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Some Fractional Inequalities of Ostrowski-Type and related Applications

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Abstract: A significant area in the region of pure and applied mathematics is the integral inequality. As it is well-known, inequalities objective to establish various mathematical methods. Today, we need to pursue exact inequalities to demonstrate the uniqueness and existence of mathematical methods. In the present research, we will develop some Ostrowski-type inequalities for the strongly quasi-convex function. We also establish some fractional weighted Ostrowski-type inequalities for differentiable strongly quasi-convex function. The remarks at the end of the results are also given. In order to show the effectiveness of our results, we present some applications to special means. The derived results generalize and refine some well-known results.

Keywords: Generalized derivatives and integral, Laplace's equation

1 Introduction

Fractional calculus is natural generalization of classical calculus which covers differentiation and integration of non-integer order. The idea of fractional calculus has been introduced virtually at the same time as the development of classical ones. Fractional calculus studies different possibilities of defining real number powers or complex number powers of the differentiation operator, the realistic use of fractional differential operators more frequent in electrical transmission line analysis is generally normal.

It has become evident that the subject of convex analysis got a reasonable space among the areas of research which has vast applications. The researchers can not ignored the link of convexity and fractional calculus in current scenario. For more on this study we refer (see [1, 2, 3, 4]). The subject of convex analysis is important for both pure and applied mathematics. The advancement of fractional calculus also sets new trends in developing inequalities of convex functions (see [5, 6, 7, 8]). Convexity is frequently hidden in many other areas of mathematics: complex analysis, functional analysis, calculus of variations, partial differential equations, graph theory, discrete mathematics, probability theory, crystallography, algebraic geometry and several other fields.

In this view, integral inequalities have played a significant role in narrating real-world problems. In this framework, Hermite-Hadamard inequalities are very dominant in convex theory, which has been proved by different ways and has several generalizations and extensions [9, 10, 11, 12, 13, 14, 15, 16]. The Hermite Hadamard inequality for convex function is as follows:
Let $\zeta : J \subseteq \mathrm{R} \to \mathrm{R}$ be a convex function. Then

$$\zeta\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \zeta(\xi)d\xi \leq \frac{\zeta(a)+\zeta(b)}{2},$$

holds for all $a,b \in J$ with $a \neq b$. For more on this study we refer the reader (see [17, 18, 19, 20, 21])

Several techniques are done by different authors to analyze unique generalizations, modifications and speculations for the Hermite-Hadamard inequality and its various forms, we specify the associated phenomena [22, 23, 24, 25, 26, 27, 28] to concerned readers.

Definition 1.(***Riemann-Liouville Fractional Integrals***) *The Riemann-Liouville fractional integrals* $J^{\delta}_{x_1+}\zeta$ *and* $J^{\delta}_{x_2-}\zeta$ *of order* δ *with* $\delta > 0$, $0 \leq x_1 < x_2$ *and let*

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$\zeta \in [x_1, x_2]$ are defined by

$$\left(J_{x_1+}^{\delta}\right)\zeta(n) = \frac{1}{\Gamma(\delta)}\int_{x_1}^{n}(n-\xi)^{\delta-1}\zeta(\xi)\,d\xi \quad (1)$$

for $n > x_1$, and

$$\left(J_{x_2-}^{\delta}\right)\zeta(n) = \frac{1}{\Gamma(\delta)}\int_{n}^{x_2}(\xi-n)^{\delta-1}\zeta(\xi)\,d\xi \quad (2)$$

for $n < x_2$ respectively, where $\Gamma(\delta) = \int_0^{\infty} e^{-\xi}\xi^{x_1-1}d\xi$ and $\left(J_{x_1+}^{0}\right)\zeta(n) = \left(J_{x_2-}^{0}\right)\zeta(n) = \zeta(n)$.

Definition 2.[29] *Let J be an interval of real numbers. A function $\zeta : J \to \mathbb{R}$ is called convex, if*

$$\zeta(\xi a + (1-\xi)b) \le \xi\zeta(a) + (1-\xi)\zeta(b), \quad (3)$$

holds $\forall\, a, b \in J$ and $\xi \in [0,1]$.

Definition 3.[30] *A function $\zeta : J \to \mathbb{R}$ is called quasi-convex, if*

$$\zeta(\xi a + (1-\xi)b) \le \max\{\zeta(a), \zeta(b)\}, \quad (4)$$

holds $\forall\, a, b \in J$ and $\xi \in [0,1]$.

Definition 4.[31] *A function $\zeta : J \to \mathbb{R}$ is called strongly quasi-convex with modulus $c \ge 0$, if*

$$\begin{aligned}&\zeta(\xi a + (1-\xi)b)\\ &\le \max\{\zeta(a), \zeta(b)\} - c\xi(1-\xi)(a-b)^2, \end{aligned}\quad (5)$$

holds $\forall\, a, b \in J$ and $\xi \in [0,1]$.

The present paper is organized as follows. In section 2, for strongly quasi-convex function, we proved Ostrowski-type inequalities. In section 3, we will derive fractional weighted Ostrowski-type inequalities for strongly quasi-convex function and related results. In section 4, we will write some applications and at last we give concluding remarks to our present article.

2 Ostrowski-type inequalities

The aim of this section is to derive some new Ostrowski-type inequalities for the class of functions whose first derivatives in absolute value are strongly quasi-convex functions.

Lemma 1.[32] *Let $\zeta : J \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on J° where $a, b \in J$ with $a < b$. If $\zeta' \in L[a,b]$, then*

$$\begin{aligned}&\zeta(x_1) - \frac{1}{b-a}\int_a^b \zeta(\omega_1)\,d\omega_1\\ &= (b-a)\int_a^b k(\xi)\zeta'(\xi a + (1-\xi)b)\,d\xi, \end{aligned}\quad (6)$$

for each $\xi \in [0,1]$, where

$$k(\xi) = \begin{cases} \xi, & \xi \in \left[0, \frac{b-x_1}{b-a}\right], \\ \xi - 1, & \xi \in \left[\frac{b-x_1}{b-a}, 1\right] \end{cases}$$

for all $x_1 \in [a,b]$.

Theorem 1. *Let $\zeta : J \subset [0,\infty] \to \mathbb{R}$ be a differentiable mapping on J° such that $\zeta' \in L[a,b]$, where $a, b \in J$ with $a < b$. If $|\zeta'|$ is strongly quasi-convex with modulus $c \ge 0$ on $[a,b]$, then*

$$\begin{aligned}&\left|\zeta(x_1) - \frac{1}{b-a}\int_a^b \zeta(\omega_1)\,d\omega_1\right|\\ &\le \frac{(b-x_1)^2}{2(b-a)}\max\left\{\left|\zeta'(x_1)\right|, \left|\zeta'(b)\right|\right\} \quad (7)\\ &-c\left(\frac{(b-x_1)^3}{3(b-a)^2} - \frac{(b-x_1)^4}{4(b-a)^3}\right)(x_1-b)^2\\ &+\frac{(x_1-a)^2}{2(b-a)}\max\left\{\left|\zeta'(x_1)\right|, \left|\zeta'(a)\right|\right\}c(x_1-a)^2 \times\\ &\left(\frac{1}{12} - \frac{(b-x_1)^2}{2(b-a)} + \frac{2(b-x_1)^3}{3(b-a)^2} - \frac{(b-x_1)^4}{4(b-a)^3}\right),\end{aligned}$$

for each $x_1 \in [a,b]$.

