrass functions and their invariants:

$$\text{g2g3} = \text{WeierstrassInvariants}[\{\omega 1/2,$$

$$\omega 2/2\}]//\text{Chop};$$

$$\wp[\text{z\_}]{:=}\text{WeierstrassP}[z, \text{g2g3}];$$

$$\boldsymbol{\wp 1[\text{z\_}]{:=}\text{WeierstrassPPrime}[z, \text{g2g3}];}$$

$$\mathbb{E}[2, \text{z\_}] = \text{If}[z == 0, \pi, \wp[z] + \pi].$$

The series are used under consideration of the Weierstrass invariants. A supplementary Weierstrass' function is introduced. The function is used for simulation of the disks.

Then the disks are summarized. A square is chosen from -1/2 till 1/2. A disk is embedded to the square with probability and this radius on Ox axis. The same on the axis *Oy*. Therefore, it is verified an imposition of the disks. In the case of imposition the disks is thrown out. The process is repeated for all disks.

3 Hex cell without the area normalization

Consider the optimal packing of the disks on the torus represented by a hexagonal lattice (on a triangular flat torus in the terminology of [24]) in the case $N = 2$. In order to be consistent with [24] consider the lattice generated by the fundamental vectors $\omega_1 = 1$ and $\omega_2 = e^{\frac{\pi i}{3}}$. Then the fundamental cell is the rhombus with the vertices $0, 1, e^{\frac{\pi i}{3}}$, and $1 + e^{\frac{\pi i}{3}}$. The first isotropy condition $e_2 = \pi$ from (18) is reduced to equation $\wp(a_2) = 0$.

The sums $S_2 = \sum_{m_1,m_2} (m_1\omega_1 + m_2\omega_2)^{-2}$ are slowly convergent if computed directly. But they can be easily calculated through the rapidly convergent series:

$$S_2 = \left(\frac{\pi}{\omega_1}\right)^2 \left(\frac{1}{3} - 8\sum_{m=1}^{\infty} \frac{mq^{2m}}{1-q^{2m}}\right), \; q = exp\left(\pi i \frac{\omega_2}{\omega_1}\right).$$

The high-order Eisenstein functions are related to the Weierstrass function $\wp(z)$ [11] by the identities [23]: $E_2 = (z) = \wp(z) + S_2$.

For instance, basic sums e_2 and take the following form:

$$e_2 = \frac{1}{N^2} \sum_{k_0=1}^{N} \sum_{k_0=1}^{N} E_2(a_{k_0} - a_{k_1}).$$

The Weierstrass function has exactly two zeros located at the points $\frac{1}{2} + \frac{i\sqrt{3}}{6}$ and $1 + \frac{i}{\sqrt{3}}$. These points coincide with the points from [24] obtained by geometrical arguments without use of the Weierstrass function. They can be also found by the numerical operator in Mathematica:

$$\omega 1 = FindRoot\,[\wp[\omega] == 0, \omega, 0.1 + 0.1i]\,[1,2]$$

$$\omega 2 = FindRoot\,[\wp[\omega] == 0, \omega, 1 + i]\,[1,2]. \qquad (1)$$

After checking, the program outputs the following values: $0.5 + 0.2886i$ and $1. + 1.1547i$.

Let's enter the values for the program to display the graph on the coordinate axes from -2 to 2 along the *Ox* and *Oy* axes. We find two zeros $\omega 1$ and $\omega 2$ (1) of the Weierstrass function corresponding the hexagonal lattice. In Figure 1, one can see that the points $\omega 1$ and $\omega 2$ are symmetrical about the center of the cell.

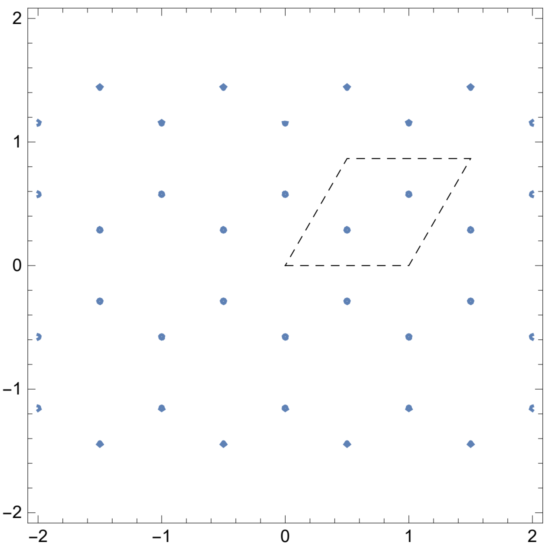

Figure 1: The roots $\omega 1, \omega 2$ calculated by (1) and their periodic images under translation.

