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New Approximate Solutions to Some of Nonlinear PDEs with Atangana-Baleanu-Caputo operator

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Abstract: The natural homotopy perturbation technique (NHPT) is an excellent analytical tool employed in this study to solve the nonlinear differential equations (NDEs). The Antagana-Baleanu sense is used to characterize the fractional derivatives (ABFD). We also discuss the convergence of the NHPT for NDEs. To show the applicability of the recommended technique, examples are presented.

Keywords: Natural transform, fractional differential equations, Homotopy permutation method, Atangana -Baleanu operator.

1 Introduction

During the last three decades, much emphasis has been placed on the study of fractional calculus and its diverse applications in physics and engineering. Fractional calculus has applications in a variety of domains, including electrical networks, dynamical systems control theory, probability and statistics, and electrochemistry. Linear or nonlinear fractional equations can be used to predict corrosion, chemical physics, optics, and signal processing. differential equations of first order Many texts provide different definitions of fractional calculus[1,2,3]. Many numerical and analytical strategies for solving linear and nonlinear FPDEs have been proposed [4,5,6,7,8,9,10,11].

In this work, we use NHPT to solve NPDEs that include the fractional operator Atangana-Baleanu-Caputo. The paper is arranged in the following way: The basic definitions for calculus and fractional integration are presented in Section 2, the methods used are analyzed in Section 3, the convergence of the method is discussed in Section 4, and examples are given in Section 5, and finally, the conclusion is introduced in Section 6.

2 Preliminary

Definition 1.[11] *Let* $v \in H^1(\varepsilon_1, \varepsilon_2), \varepsilon_1 > \varepsilon_2$*, the ABC sense for* $0 < \delta < 1$ *is*

$$ {}^{\mathscr{ABC}}\mathscr{D}_t^{\delta} v(t) = \frac{\mathscr{B}(\delta)}{1-\delta} \int_0^t \mathrm{E}_a \left(-\frac{\delta (t-s)^{\delta}}{1-\delta} \right) v'(s) \mathrm{d}s, t \geq 0 \tag{1} $$

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where $\mathscr{B}(\delta)$ is a normalization function such that $\mathscr{B}(0)=\mathscr{B}(1)=1$ and $v'(8)$ is the derivative of v.

Definition 2.*[12] The natural transform is defined over the set of function*

$$\mathscr{A}=\left\{v(t)\,|\exists \mathscr{M},\tau_1,\tau_2>0,|\,v(t)\,|<\mathscr{M}e^{\frac{|t|}{\tau_j}},\ if\ t\in(-1)^j\times[0,\infty)\right\}$$

by the following formula

$$\mathscr{N}[v(t)]=\mathscr{R}(u,s)=\int_0^\infty v(ut)e^{-st}dt,u\in(\tau_1,\tau_2) \tag{2}$$

Definition 3.*[?] The inverse natural transform of a function is defined by*

$$\mathscr{N}^{-1}[\mathscr{R}(u,s)]=v(t)=\frac{1}{2i\pi}\int_{p-\infty}^{p+\infty}e^{\frac{st}{u}}\mathscr{R}(u,s)dt,u,s>0 \tag{3}$$

where s and u are natural transform variables a and p is real constant.
The LT can be obtained by the NT through the following relationship

$$\mathscr{R}(u,s)=\frac{1}{u}\int_0^\infty v(t)e^{-\frac{st}{u}}dt=\frac{1}{u}\mathscr{F}\left(\frac{s}{u}\right) \tag{4}$$

Lemma 1.*Let $\mathscr{N}[v(t)]$ is the natural transform of $v(t)$, then the natural transform of the fractional derivative with ABC of $v(t)$ for $\delta\in(0,1)$ is*

$$\mathscr{N}\left({}^{\mathscr{A}\mathscr{B}\mathscr{C}}\mathscr{D}_t^\delta v(t)\right)=\frac{\mathscr{B}(\delta)}{1-\delta+\delta\left(\frac{u}{8}\right)}\delta\left(\mathscr{N}(u,s)-\frac{1}{s}v(0)\right), \tag{5}$$