Proof. Applying Lemma 2, properties of modulus and strong quasi-convexity of $|\zeta'|$ yields that

$$\begin{aligned}&\left|\zeta(x_1) - \frac{1}{b-a}\int_a^b \zeta(\omega_1)\,d\omega_1\right|\\ &\le (b-a)\int_0^{\frac{b-x_1}{b-a}} \xi\left|\zeta'(\xi a + (1-\xi)b)\right|d\xi\\ &+(b-a)\int_{\frac{b-x_1}{b-a}}^{1} |\xi-1|\left|\zeta'(\xi a + (1-\xi)b)\right|d\xi\\ &\le (b-a)\int_0^{\frac{b-x_1}{b-a}} \xi\left(A_1 - c\xi(1-\xi)(x_1-b)^2\right)d\xi\\ &+(b-a)\int_{\frac{b-x_1}{b-a}}^{1}(1-\xi)\left(A_2 - c\xi(1-\xi)(x_1-a)^2\right)d\xi\\ &\le (b-a)A_1 \times\\ &\quad\int_0^{\frac{b-x_1}{b-a}} \xi\,d\xi - c(b-a)(x_1-b)^2\int_0^{\frac{b-x_1}{b-a}} \xi^2(1-\xi)\,d\xi\\ &+(b-a)A_2 \times\\ &\int_{\frac{b-x_1}{b-a}}^{1}(1-\xi)\,d\xi - c(b-a)(x_1-a)^2\int_{\frac{b-x_1}{b-a}}^{1}\xi(1-\xi)^2\,d\xi\\ &\le \frac{(b-x_1)^2}{2(b-a)}A_1 \times\\ &\quad -c\left(\frac{(b-x_1)^3}{3(b-a)^2} - \frac{(b-x_1)^4}{4(b-a)^3}\right)(x_1-b)^2\\ &\qquad+\frac{(x_1-a)^2}{2(b-a)}A_2 \times\\ &c\left(\frac{1}{12} - \frac{(b-x_1)^2}{2(b-a)} + \frac{2(b-x_1)^3}{3(b-a)^2} - \frac{(b-x_1)^4}{4(b-a)^3}\right)(x_1-a)^2,\end{aligned}$$

where

$$A_1 = \max\left\{\left|\zeta'(x_1)\right|, \left|\zeta'(b)\right|\right\} \quad \text{and}$$

$$A_2 = \max\left\{\left|\zeta'(x_1)\right|, \left|\zeta'(b)\right|\right\}$$

The proof is complete.

*Remark.*For $c = 0$ in Theorem 1, then we get Theorem 2 of [33].

Theorem 2. *Consider $\zeta : J \subset [0,\infty] \to \mathbb{R}$ be a differentiable mapping on J° such that $\zeta' \in L[a,b]$, where $a,b \in J$ with $a < b$. If $\left|\zeta'\right|^q$ is strongly quasi-convex with modulus $c \geq 0$ on $[a,b]$, then*

$$\begin{aligned}
&\left|\zeta(x_1) - \frac{1}{b-a}\int_a^b \zeta(\omega_1)d\omega_1\right| \\
&\leq \left(\frac{(b-x_1)^{p+1}}{(b-a)(p+1)}\right)^{\frac{1}{p}} \times \\
&\quad \left[\max\left\{\left|\zeta'(x_1)\right|^q, \left|\zeta'(b)\right|^q\right\}\right. \\
&\quad\quad \left. -c\left(\frac{(b-x_1)^2}{2(b-a)^2} - \frac{(b-x_1)^3}{3(b-a)^3}\right)(x_1-b)^2\right]^{\frac{1}{q}} \\
&+ \frac{(x_1-a)^{p+1}}{(b-a)(p+1)} \times \\
&\quad \left[\max\left\{\left|\zeta'(x_1)\right|^q, \left|\zeta'(a)\right|^q\right\}\right. \\
&\quad\quad \left. -c\left(\frac{1}{6} - \frac{(b-x_1)^2}{2(b-a)^2} + \frac{(b-x_1)^3}{3(b-a)^3}\right)(x_1-a)^2\right]^{\frac{1}{q}},
\end{aligned} \tag{8}$$

for each $x_1 \in [a,b]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By Lemma 2, properties of modulus, Hölder's integral inequality and strong quasi-convexity of $\left|\zeta'\right|^q$ yields that

$$\begin{aligned}
&\left|\zeta(x_1) - \frac{1}{b-a}\int_a^b \zeta(\omega_1)d\omega_1\right| \\
&\leq (b-a)\int_0^{\frac{b-x_1}{b-a}} \xi\left|\zeta'(\xi a + (1-\xi)b)\right|d\xi \\
&\quad + (b-a)\int_{\frac{b-x_1}{b-a}}^1 |\xi - 1|\left|\zeta'(\xi a + (1-\xi)b)\right|d\xi \\
&\leq (b-a)\left(\int_0^{\frac{b-x_1}{b-a}} \xi^p d\xi\right)^{\frac{1}{p}} \times \\
&\quad \left(\int_0^{\frac{b-x_1}{b-a}} \left|\zeta'(\xi a + (1-\xi)b)\right|^q d\xi\right)^{\frac{1}{q}} \\
&\quad + (b-a)\left(\int_{\frac{b-x_1}{b-a}}^1 (1-\xi)^p d\xi\right)^{\frac{1}{p}} \times \\
&\quad \left(\int_{\frac{b-x_1}{b-a}}^1 \left|\zeta'(\xi a + (1-\xi)b)\right|^q d\xi\right)^{\frac{1}{q}}
\end{aligned} \tag{9}$$

$$\begin{aligned}
&= \frac{(b-x_1)^{\frac{p+1}{p}}}{(b-a)^{\frac{1}{p}}(p+1)^{\frac{1}{p}}} \times \\
&\quad \left[\max\left\{\left|\zeta'(x_1)\right|^q, \left|\zeta'(b)\right|^q\right\}\right. \\
&\quad\quad \left. -c\left(\frac{(b-x_1)^2}{2(b-a)^2} - \frac{(b-x_1)^3}{3(b-a)^3}\right)(x_1-b)^2\right]^{\frac{1}{q}} \\
&+ \frac{(x_1-a)^{\frac{p+1}{p}}}{(b-a)^{\frac{1}{p}}(p+1)^{\frac{1}{p}}} \times \\
&\quad \left[\max\left\{\left|\zeta'(x_1)\right|^q, \left|\zeta'(a)\right|^q\right\}\right. \\
&\quad\quad \left. -c\left(\frac{1}{6} - \frac{(b-x_1)^2}{2(b-a)^2} + \frac{(b-x_1)^3}{3(b-a)^3}\right)(x_1-a)^2\right]^{\frac{1}{q}},
\end{aligned} \tag{10}$$

which completes the proof.

Remark. For $c = 0$ in Theorem 2, then we get Theorem 3 of [33].