To describe the fundamental domain of the standard triangular torus with the standard basis [28] by translations we may fix the first circle at $(0,0)$ and we must place the second circle in a location where it is tangent to the first circle in at least 3 ways. Using the symmetries of the lattice, we may assume that this circle is located in the triangle with vertices $(0,0), (1,0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Inside this fundamental domain, if we place this circle anywhere except at $(\frac{1}{2}, \frac{\sqrt{3}}{6})$, then a maximum of two tangencies are formed. Therefore, we must place the second circle at $(\frac{1}{2}, \frac{\sqrt{3}}{6})$. Using inequality (1) on the three edges of the packing graph it and it is easy to show that this packing is maximally dense [8].

We have written a program for Mathematica to illustrate this result:

```
disks = Graphics[{{Opacity[0.8], Gray, Disk[{Re[w1], Im[w1]}, r0]},
   {Opacity[0.3], Gray, Disk[{0, 0}, r0], Disk[{Re[w1], Im[w1]}, r0],
    Disk[{Re[ω2], Im[ω2]}, r0], Disk[{Re[ω1 + ω2], Im[ω1 + ω2]}, r0]}}];
 Show[disks, cell]
```

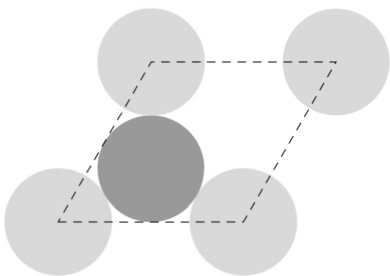

Figure 2: There are two disks with the centers of the full disk at the point w1 with the coordinates of the real and imaginary parts of the value 0.5 + 0.288675i. We described above in the program in the hexagonal lattice. The radius calculated by formula $r_0 = Abs[\omega 1]2$ and equals to 0.28867513459481287.

4 Square cell

Let us consider the hexagonal lattice with the fundamental translation vectors

$$\omega 1 = 1; \omega 2 = i.$$

Also we consider Weierstrass functions and their invariants:

$$\text{g2g3} = \text{WeierstrassInvariants}[\{\omega 1/2,$$

$$\omega 2/2\}]//\text{Chop};$$

$$\wp[\text{z\_}]:=\text{WeierstrassP}[z, \text{g2g3}];$$

$$\boldsymbol{\wp 1[\text{z\_}]:=\text{WeierstrassPPrime}[z, \text{g2g3}];}$$

$$\wp[2, \text{z\_}] = \text{If}[z == 0, \pi, [z] + \pi];$$

We are going to use the Eisenstein structural sums for multivariable functions and calculate the Eisenstein function:

$$\omega 1 = FindRoot\,[\wp[\omega] == 0, \omega, 0.1 + 0.1i]\,[1, 2]$$

After checking, the program outputs the following value: $0.5 + 0.5i$.

Again let us enter the values for the program to display the graph on the coordinate axes from -2 to 2 along the Ox and Oy axes. We introduce contours with the absolute value of the Weierstrass functions and obtain two points in the square lattice. In the picture you can see that the points are symmetrical about the origin.

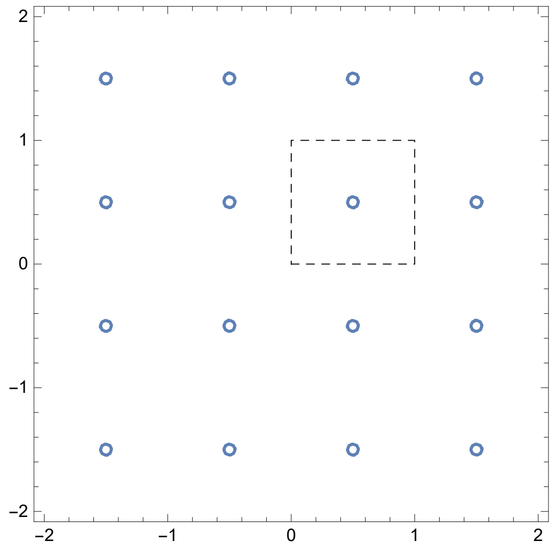

Figure 3: The root $\omega 1$ calculated for the square cell and their periodic images under translation.