*Proof.*From [11] , Laplace transform of Atangana-Baleanu-Caputo operator of $f(t)$ is

$$\mathscr{L}\left({}^{\mathscr{A}\mathscr{B}\mathscr{C}}\mathscr{D}_t^\delta v(t)\right)=\frac{\mathscr{B}(\delta)}{1-\delta}\frac{s^\delta\mathscr{F}(s)-s^{\delta-1}v(0)}{s^\delta+\frac{\delta}{1-\delta}} \tag{6}$$

after a few simple steps, the following relationship can be obtained

$$\mathscr{L}\left({}^{\mathscr{A}\mathscr{B}\mathscr{C}}\mathscr{D}_t^\delta v(t)\right)=\frac{\mathscr{B}(\delta)}{1-\delta+\delta s^{-a}}\left(\mathscr{F}(s)-\frac{1}{s}v(0)\right), \tag{7}$$

from relation (3), the result is,

$$\begin{aligned}\mathscr{N}\left({}^{\mathscr{A}\mathscr{B}\mathscr{C}}\mathscr{D}_t^\delta v(t)\right)&=\frac{\mathscr{B}(\delta)}{1-\delta+\delta\left(\frac{u}{s}\right)^\delta}\left(\frac{1}{u}\mathscr{F}\left(\frac{s}{u}\right)-\frac{1}{s}v(0)\right)\\&=\frac{\mathscr{B}(\delta)}{1-\delta+\delta\left(\frac{u}{s}\right)^\delta}\left(\mathscr{N}(u,s)-\frac{1}{s}v(0)\right)\end{aligned} \tag{8}$$

3 Analysis of the Method

Suppose that fractional PDE with ABC operator

$${}^{\mathscr{A}\mathscr{B}\mathscr{C}}\mathscr{D}_t^\delta v(x,t)+\mathscr{L}[v(x,t)]+\mathscr{M}[v(x,t)]=g(x,t) \tag{9}$$

with initial conditions $v(x,0)=v_0(x)$,

Applying the natural transform to (9) subject to the given initial condition,

$$\frac{\mathscr{B}(\delta)}{1-\delta+\delta\left(\frac{u}{s}\right)^\delta}\left(\mathscr{N}(u,s)-\frac{1}{s}v(0)\right)=\mathscr{N}(g(x,t)-\mathscr{L}[v(x,t)]-\mathscr{M}[v(x,t)]) \tag{10}$$

by substituting initial condition of eq.(9),

$$\bar{v} = \frac{1}{8}v_0(x) - \frac{1-\delta+\delta\left(\frac{u}{8}\right)^{\delta}}{\mathscr{B}(\delta)}\mathscr{N}(\mathscr{L}[v]+\mathscr{M}[v]-g), \tag{11}$$

applying the inverse of the NT to both sides of the Eq.(11),

$$v = v_0(x) + \mathscr{N}^{-1}(g) - \mathscr{N}^{-1}\left(\frac{1-\delta+\delta\left(\frac{u}{g}\right)^{\delta}}{\mathscr{B}(\delta)}\mathscr{N}(\mathscr{L}[v]+\mathscr{M}[v])\right) \tag{12}$$

By applying HPM,

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t), \quad \mathscr{N}[u(x,t)] = \sum_{n=0}^{\infty} p^n \mathscr{H}_n(v) \tag{13}$$

Substituting Eq.(13) into Eq.(12):

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = \mathscr{G}(x,t) - p\left(\mathscr{N}^{-1}\left(\frac{1-\delta+\delta\left(\frac{u}{\delta}\right)^{\delta}}{\mathscr{B}(\delta)}\mathscr{N}\left(\sum_{\mathrm{n}=0}^{\infty}\sqrt{}^{n}\mathscr{L}[v_n]+\sum_{\mathrm{n}=0}^{\infty}\sqrt{}^{\mathrm{n}}\mathscr{H}_{\mathrm{n}}(v)\right)\right)\right) \tag{14}$$