Corollary 1.*Considering $x_1 = \frac{a+b}{2}$ in Theorem 2, we get*

$$\begin{aligned}
&\left|\zeta\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \zeta(\omega_1)d\omega_1\right| \\
&\leq \frac{(b-a)}{2^{\frac{p+1}{p}}(p+1)^{\frac{1}{p}}} \times \\
&\left\{\left[\max\left\{\left|\zeta'\left(\frac{a+b}{2}\right)\right|^q, \left|\zeta'(b)\right|^q\right\} - \frac{c}{48}(b-a)^2\right]^{\frac{1}{q}}\right. \\
&\quad \left. + \left[\max\left\{\left|\zeta'\left(\frac{a+b}{2}\right)\right|^q, \left|\zeta'(a)\right|^q\right\} - \frac{c}{48}(b-a)^2\right]^{\frac{1}{q}}\right\}.
\end{aligned} \tag{11}$$

Theorem 3. *Consider $\zeta : J \subset [0,\infty] \to \mathbb{R}$ be a differentiable mapping on J° such that $\zeta' \in L[a,b]$, where $a,b \in J$ with $a < b$. If $\left|\zeta'\right|^q$ is strongly quasi-convex with modulus $c \geq 0$ on $[a,b]$, $q \geq 1$, and $\left|\zeta'(x_1)\right| \leq M$, $x_1 \in [a,b]$, then*

$$\begin{aligned}
&\left|\zeta(x_1) - \frac{1}{b-a}\int_a^b \zeta(\omega_1)d\omega_1\right| \\
&\leq \frac{(b-x_1)^2}{2(b-a)} \times \\
&\quad \left(\max\left\{\left|\zeta'(x_1)\right|^q, \left|\zeta'(b)\right|^q\right\}\right. \\
&\quad\quad \left. -c\left(\frac{2(b-x_1)}{3(b-a)} - \frac{(b-x_1)^2}{2(b-a)^2}\right)(x_1-b)^2\right)^{\frac{1}{q}} \\
&\quad\quad + \frac{(x_1-a)^2}{2(b-a)}\left(\max\left\{\left|\zeta'(x_1)\right|^q, \left|\zeta'(a)\right|^q\right\}\right. \\
&\quad -c\left(\frac{(b-a)^2}{6(x_1-a)^2} - \frac{(b-x_1)^2}{(x_1-a)^2}\right. \\
&\quad\quad \left.\left. + \frac{4(b-x_1)^3}{3(b-a)(x_1-a)^2} - \frac{(b-x_1)^4}{2(b-a)^2(x_1-a)^2}\right)(x_1-a)^2\right)^{\frac{1}{q}},
\end{aligned}$$

for each $x_1 \in [a,b]$.

Proof. Take $q \geq 1$. Applying Lemma 2, properties of modulus and power mean integral inequality yields that

$$\begin{aligned}&\left|\zeta(x_1)-\frac{1}{b-a}\int_a^b \zeta(\omega_1)d\omega_1\right|\\&\leq (b-a)\int_0^{\frac{b-x_1}{b-a}} \xi\left|\zeta'(\xi a+(1-\xi)b)\right|d\xi\\&\quad+(b-a)\int_{\frac{b-x_1}{b-a}}^1 |\xi-1|\left|\zeta'(\xi a+(1-\xi)b)\right|d\xi\\&\leq (b-a)\left(\int_0^{\frac{b-x_1}{b-a}} \xi d\xi\right)^{1-\frac{1}{q}}\times\\&\quad\left(\int_0^{\frac{b-x_1}{b-a}} \xi\left|\zeta'(\xi a+(1-\xi)b)\right|^q d\xi\right)^{\frac{1}{q}}\\&+(b-a)\left(\int_{\frac{b-x_1}{b-a}}^1 (1-\xi)d\xi\right)^{1-\frac{1}{q}}\times\\&\quad\left(\int_{\frac{b-x_1}{b-a}}^1 (1-\xi)\left|\zeta'(\xi a+(1-\xi)b)\right|^q d\xi\right)^{\frac{1}{q}}.\end{aligned}$$

Since $\left|\zeta'\right|^q$ is strongly quasi-convex, we have

$$\begin{aligned}&\int_0^{\frac{b-x_1}{b-a}} \xi\left|\zeta'(\xi a+(1-\xi)b)\right|^q d\xi\\&\leq \int_0^{\frac{b-x_1}{b-a}} \xi\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(b)\right|^q\right\}\right.\\&\qquad\left.-c\xi(1-\xi)(x_1-b)^2\right)d\xi\\&=\frac{(b-x_1)^2}{2(b-a)^2}\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(b)\right|^q\right\}\\&\qquad-c\left(\frac{(b-x_1)^3}{3(b-a)^3}-\frac{(b-x_1)^4}{4(b-a)^4}\right)(x_1-b)^2 \qquad (12)\\&=\frac{(b-x_1)^2}{2(b-a)^2}\times\\&\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(b)\right|^q\right\}\right.\\&\qquad\left.-c\left(\frac{2(b-x_1)}{3(b-a)}-\frac{(b-x_1)^2}{2(b-a)^2}\right)(x_1-b)^2\right),\end{aligned}$$

and

$$\begin{aligned}&\int_{\frac{b-x_1}{b-a}}^1 (1-\xi)\left|\zeta'(\xi a+(1-\xi)b)\right|^q d\xi\\&\leq \int_{\frac{b-x_1}{b-a}}^1 (1-\xi).\left(\max\left\{\left|\zeta'(a)\right|^q,\left|\zeta'(x_1)\right|^q\right\}\right.\\&\qquad\left.-c\xi(1-\xi)(a-x_1)^2\right)d\xi\\&=\frac{(x_1-a)^2}{2(b-a)^2}\max\left\{\left|\zeta'(a)\right|^q,\left|\zeta'(x_1)\right|^q\right\}\\&\quad-c\left(\frac{1}{12}-\frac{(b-x_1)^2}{2(b-a)^2}+\frac{2(b-x_1)^3}{3(b-a)^3}-\frac{(b-x_1)^4}{4(b-a)^4}\right)(a-x_1)^2\\&=\frac{(x_1-a)^2}{2(b-a)^2}\left(\max\left\{\left|\zeta'(a)\right|^q,\left|\zeta'(x_1)\right|^q\right\}\right.\\&\qquad-c\left(\frac{(b-a)^2}{6(x_1-a)^2}-\frac{(b-x_1)^2}{(x_1-a)^2}+\frac{4(b-x_1)^3}{3(b-a)(x_1-a)^2}\right.\\&\qquad\left.\left.-\frac{(b-x_1)^4}{2(b-a)^2(x_1-a)^2}\right)(a-x_1)^2\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\left|\zeta(x_1)-\frac{1}{b-a}\int_a^b \zeta(\omega_1)d\omega_1\right|\\&\leq\frac{(b-x_1)^2}{2(b-a)}\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(b)\right|^q\right\}\right.\\&\qquad\left.-c\left(\frac{2(b-x_1)}{3(b-a)}-\frac{(b-x_1)^2}{2(b-a)^2}\right)(x_1-b)^2\right)^{\frac{1}{q}}\\&+\frac{(x_1-a)^2}{2(b-a)}\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(a)\right|^q\right\}\right.\\&\quad-c\left(\frac{(b-a)^2}{6(x_1-a)^2}-\frac{(b-x_1)^2}{(x_1-a)^2}+\frac{4(b-x_1)^3}{3(b-a)(x_1-a)^2}\right.\\&\qquad\left.\left.-\frac{(b-x_1)^4}{2(b-a)^2(x_1-a)^2}\right)(x_1-a)^2\right)^{\frac{1}{q}}.\end{aligned}$$

This completes the proof.

*Remark.*For $c=0$ in Theorem 2, then we get Theorem 4 of [33].