Draw an optimal packing graph for two disks in the square lattice. The packing radius is equal to the absolute value of $\omega 1$ divided by two:

```
disks = Graphics[{{Opacity[0.8], Gray, Disk[{Re[ω1], Im[ω1]}, r0]},
  {Opacity[0.3], Gray, Disk[{0, 0}, r0], Disk[{Re[ω1], Im[ω1]}, r0],
   Disk[{Re[ω2], Im[ω2]}, r0], Disk[{Re[ω1 + ω2], Im[ω1 + ω2]}, r0]}}];
Show[disks, cell]
```

It is interesting that the same method holds for two disks of two different radii. Since the same equation

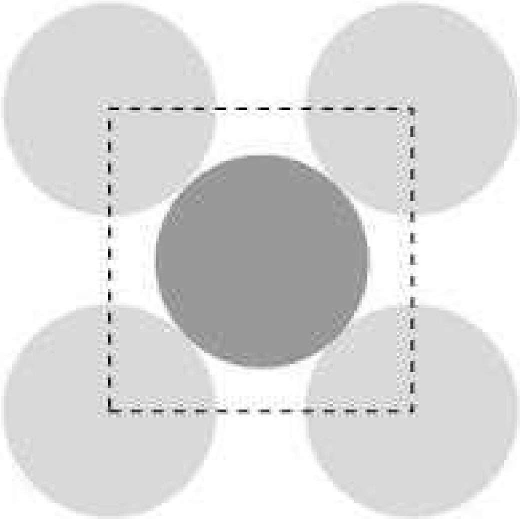

Figure 4: There are two disks with the centers of the full disk at the point w1 with the coordinates of the real and imaginary parts of the value 0.5+0.5i. We described above in the program in the square lattice. The radius calculated by formula $r_0 = Abs[\omega 1]2$ and approximately equals to 0.353553.

$\wp(a_2) = 0$ has to be solved to determine the isotropic structure.

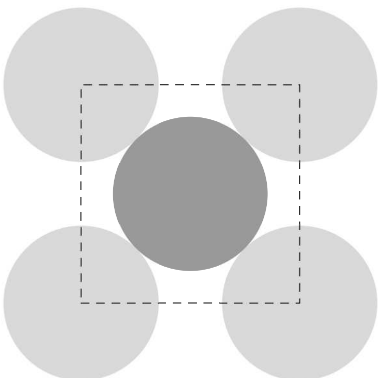

Figure 5: There are two disks with the centers of the full disk at the point w1 with the coordinates of the real and imaginary parts of the value 0.5+0.5i. We described above in the program in the square lattice. The radius calculated by formula $r_0 = Abs[\omega 1]2$ It equals to 0.459619399359373 for disk in center and equals to 0.2525381315161392 for disks in angles.

5 Conclusion

There exist locally maximally dense packing of equal circles which contain circles that are free to move (i.e. they are not held fixed by their neighbors), but the common diameter of all the circles cannot increase. For example, this occurs in the globally maximally dense arrangement of 7 circles packed into a hard-boundary square where one circle is is free to move. This result is attributed to Schaer in 1965 by Goldberg. If we remove any circle that are free to move, called free circles (also known as floaters or rattlers), from such an arrangement, then we obtain a locally maximally dense packing for fewer circles in the flat torus. Our search will continue sequentially from 1 to 6 circles and in this article we will consider the case of n equal to two, and we will be able to say for any locally maximum dense arrangement, is there room for another such circle, which can be a free circle in a locally maximum dense arrangement more circles packed onto a flat torus.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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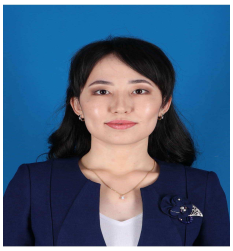

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