By comparing both sides of the equation, the following result is obtained,

$$p^0 : v_0(x,t) = \mathscr{G}(x,t)$$

$$p^1 : v_1(x,t) = -\mathscr{N}^{-1}\left(\frac{1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}}{\mathscr{B}(\delta)}\mathscr{N}(\mathscr{L}[v_0]+\mathscr{H}_0(v))\right)$$

$$\vdots$$

$$p^n : v_n(x,t) = -\mathscr{N}^{-1}\left(\frac{1-\delta+\delta\left(\frac{u}{8}\right)^{\delta}}{\mathscr{B}(\delta)}\mathscr{N}(\mathscr{L}[v_{n-1}]+\mathscr{H}_{n-1}(v))\right) \tag{15}$$

Using the parameter p, the solution is

$$v(x,t) = \sum^{\infty} p^n v_n(x,t) \tag{16}$$

Setting $p = 1$ results in the solution of Eq.(17)

$$v(x,t) = \lim_{p\to 1}\sum_{\mathrm{n}=0}^{\infty} p^n v_n(x,t) = \sum_{\mathrm{n}=0}^{\infty} v_n(x,t) \tag{17}$$

4 Convergence Analysis

In this section, we discuss the convergence of the natural variational iteration method, relying on the previous section by the method of Cauchy series for a sequence of partial sums, we will see the preliminary results in the following theorems.

Theorem 1.*Let* H *is a Hilbert space , then the series* $v(x,t) = \sum_{\mathrm{n}=0}^{\infty} v_n(x,t)$ *defined in (18) converges to* $\mathscr{S} \in$ H, *if* $\exists 0 < \delta < 1$ *such that* $\text{——} v_{n+1}|| < \delta \|v_n\|, \mathrm{n} = 0,1,2,3,\ldots$

*Proof.*Define the sequence of partial sums $\{\delta\}_{n=0}^{\infty}$ as,

$$\begin{aligned}\mathscr{S}_0 &= v_0\\ \mathscr{S}_1 &= v_0 + v_1\\ \mathscr{S}_2 &= v_0 + v_1 + v_2\\ &\vdots\\ \mathscr{S}_n &= v_0 + v_1 + v_2 + \cdots + v_n\end{aligned} \tag{18}$$

now, we prove that $\{\mathscr{S}\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space H.

Proof.

$$\|\delta_{n+1}-\mathscr{S}_n\| = \left\|\sum_{n=0}^{n+1} v_i - \sum_{n=0}^{n} v_i\right\| = \|v_{n+1}\|$$
$$\leq \delta||v_n\|\leq \delta^2||v_{n-1}|| \leq \delta^3||v_{n-2}\||\leq \cdots \leq \delta^{n+1}||v_0|\,|\,. \tag{19}$$

For all $n,m \in N, n \geq m$,

$$\|\mathscr{S}_n-\mathscr{S}_m\| = \|(\delta_n-\delta_{n-1})+(\delta_{n-1}-\delta_{n-2})+\cdots+(\delta_{m+1}-\mathscr{S}_m)\|\,|$$
$$\leq \|\delta_n-\delta_{n-1}\|+\|\delta_{n-1}-\delta_{n-2}\|+\cdots+\|\delta_{m+1}-\delta_m\|$$
$$\leq \delta^n||v_0|| \leq \delta^{n-1}||v_0|| \leq \delta^{n-2}||v_0|| \leq \cdots \leq \delta^{m+1}||v_0|.$$
$$\leq \delta^{m+1}||v_0||\left(\delta^{n-m-1}+\delta^{n-m-2}+\cdots+1\right)$$
$$= \frac{1-\delta^{n-m}}{1-\delta}\delta^{m+1}||v_0||, \tag{20}$$

Since $\left(\delta^{n-m-1}+\delta^{n-m-2}+\cdots+1\right)$ is a geometric series and $0<\delta<1$, then $\lim_{n,m\to\infty}\|\mathscr{S}_n-\mathscr{S}_m\|=0$. Therefore is $\{\delta\}_{n=0}^{\infty}$ a Cauchy sequence in the Hilbert space H and therefore produces that the series solution $v(x,t)=\sum_{n=0}^{\infty} v_n(x,t)$, defined in (18) converges.