Corollary 2.*Considering* $x_1=\frac{a+b}{2}$ *in Theorem 2, we get*

$$\begin{aligned}&\left|\zeta\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b \zeta(\omega_1)d\omega_1\right|\\&\leq\frac{(b-a)}{8}\left[\left(\max\left\{\left|\zeta'\left(\frac{a+b}{2}\right)\right|^q,\left|\zeta'(b)\right|^q\right\}-\frac{5c}{96}(b-a)^2\right)^{\frac{1}{q}}\right.\\&\quad\left.+\left(\max\left\{\left|\zeta'\left(\frac{a+b}{2}\right)\right|^q,\left|\zeta'(a)\right|^q\right\}-\frac{5c}{96}(b-a)^2\right)^{\frac{1}{q}}\right].\end{aligned}$$

3 Fractional weighted Ostrowski-type inequalities

The current section is devoted for fractional weighted Ostrowski-type inequalities via strongly quasi-convex function.

Lemma 2.[34] *Let* $\zeta: J\to\mathbb{R}$*be a differentiable function on* J°, $0\leq a<b$, *and* $\omega_1: J\to\mathbb{R}$ *a continuous function. If*

ζ', $\omega_1 \in L(J)$, then

$$J^{\delta}_{x_1^-}\omega_1\zeta(a)+J^{\delta}_{x_1^+}\omega_1\zeta(b)-\left[J^{\delta}_{x_1^-}\omega_1(a)+J^{\delta}_{x_1^+}\omega_1(a)\right]\zeta(x_1)$$
$$=\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta)}\int_0^1 k_1(\xi)\zeta'(\xi b+(1-\xi)x_1)d\xi$$
$$-\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta)}\int_0^1 k_2(\xi)\zeta'(\xi a+(1-\xi)x_1)d\xi,$$

where

$$k_1(\xi):=\int_\xi^1(1-\alpha)^{\delta-1}\omega_1(\alpha b+(1-\alpha)x_1)d\alpha, \quad (13)$$

and

$$k_2(\xi):=\int_\xi^1(1-\alpha)^{\delta-1}\omega_1(\alpha a+(1-\alpha)x_1)d\alpha. \quad (14)$$

Theorem 4. *Consider $\zeta: J\to\mathbb{R}$ be a differentiable function with $\zeta'\in L(J)$ where $0\le a<b$, and let $\omega_1: J\to\mathbb{R}$ be a continuous function. If $|\zeta'|$ is strongly quasi-convex with modulus $c\ge 0$, then*

$$|J^{\delta}_{x_1^-}\omega_1\zeta(a)+J^{\delta}_{x_1^+}\omega_1\zeta(b)-\left[J^{\delta}_{x_1^-}\omega_1(a)+J^{\delta}_{x_1^+}\omega_1(a)\right]\zeta(x_1)|$$
$$\le\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta+2)}\max\left\{|\zeta'(x_1)|,|\zeta'(b)|\right\}\parallel\omega_1\parallel_{[x_1,b],\infty}$$
$$-\left(\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta+3)}-\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta+4)}\right)c(x_1-b)^2\parallel\omega_1\parallel_{[x_1,b],\infty}$$
$$+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta+2)}\max\left\{|\zeta'(x_1)|,|\zeta'(a)|\right\}\parallel\omega_1\parallel_{[a,x_1],\infty}$$
$$-\left(\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta+3)}-\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta+4)}\right)c(x_1-a)^2\parallel\omega_1\parallel_{[x_1,b],\infty}.$$

Proof. Applying properties of absolute value, Lemma 2 and strong quasi-convexity of $|\zeta'|$ yields that

$$|J^{\delta}_{x_1^-}\omega_1\zeta(a)+J^{\delta}_{x_1^+}\omega_1\zeta(b)-\left[J^{\delta}_{x_1^-}\omega_1(a)+J^{\delta}_{x_1^+}\omega_1(a)\right]\zeta(x_1)|$$
$$\le\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta)}\int_0^1|k_1(\xi)||\zeta'(\xi b+(1-\xi)x_1)|d\xi$$
$$+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta)}\int_0^1|k_2(\xi)||\zeta'(\xi a+(1-\xi)x_1)|d\xi$$

$$\le\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta)}\int_0^1|k_1(\xi)|\left(A_1-c\xi(1-\xi)(x_1-b)^2\right)d\xi$$
$$+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta)}\int_0^1|k_2(\xi)\left(A_2-c\xi(1-\xi)(x_1-a)^2\right)d\xi$$
$$\le\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta)}\times$$
$$\left[A_1\int_0^1\left|\int_\xi^1(1-\alpha)^{\delta-1}\omega_1(\alpha b+(1-\alpha)x_1)d\alpha\right|d\xi\right.$$
$$-c(x_1-b)^2\times$$
$$\left.\int_0^1\xi(1-\xi)\left|\int_\xi^1(1-\alpha)^{\delta-1}\omega_1(\alpha b+(1-\alpha)x_1)d\alpha\right|d\xi\right]$$
$$+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta)}\left[A_2\times\right.$$
$$\int_0^1\left|\int_\xi^1(1-\alpha)^{\delta-1}\omega_1(\alpha a+(1-\alpha)x_1)d\alpha\right|d\xi$$
$$-c(x_1-a)^2\times$$
$$\left.\int_0^1\xi(1-\xi)\left|\int_\xi^1(1-\alpha)^{\delta-1}\omega_1(\alpha a+(1-\alpha)x_1)d\alpha\right|d\xi\right]$$
$$\le\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta)}\left[A_1\parallel\omega_1\parallel_{[x_1,b],\infty}\int_0^1\left|\int_\xi^1(1-\alpha)^{\delta-1}d\alpha\right|d\xi\right.$$
$$\left.-c(x_1-b)^2\parallel\omega_1\parallel_{[x_1,b],\infty}\int_0^1\xi(1-\xi)\left|\int_\xi^1(1-\alpha)^{\delta-1}d\alpha\right|d\xi\right]$$
$$+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta)}\left[A_2\parallel\omega_1\parallel_{[a,x_1],\infty}\int_0^1\left|\int_\xi^1(1-\alpha)^{\delta-1}d\alpha\right|d\xi\right.$$
$$\left.-c(x_1-a)^2\parallel\omega_1\parallel_{[a,x_1],\infty}\int_0^1\xi(1-\xi)\left|\int_\xi^1(1-\alpha)^{\delta-1}d\alpha\right|d\xi\right]$$
$$\le\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta+2)}A_1\parallel\omega_1\parallel_{[x_1,b],\infty}$$
$$-\left(\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta+3)}-\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta+4)}\right)c(x_1-b)^2\parallel\omega_1\parallel_{[x_1,b],\infty}$$
$$+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta+2)}A_2\parallel\omega_1\parallel_{[a,x_1],\infty}$$
$$\left(\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta+3)}-\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta+4)}\right)c(x_1-a)^2\parallel\omega_1\parallel_{[a,x_1],\infty},$$

where

$$A_1=\max\left\{|\zeta'(x_1)|,|\zeta'(b)|\right\} \quad \text{and}$$
$$A_2=\max\left\{|\zeta'(x_1)|,|\zeta'(b)|\right\}.$$

This completes the proof.