Theorem 2.*Suppose that the series solution $\sum_{\mathrm{n}=0}^{\infty} v_n(x,t)$ mentioned in (18) is convergent to the solution $v(x,t)$. If $\sum_{\mathrm{n}=0}^{\infty} v_n(x,t)$ is used as an approximation to the solution* v(x,t) *of problem (9) then the maximum error , $E_m(x,t)$ is estimated as*

$$E_m(x,t) \leq \frac{1}{1-\delta}\delta^{m+1}||v_0||. \tag{21}$$

*Proof.*From theorem 1 , inequality (21)

$$\|\delta_n-\mathscr{S}_m\| \leq \frac{1-\delta^{n-m}}{1-\delta}\delta^{m+1}||v_0||, \tag{22}$$

for $n \geq m$, now, as $n\to\infty$ then $\mathscr{S}_n \to v(x,t)$ so,

$$\left\|v(x,t)-\sum_{\mathrm{k}=0}^{\mathrm{m}} v_k(x,t)\right\| \leq \frac{1-\delta^{n-m}}{1-\delta}\delta^{m+1}||v_0||, \tag{23}$$

Also, since $0\leq\delta\leq 1$ we have $(1-\delta_{n-m})<1$ Therefor the above inequality becomes

$$\left\|v(x,t)-\sum_{\mathrm{k}=0}^{\mathrm{m}} v_k(x,t)\right\| \leq \frac{1}{1-\delta}\delta^{m+1}||v_0|| \tag{24}$$

5 Implementations

We will solve two linear and non-linear equations and show tables of solution values and graphs to solve the two equations, we will suppose that $B(\kappa)=1$.

*Example 1.*Let us consider the following PDE with the Atangana-Baleanu-Caputo sense

$${}^{\mathscr{ABC}}\mathscr{D}_\tau^\delta \varphi(\mu,\tau) = -\frac{\partial}{\partial\mu}\left(\frac{12}{\mu}\varphi-\mu\right)\varphi+\frac{\partial^2}{\partial\mu^2}\varphi^2, \quad , \quad 0<\delta\leq 1 \tag{25}$$

subject to the initial condition $\varphi(\mu,0)=\mu^2$ By using the NT to both sides of (26),

$$\mathscr{N}\left[{}^{\mathscr{ABC}}\mathscr{D}_\tau^\delta\varphi(\mu,\tau)=\frac{12}{\mu^2}\varphi^2-\frac{12}{\mu}\varphi_\mu\varphi+\varphi+\left(\varphi^2\right)_{\mu\mu}\right] \tag{26}$$

Taking INT of (27):

$$\varphi(\mu,\tau)=\mu^2+\mathscr{N}^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{\delta}\right)^\delta\right)\mathscr{N}\left[\frac{12}{\mu^2}\varphi^2-\frac{12}{\mu}\varphi_\mu\varphi+\left(\varphi^2\right)_{\mu\mu}+\varphi\right]\right] \tag{27}$$

By applying HPM on Eq.(28),

$$\sum_{n=0}^{\infty}p^n\varphi_n=\mu^2-p\mathscr{N}^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{g}\right)^\delta\right)\mathscr{N}\left[\sum_{n=0}^{\infty}p^{\mathrm{n}}\mathscr{A}_{\mathrm{n}}-\sum_{\mathrm{n}=0}^{\infty}p^{\mathrm{n}}\mathscr{B}_{\mathrm{n}}+\sum_{\mathrm{n}=0}^{\infty}\sqrt{}^{\mathrm{n}}\mathscr{C}_{\mathrm{n}}+\sum_{n=0}^{\infty}p^n\varphi_n\right]\right] \tag{28}$$