Corollary 3. *Choosing $x_1=\frac{a+b}{2}$ in Theorem 4, we obtain*

$$\left|J^{\delta}_{\frac{a+b}{2}^-}\omega_1\zeta(a)+J^{\delta}_{\frac{a+b}{2}^+}\omega_1\zeta(b)\right.$$
$$\left.-\left[J^{\delta}_{\frac{a+b}{2}^-}\omega_1(a)+J^{\delta}_{\frac{a+b}{2}^+}\omega_1(a)\right]\zeta\left(\frac{a+b}{2}\right)\right|$$
$$\le\frac{(b-a)^{\delta+1}}{2^{\delta+1}\Gamma(\delta+2)}\parallel\omega_1\parallel_{[a,b],\infty}\max\left\{|\zeta'(x_1)|,|\zeta'(b)|\right\}$$
$$-\left(\frac{(b-a)^{\delta+1}}{2^{\delta+1}\Gamma(\delta+3)}-\frac{(b-a)^{\delta+1}}{2^{\delta+1}\Gamma(\delta+4)}\right)c\parallel\omega_1\parallel_{[a,b],\infty}\left(\frac{b-a}{2}\right)^2$$
$$+\frac{(b-a)^{\delta+1}}{2^{\delta+1}\Gamma(\delta+2)}\max\left\{|\zeta'(x_1)|,|\zeta'(a)|\right\}\parallel\omega_1\parallel_{[a,b],\infty}$$
$$-\left(\frac{(b-a)^{\delta+1}}{2^{\delta+1}\Gamma(\delta+3)}-\frac{(b-a)^{\delta+1}}{2^{\delta+1}\Gamma(\delta+4)}\right)c\left(\frac{b-a}{2}\right)^2\parallel\omega_1\parallel_{[a,b],\infty}.$$

Corollary 4. *Substituting $\omega_1(u) = \frac{1}{b-a}$ in Theorem 4, we get*

$$\left|J^{\delta}_{\frac{a+b}{2}^-}\omega_1\zeta(a) + J^{\delta}_{\frac{a+b}{2}^+}\omega_1\zeta(b) - \frac{(b-x_1)^{\delta}+(x_1-a)^{\delta}}{\Gamma(\delta+1)}\zeta(x_1)\right|$$
$$\leq \frac{(b-x_1)^{\delta+1}}{\Gamma(\delta+2)}\max\left\{\left|\zeta'(x_1)\right|,\left|\zeta'(b)\right|\right\}$$
$$-\left(\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta+3)} - \frac{(b-x_1)^{\delta+1}}{\Gamma(\delta+4)}\right)c(x_1-b)^2$$
$$+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta+2)}\max\left\{\left|\zeta'(x_1)\right|,\left|\zeta'(a)\right|\right\}$$
$$-\left(\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta+3)} - \frac{(x_1-a)^{\delta+1}}{\Gamma(\delta+4)}\right)c(x_1-a)^2.$$

Corollary 5. *For $\alpha = 1$ in Theorem 4, we get*

$$\left|\int_a^b \omega_1(u)\zeta(u)du - \left(\int_a^b \omega_1(u)du\right)\zeta(x_1)\right|$$
$$\leq \frac{(b-x_1)^2}{2}\times$$
$$\left(\max\left\{\left|\zeta'(x_1)\right|,\left|\zeta'(b)\right|\right\} - \frac{c}{24}(x_1-b)^2\right)\parallel \omega_1 \parallel_{[x_1,b],\infty}$$
$$+\frac{(x_1-a)^2}{2}\times$$
$$\left(\max\left\{\left|\zeta'(x_1)\right|,\left|\zeta'(a)\right|\right\} - \frac{c}{24}(x_1-a)^2\right)\parallel \omega_1 \parallel_{[a,x_1],\infty}.$$

Moreover, if we we choose $x_1 = \frac{a+b}{2}$, we obtain

$$\left|\int_a^b \omega_1(u)\zeta(u)du - \left(\int_a^b \omega_1(u)du\right)\zeta\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^2}{8}\times$$
$$\left(\max\left\{\left|\zeta'\left(\frac{a+b}{2}\right)\right|,\left|\zeta'(b)\right|\right\} - \frac{c}{96}(b-a)^2\right)\parallel \omega_1 \parallel_{[a,b],\infty}$$
$$+\frac{(b-a)^2}{8}\times$$
$$\left(\max\left\{\left|\zeta'(\frac{a+b}{2})\right|,\left|\zeta'(a)\right|\right\} - \frac{c}{96}(b-a)^2\right)\parallel \omega_1 \parallel_{[a,b],\infty}.$$

Remark. For $c = 0$ in Theorem 4, then we get Theorem 8 of [34].

Theorem 5. *Let $\zeta : J \to \mathbb{R}$ be a differentiable function with $\zeta' \in L(J)$ where $0 \leq a < b$, and consider $\omega_1 : J \to \mathbb{R}$ be a continuous function. If $\left|\zeta'\right|^q$ is strongly quasi-convex with modulus $c \geq 0$, $q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$\left|J^{\delta}_{x_1^-}\omega_1\zeta(a) + J^{\delta}_{x_1^+}\omega_1\zeta(b) - \left[J^{\delta}_{x_1^-}\omega_1(a) + J^{\delta}_{x_1^+}\omega_1(a)\right]\zeta(x_1)\right|$$
$$\leq \frac{(b-x_1)^{\delta+1}}{(\delta p+1)^{\frac{1}{p}}\Gamma(\delta+1)}\parallel \omega_1 \parallel_{[x_1,b],\infty}\times$$
$$\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(b)\right|^q\right\} - \frac{c}{6}(x_1-b)^2\right)^{\frac{1}{q}}$$
$$+\frac{(x_1-a)^{\delta+1}}{(\delta p+1)^{\frac{1}{p}}\Gamma(\delta+1)}\parallel \omega_1 \parallel_{[a,x_1],\infty}\times$$
$$\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(a)\right|^q\right\} - \frac{c}{6}(x_1-a)^2\right)^{\frac{1}{q}}.$$

Proof. Applying Lemma 2, properties of modulus, Hölder's inequality and strong quasi-convexity of $\left|\zeta'\right|^q$ yields that

$$|J^{\delta}_{x_1^-}\omega_1\zeta(a) + J^{\delta}_{x_1^+}\omega_1\zeta(b) - \left[J^{\delta}_{x_1^-}\omega_1(a) + J^{\delta}_{x_1^+}\omega_1(a)\right]\zeta(x_1)|$$
$$\leq \frac{(b-x_1)^{\delta+1}}{\Gamma(\delta)}\left(\int_0^1 |k_1(\xi)|^p d\xi\right)^{\frac{1}{p}}\times$$
$$\left(\int_0^1 \left|\zeta'(\xi b+(1-\xi)x_1)\right|^q d\xi\right)^{\frac{1}{q}}$$
$$+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta)}\left(\int_0^1 |k_2(\xi)|^p d\xi\right)^{\frac{1}{p}}\times$$
$$\left(\int_0^1 \left|\zeta'(\xi a+(1-\xi)x_1)\right|^q d\xi\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-x_1)^{\delta+1}}{\Gamma(\delta)}\times$$
$$\left(\int_0^1 \left|\int_{\xi}^1 (1-\alpha)^{\delta-1}\omega_1(\alpha b+(1-\alpha)x_1)d\alpha\right|^p d\xi\right)^{\frac{1}{p}}\times$$
$$\left(\int_0^1 \left[B_1 - c\xi(1-\xi)(x_1-b)^2\right]d\xi\right)^{\frac{1}{q}}$$
$$+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta)}\times$$
$$\left(\int_0^1 \left|\int_{\xi}^1 (1-\alpha)^{\delta-1}\omega_1(\alpha a+(1-\alpha)x_1)d\alpha\right|^p d\xi\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1 \left[B_2 - c\xi(1-\xi)(x_1-a)^2\right]d\xi\right)^{\frac{1}{q}}$$
$$\leq \frac{(b-x_1)^{\delta+1}}{\Gamma(\delta)}\parallel \omega_1 \parallel_{[x_1,b],\infty}\left(\int_0^1\left|\int_{\xi}^1 (1-\alpha)^{\delta-1}d\alpha\right|^p d\xi\right)^{\frac{1}{p}}$$

$$\times\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(b)\right|^q\right\}-\frac{c}{6}(x_1-b)^2\right)^{\frac{1}{q}}$$
$$+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta)}\parallel\omega_1\parallel_{[a,x_1],\infty}\left(\int_0^1\left|\int_\xi^1(1-\alpha)^{\delta-1}d\alpha\right|^p d\xi\right)^{\frac{1}{p}}$$
$$\times\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(a)\right|^q\right\}-\frac{c}{6}(x_1-a)^2\right)^{\frac{1}{q}}$$
$$\leq\frac{(b-x_1)^{\delta+1}}{(\delta p+1)^{\frac{1}{p}}\Gamma(\delta+1)}\parallel\omega_1\parallel_{[x_1,b],\infty}\times$$
$$\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(b)\right|^q\right\}-\frac{c}{6}(x_1-b)^2\right)^{\frac{1}{q}}$$
$$+\frac{(x_1-a)^{\delta+1}}{(\delta p+1)^{\frac{1}{p}}\Gamma(\delta+1)}\parallel\omega_1\parallel_{[a,x_1],\infty}\times$$
$$\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(a)\right|^q\right\}-\frac{c}{6}(x_1-a)^2\right)^{\frac{1}{q}},$$

where

$$B_1=\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(b)\right|^q\right\}$$
$$B_2=\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(a)\right|^q\right\}$$

This completes the proof.

Corollary 6. *Choosing $x_1=\frac{a+b}{2}$ in Theorem 5, we obtain*

$$\left|J^\delta_{\frac{a+b}{2}^-}\omega_1\zeta(a)+J^\delta_{\frac{a+b}{2}^+}\omega_1\zeta(b)-\left[J^\delta_{\frac{a+b}{2}^-}\omega_1(a)+J^\delta_{\frac{a+b}{2}^+}\omega_1(a)\right]\zeta\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^{\delta+1}}{2^{\delta+1}(\delta p+1)^{\frac{1}{p}}\Gamma(\delta+1)}\parallel\omega_1\parallel_{[a,b],\infty}\times$$
$$\left(\left(\max\left\{\left|\zeta'\left(\frac{a+b}{2}\right)\right|^q,\left|\zeta'(b)\right|^q\right\}-\frac{c}{24}(b-a)^2\right)^{\frac{1}{q}}+\left(\max\left\{\left|\zeta'\left(\frac{a+b}{2}\right)\right|^q,\left|\zeta'(a)\right|^q\right\}-\frac{c}{24}(b-a)^2\right)^{\frac{1}{q}}\right).$$

Corollary 7. *Putting $\omega_1(u)=\frac{1}{b-a}$ in Theorem 5, we get*

$$\left|J^\delta_{\frac{a+b}{2}^-}\omega_1\zeta(a)+J^\delta_{\frac{a+b}{2}^+}\omega_1\zeta(b)-\frac{(b-x_1)^\delta+(x_1-a)^\delta}{\Gamma(\delta+1)}\zeta(x_1)\right|$$
$$\leq\frac{(b-x_1)^{\delta+1}}{(\delta p+1)^{\frac{1}{p}}\Gamma(\delta+1)}\times\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(b)\right|^q\right\}-\frac{c}{6}(x_1-b)^2\right)^{\frac{1}{q}}$$
$$+\frac{(x_1-a)^{\delta+1}}{(\delta p+1)^{\frac{1}{p}}\Gamma(\delta+1)}\times\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(a)\right|^q\right\}-\frac{c}{6}(x_1-a)^2\right)^{\frac{1}{q}}.$$

Corollary 8. *For $\alpha=1$ in Theorem 5, we get*

$$\left|\int_a^b\omega_1(u)\zeta(u)du-\left(\int_a^b\omega_1(u)du\right)\zeta(x_1)\right|$$
$$\leq\frac{(b-x_1)^2}{(p+1)^{\frac{1}{p}}}\parallel\omega_1\parallel_{[x_1,b],\infty}\times\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(b)\right|^q\right\}-\frac{c}{6}(x_1-b)^2\right)^{\frac{1}{q}}$$
$$+\frac{(x_1-a)^2}{(p+1)^{\frac{1}{p}}}\parallel\omega_1\parallel_{[a,x_1],\infty}\times\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(a)\right|^q\right\}-\frac{c}{6}(x_1-a)^2\right)^{\frac{1}{q}}.$$

Moreover, if we we choose $x_1=\frac{a+b}{2}$, we obtain

$$\left|\int_a^b\omega_1(u)\zeta(u)du-\left(\int_a^b\omega_1(u)du\right)\zeta\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\parallel\omega_1\parallel_{[a,b],\infty}\times$$
$$\left(\left(\max\left\{\left|\zeta'\left(\frac{a+b}{2}\right)\right|^q,\left|\zeta'(b)\right|^q\right\}-\frac{c}{24}(b-a)^2\right)^{\frac{1}{q}}+\left(\max\left\{\left|\zeta'\left(\frac{a+b}{2}\right)\right|^q,\left|\zeta'(a)\right|^q\right\}-\frac{c}{24}(b-a)^2\right)^{\frac{1}{q}}\right).$$

Corollary 9. *In Corollary 8, for $\omega_1(u)=\frac{1}{b-a}$*

$$\left|\frac{1}{b-a}\int_a^b\zeta(u)du-\zeta(x_1)\right|$$
$$\leq\frac{(b-x_1)^2}{(b-a)(p+1)^{\frac{1}{p}}}\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(b)\right|^q\right\}-\frac{c}{6}(x_1-b)^2\right)^{\frac{1}{q}}$$
$$+\frac{(x_1-a)^2}{(b-a)(p+1)^{\frac{1}{p}}}\left(\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(a)\right|^q\right\}-\frac{c}{6}(x_1-a)^2\right)^{\frac{1}{q}}.$$

Remark. For $c=0$ in Theorem 5, then we get Theorem 9 of [34].