By comparing both sides of the Eq.(29), the following result is obtained,

$$p^0:\varphi_0=\mu^2$$

$$p^1:\varphi_1=\mathscr{N}^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{\int}\right)^\delta\right)\mathscr{N}\left[p^0\mathscr{A}_0-p^0\mathscr{B}_0+p^0\mathscr{C}_0+p^0\varphi_0\right]\right]$$

$$p^2:\varphi_2=\mathscr{N}^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{\}}\right)^\delta\right)\mathscr{N}\left[p^1\mathscr{A}_1-p^1\mathscr{B}_1+p^1\mathscr{C}_1+p^1\varphi_1\right]\right]$$

By the above algorithms,

$$\varphi_0=\mu^2$$

$$\varphi_1=\mu^2\left(1-\delta+\delta\frac{\tau^\delta}{\Gamma(\delta+1)}\right)$$

$$\varphi_2=\mu^2\left(\left(1-2\delta+\delta^2\right)+\left(2\delta-2\delta^2\right)\frac{\tau^a}{\Gamma(\delta+1)}+\delta^2\frac{\tau^{2a}}{\Gamma(2\delta+1)}\right)$$

and so on.
Therefore, the series solution $\varphi(\mu,\tau)$ is given by

$$\varphi(\mu,\tau)=\mu^2\left[\left(3-3\delta+\delta^2\right)+\left(3\delta-2\delta^2\right)\frac{\tau^a}{\Gamma(\delta+1)}+\delta^2\frac{\tau^{2a}}{\Gamma(2\delta+1)}+\cdots\right] \tag{29}$$

If we $\delta\to 1$ in (30):

$$\varphi(\mu,\tau)=\mu^2\left(1+\frac{\tau}{1!}+\frac{\tau^2}{2!}+\cdots\right)=\mu^2\sum_{k=0}^{\infty}\frac{\tau^k}{k!}=\mu^2e^\tau \tag{30}$$

*Example 2.*Consider the gas dynamics equation with the ABC sense

$${}^{\mathscr{ABC}}\mathscr{D}_\tau^\delta\varphi(\mu,\tau)+\frac{1}{2}\left(\varphi^2\right)_\mu-\varphi+\varphi^2=0, \tag{31}$$

subject to the initial condition $\varphi(\mu,0)=\mathrm{e}^{-\mu}$ By using the NT to both sides of (32),

$$\mathscr{N}\left[{}^{\mathscr{ABC}}\mathscr{D}_\tau^\delta\varphi(\mu,\tau)=-\frac{1}{2}\left(\varphi^2\right)_\mu+\varphi-\varphi^2\right] \tag{32}$$

Taking the inverse NT of (33):

$$\varphi(\mu,\tau)=\mu^2+\mathscr{N}^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{s}\right)^\delta\right)\mathscr{N}\left[-\frac{1}{2}\left(\varphi^2\right)_\mu-\varphi^2+\varphi\right]\right] \tag{33}$$

By applying HPM (34),

$$\sum_{n=0}^{\infty} p^n \varphi_n = \mu^2 - p\mathscr{N}^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{s}\right)^{\delta}\right)\mathscr{N}\left[\sum_{n=0}^{\infty} p^{\mathrm{n}}\mathscr{A}_n - \sum_{\mathrm{n}=0}^{\infty}\sqrt[\mathrm{n}]{}\mathscr{B}_{\mathrm{n}} + \sum_{n=0}^{\infty}\sqrt[n]{}\varphi_n\right]\right] \tag{34}$$