Theorem 6. *Consider $\zeta:J\to\mathbb{R}$ be a differentiable function with $\zeta'\in L(J)$ where $0\leq a<b$, and let $\omega_1:J\to\mathbb{R}$ be a continuous function. If $\left|\zeta'\right|^q$ is strongly quasi-convex with modulus $c\geq 0$, where $q\geq 1$, then*

$$\left|J^\delta_{x_1^-}\omega_1\zeta(a)+J^\delta_{x_1^+}\omega_1\zeta(b)-\left[J^\delta_{x_1^-}\omega_1(a)+J^\delta_{x_1^+}\omega_1(a)\right]\zeta(x_1)\right|$$
$$\leq\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta+2)}\parallel\omega_1\parallel_{[x_1,b],\infty}$$
$$\times\left(B_1-c\left(\frac{\delta+1}{(\delta+2)}-\frac{\delta+1}{(\delta+3)}\right)(x_1-b)^2\right)^{\frac{1}{q}}$$
$$+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta+2)}\parallel\omega_1\parallel_{[a,x_1],\infty}$$
$$\times\left(B_2-c\left(\frac{\delta+1}{(\delta+2)}-\frac{\delta+1}{(\delta+3)}\right)(x_1-a)^2\right)^{\frac{1}{q}},$$

where

$$B_1=\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(b)\right|^q\right\}$$
$$B_2=\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(a)\right|^q\right\}$$

*Proof.*By properties of modulus, applying Lemma 2, power mean inequality and strong quasi-convexity of $\left|\zeta'\right|^q$ yields that

$$\begin{aligned}
&\left|J^{\delta}_{x_1^-}\omega_1\zeta(a)+J^{\delta}_{x_1^+}\omega_1\zeta(b)-\left[J^{\delta}_{x_1^-}\omega_1(a)+J^{\delta}_{x_1^+}\omega_1(a)\right]\zeta(x_1)\right|\\
&\le\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta)}\left(\int_0^1|k_1(\xi)|d\xi\right)^{1-\frac{1}{q}}\times\\
&\qquad\left(\int_0^1|k_1(\xi)||\zeta'(\xi b+(1-\xi)x_1)|^q d\xi\right)^{\frac{1}{q}}\\
&+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta)}\left(\int_0^1|k_2(\xi)|d\xi\right)^{1-\frac{1}{q}}\times\\
&\qquad\left(\int_0^1|k_2(\xi)||\zeta'(\xi a+(1-\xi)x_1)|^q d\xi\right)^{\frac{1}{q}}\\
&\le\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta)}\left(\int_0^1|k_1(\xi)|d\xi\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1|k_1(\xi)|\left[B_1-c\xi(1-\xi)(x_1-b)^2\right]d\xi\right)^{\frac{1}{q}}\\
&+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta)}\left(\int_0^1|k_2(\xi)|d\xi\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1|k_2(\xi)|\left[B_2-c\xi(1-\xi)(x_1-a)^2\right]d\xi\right)^{\frac{1}{q}}\\
&\le\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta)}\parallel\omega_1\parallel_{[x_1,b],\infty}\left[\left(\int_0^1(A(\xi))d\xi\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\int_0^1(A(\xi))\left[B_1-c\xi(1-\xi)(x_1-b)^2\right]d\xi\right)^{\frac{1}{q}}\\
&+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta)}\parallel\omega_1\parallel_{[a,x_1],\infty}\left[\left(\int_0^1(A(\xi))d\xi\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\int_0^1A(\xi)\left[B_2-c\xi(1-\xi)(x_1-a)^2\right]d\xi\right)^{\frac{1}{q}}\\
&\le\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta+2)}\parallel\omega_1\parallel_{[x_1,b],\infty}\\
&\times\left(B_1-c\left(\frac{\delta+1}{(\delta+2)}-\frac{\delta+1}{(\delta+3)}\right)(x_1-b)^2\right)^{\frac{1}{q}}\\
&+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta+2)}\parallel\omega_1\parallel_{[a,x_1],\infty}\\
&\times\left(B_2-c\left(\frac{\delta+1}{(\delta+2)}-\frac{\delta+1}{(\delta+3)}\right)(x_1-a)^2\right)^{\frac{1}{q}}.
\end{aligned}$$

where

$$A(\xi)=\int_\xi^1(1-\alpha)^{\delta-1}d\alpha$$
$$B_1=\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(b)\right|^q\right\}$$
$$B_2=\max\left\{\left|\zeta'(x_1)\right|^q,\left|\zeta'(a)\right|^q\right\}$$

This completes the proof.

Corollary 10.*Choosing $x_1=\frac{a+b}{2}$ in Theorem 6, we obtain*

$$\begin{aligned}
&\left|J^{\delta}_{\frac{a+b}{2}^-}\omega_1\zeta(a)+J^{\delta}_{\frac{a+b}{2}^+}\omega_1\zeta(b)\right.\\
&\qquad\left.-\left[J^{\delta}_{\frac{a+b}{2}^-}\omega_1(a)+J^{\delta}_{\frac{a+b}{2}^+}\omega_1(a)\right]\zeta\left(\frac{a+b}{2}\right)\right|\\
&\le\frac{(b-a)^{\delta+1}}{2^{\delta+1}\Gamma(\delta+2)}\parallel\omega_1\parallel_{[a,b],\infty}\times\\
&\left[\left(C_1-\frac{c}{4}\left(\frac{\delta+1}{(\delta+2)}-\frac{\delta+1}{(\delta+3)}\right)(b-a)^2\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(C_2-\frac{c}{4}\left(\frac{\delta+1}{(\delta+2)}-\frac{\delta+1}{(\delta+3)}\right)(b-a)^2\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$C_1=\max\left\{\left|\zeta'\left(\frac{a+b}{2}\right)\right|^q,\left|\zeta'(b)\right|^q\right\}$$
$$C_2=\max\left\{\left|\zeta'\left(\frac{a+b}{2}\right)\right|^q,\left|\zeta'(a)\right|^q\right\}$$

Corollary 11.*Putting $\omega_1(u)=\frac{1}{b-a}$ in Theorem 6, we get*

$$\begin{aligned}
&\left|J^{\delta}_{\frac{a+b}{2}^-}\zeta(a)+J^{\delta}_{\frac{a+b}{2}^+}\zeta(b)-\frac{(b-x_1)^{\delta}+(x_1-a)^{\delta}}{\Gamma(\delta+1)}\zeta(x_1)\right|\\
&\le\frac{(b-x_1)^{\delta+1}}{\Gamma(\delta+2)}\times\\
&\qquad\left(B_1-c\left(\frac{\delta+1}{(\delta+2)}-\frac{\delta+1}{(\delta+3)}\right)(x_1-b)^2\right)^{\frac{1}{q}}\\
&+\frac{(x_1-a)^{\delta+1}}{\Gamma(\delta+2)}\times\\
&\qquad\left(B_2-c\left(\frac{\delta+1}{(\delta+2)}-\frac{\delta+1}{(\delta+3)}\right)(x_1-a)^2\right)^{\frac{1}{q}},
\end{aligned}$$

where B_1 and B_2 are defined as in the proof of Theorem 6. Moreover, for $x_1=\frac{a+b}{2}$, we get

$$\begin{aligned}
&\left|\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)^{\delta}}\left(J^{\delta}_{\frac{a+b}{2}^-}\zeta(a)+J^{\delta}_{\frac{a+b}{2}^+}\zeta(b)\right)-\zeta\left(\frac{a+b}{2}\right)\right|\\
&\le\frac{(b-a)}{4(\delta+1)}\left[\left(C_1-\frac{c}{4}\left(\frac{\delta+1}{(\delta+2)}-\frac{\delta+1}{(\delta+3)}\right)(b-a)^2\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(C_2-\frac{c}{4}\left(\frac{\delta+1}{(\delta+2)}-\frac{\delta+1}{(\delta+3)}\right)(b-a)^2\right)^{\frac{1}{q}}\right].
\end{aligned}$$

where C_1 and C_2 are defined as in the proof of Corollary 10.

Corollary 12. *For $\alpha = 1$ in Theorem 6, we get*

$$\left| \int_a^b \omega_1(u)\zeta(u)du - \left(\int_a^b \omega_1(u)du\right)\zeta(x_1)\right|$$
$$\leq \frac{(b-x_1)^{\delta+1}}{2} \parallel \omega_1 \parallel_{[x_1,b],\infty} \times \left(B_1 - c\left(\frac{\delta+1}{(\delta+2)} - \frac{\delta+1}{(\delta+3)}\right)(x_1-b)^2\right)^{\frac{1}{q}}$$
$$+ \frac{(x_1-a)^{\delta+1}}{2} \parallel \omega_1 \parallel_{[a,x_1],\infty} \times \left(B_2 - c\left(\frac{\delta+1}{(\delta+2)} - \frac{\delta+1}{(\delta+3)}\right)(x_1-a)^2\right)^{\frac{1}{q}},$$

where B_1 and B_2 are defined as in the proof of Theorem 6. Moreover, if we we choose $x_1 = \frac{a+b}{2}$, we obtain

$$\left| \int_a^b \omega_1(u)\zeta(u)du - \left(\int_a^b \omega_1(u)du\right)\zeta\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^{\delta+1}}{2^{\delta+2}} \parallel \omega_1 \parallel_{[a,b],\infty} \times \left[\left(C_1 - \frac{c}{4}\left(\frac{\delta+1}{(\delta+2)} - \frac{\delta+1}{(\delta+3)}\right)(b-a)^2\right)^{\frac{1}{q}} + \left(C_2 - \frac{c}{4}\left(\frac{\delta+1}{(\delta+2)} - \frac{\delta+1}{(\delta+3)}\right)(b-a)^2\right)^{\frac{1}{q}}\right],$$

where C_1 and C_2 are defined as in the proof of Corollary 10.

Remark. For $c = 0$ in Theorem 6, then we obtain Theorem 10 of [34].

4 Applications

Let us recall the following special means for arbitrary positive real numbers a and b, where $a < b$. we take

(1) Arithmetic mean:

$$A(a,b) = \frac{a+b}{2}, \ a,b \in \mathbb{R}.$$

(2) Logarithmic mean:

$$L(a,b) = \frac{a-b}{\ln|a| - \ln|b|}, \ |a| \neq |b|, \ a,b \neq 0, \ a,b \in \mathbb{R}.$$

(3) n-Logarithmic mean:

$$L_n(a,b) = \left[\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}}, \ n \in \mathbb{Z}\ \{0,-1\}.$$

Now, using section 2, we will write some applications to special means.

Proposition 1. *Let $a,b \in \mathbb{R}$, $0 < a < b$ and $p > 1$, then*

$$\left|A^{-1}(a,b) - L^{-1}(a,b)\right| \leq \frac{(b-a)}{2^{\frac{p+1}{p}}(p+1)^{\frac{1}{p}}} \times$$
$$\left\{\left[\max\left(\left|\left(\frac{a+b}{2}\right)\right|^{-\frac{2p}{p-1}}, |b|^{-\frac{2p}{p-1}}\right) - \frac{c}{48}(b-a)^2\right]^{\frac{p-1}{p}} + \left[\max\left(\left|\left(\frac{a+b}{2}\right)\right|^{-\frac{2p}{p-1}}, |a|^{-\frac{2p}{p-1}}\right) - \frac{c}{48}(b-a)^2\right]^{\frac{p-1}{p}}\right\}.$$

Proof. Choosing $\zeta(x_1) = \frac{1}{x_1}$, $x_1 > 0$ in corollary 1, we get the desired result.

Proposition 2. *Let $a,b \in \mathbb{R}$, $0 < a < b$, $q \geq 1$, $n \in \mathbb{N}$ and $n \geq 2$, then*

$$|A^n(a,b) - L_n^n(a,b)| \leq \frac{n(b-a)}{8} \times$$
$$\left[\left(\max\left\{\left|\frac{a+b}{2}\right|^{(n-1)q}, |b|^{(n-1)q}\right\} - \frac{5c}{96n}(b-a)^2\right)^{\frac{1}{q}} + \left(\max\left\{\left|\frac{a+b}{2}\right|^{(n-1)q}, |a|^{q}\right\} - \frac{5c}{96n}(b-a)^2\right)^{\frac{1}{q}}\right].$$

Proof. Taking $\zeta(x_1) = x_1^n$, $x_1 \in \mathbb{R}$ in corollary 2, we obtain the desired result.

Proposition 3. *Let $a,b \in \mathbb{R}$, $0 < a < b$, $n \in \mathbb{N}$ and $n \geq 2$, then*

$$\left|L_{n+1}^{n+1}(a,b) - A^{n+1}(a,b)\right| \leq \frac{nb(b-a)}{4}\left(A\left(A^{n-1}(a,b), b^{n-1}\right) - \frac{c}{96n}(b-a)^2\right).$$

Proof. Choosing $\zeta(x_1) = x_1^n$, $x_1 > 0$ for $x_1 = \frac{a+b}{2}$ and $\omega_1(x_1) = x_1$ in corollary 5, we obtain the desired result.

Proposition 4. *Let $a,b \in \mathbb{R}$, $0 < a < b$, $n \in \mathbb{N}$ and $n \geq 2$, then*

$$\left|L_{n-1}^{n-1}(a,b) - \frac{L_n^n(a,b)}{A(a,b)}\right| \leq \frac{b^n(b-a)}{4}\left(A\left(\frac{1}{a^2}, \frac{1}{A^2(a,b)}\right) - \frac{c}{96}(b-a)^2\right).$$

Proof. Considering $\zeta(x_1) = \frac{1}{x_1}$, $x_1 > 0$ for $x_1 = \frac{a+b}{2}$ and $\omega_1(x_1) = x_1^n$ in corollary 5, we obtain the desired result.

Proposition 5. *Let $a,b \in \mathbb{R}$, $0 < a < b$, $p > 1$, $n \in \mathbb{N}$ and $n \geq 2$, then*

$$\left|L_{n+1}^{n+1}(a,b) - A^{n+1}(a,b)\right| \leq \frac{nb(b-a)}{2(p+1)^{\frac{1}{p}}}\left(A\left(A^{n-1}(a,b), b^{n-1}\right) - \frac{c}{24n}(b-a)^2\right).$$

Proof. Putting $\zeta(x_1) = x_1^n$, $x_1 > 0$ for $x_1 = \frac{a+b}{2}$ and $\omega_1(x_1) = x_1$ in corollary 8, we get the desired result.

Proposition 6. *Let $a, b \in \mathbb{R}$, $0 < a < b$, $p > 1$, $n \in \mathbb{N}$ and $n \geq 2$, then*

$$\left| L_{n-1}^{n-1}(a,b) - \frac{L_n^n(a,b)}{A(a,b)} \right| \leq \frac{b^n(b-a)}{2(p+1)^{\frac{1}{p}}} \left(A\left(\frac{1}{a^2}, \frac{1}{A^2(a,b)} \right) - \frac{c}{24}(b-a)^2 \right).$$

Proof. Taking $\zeta(x_1) = \frac{1}{x_1}, x_1 > 0$ for $x_1 = \frac{a+b}{2}$ and $\omega_1(x_1) = x_1^n$ in corollary 8, we get the desired result.

5 Conclusions

In this study, we discussed some Ostrowski-type inequalities for the strongly quasi-convex function and then we derive fractional weighted Ostrowski-type inequalities using differentiable strongly quasi-convex function. Further, several results with a bounded first derivative are provided. Also, some applications to special means are presented. In the future, from our results, the concerned reviewers can find many novel inequalities from several areas of applied and pure sciences. Furthermore, they can establish applications to special means for various quasi-convex functions using our technique.

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Conflicts of Interests

The authors declare that they have no conflicts of interests.

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