Then, we get

$$p^0 : \varphi_0 = \mathrm{e}^{-\mu}$$

$$p^1 : \varphi_1 = \mathscr{N}^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{g}\right)^{\delta}\right)\mathscr{N}\left[p^0\mathscr{A}_0 - p^0\mathscr{B}_0 + p^0\varphi_0\right]\right]$$

$$p^2 : \varphi_2 = \mathscr{N}^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{g}\right)^{\delta}\right)\mathscr{N}\left[p^1\mathscr{A}_1 - p^1\mathscr{B}_1 + p^1\varphi_1\right]\right]$$

Then

$$\varphi_0 = \mathrm{e}^{-\mu}$$

$$\varphi_1 = \mathrm{e}^{-\mu}\left(1-\delta+\delta\frac{\tau^{\delta}}{\Gamma(\delta+1)}\right)$$

$$\varphi_2 = \mathrm{e}^{-\mu}\left(\left(1-2\delta+\delta^2\right)+\left(2\delta-2\delta^2\right)\frac{\tau^a}{\Gamma(\delta+1)}+\delta^2\frac{\tau^{2a}}{\Gamma(2\delta+1)}\right)$$

and so on.
Therefore, the series solution $\varphi(\mu,\tau)$ is given by

$$\varphi(\mu,\tau) = \mathrm{e}^{-\mu}\left[\left(3-3\delta+\delta^2\right)+\left(3\delta-2\delta^2\right)\frac{\tau^a}{\Gamma(\delta+1)}+\delta^2\frac{\tau^{2a}}{\Gamma(2\delta+1)}+\cdots\right] \tag{35}$$

If $\delta \to 1$ in (36):n

$$\varphi(\mu,\tau) = \mathrm{e}^{-\mu}\left(1+\frac{\tau}{1!}+\frac{\tau^2}{2!}+\cdots\right) = \mathrm{e}^{-\mu}\sum_{\mathrm{k}=0}^{\infty}\frac{\tau^{\mathrm{k}}}{k!} = \exp(-\mu+\tau) \tag{36}$$

*Example 3.*Consider the nonlinear system of time-fractional PDEs in the ABC operator:

$$\begin{aligned} {}^{\mathscr{A}\mathscr{B}\mathscr{C}}D_\tau^{\delta}\varphi(\mu,\tau) - \psi_\mu + \psi + \varphi = 0, 0 < \delta \le 1 \\ {}^{\mathscr{A}\mathscr{B}\mathscr{C}}D_\tau^{\gamma}\psi(\mu,\tau) - \varphi_\mu + \psi + \varphi = 0, 0 < \gamma \le 1 \end{aligned} \tag{37}$$

where $0 < \delta, \gamma \le 1$ and the initial conditions are

$$\begin{aligned} \varphi(\mu,0) &= \sinh(\mu) \\ \psi(\mu,0) &= \cosh(\mu) \end{aligned} \tag{38}$$

Taking the NT on both sides of (38),

$$\begin{aligned} \mathscr{N}\left\{{}^{\mathscr{A}\mathscr{B}\mathscr{C}}\mathscr{D}_t^{\delta}\varphi(\mu,\tau)\right\} &= \varphi(\mu,0)+\left(1-\delta+\delta\left(\frac{u}{S}\right)^{\delta}\right)\mathscr{N}\left\{\psi_\mu-\psi-\varphi\right\} \\ \mathscr{N}\left\{{}^{\mathscr{A}\mathscr{B}\mathscr{C}}\mathscr{D}_t^{\gamma}\psi(\mu,\tau)\right\} &= \psi(\mu,0)+\left(1-\lambda+\lambda\left(\frac{u}{s}\right)^{\lambda}\right)\mathscr{N}\left\{\varphi_\mu-\psi-\varphi\right\} \end{aligned} \tag{39}$$

Operating with the NT on both sides of (40) gives

$$\begin{aligned} \varphi(\mu,\tau) &= \sinh(\mu)+\mathscr{N}^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{S}\right)^{\delta}\right)\mathscr{N}\left\{\psi_\mu-\psi-\varphi\right\}\right] \\ \psi(\mu,\tau) &= \cosh(\mu)+\mathscr{N}^{-1}\left[\left(1-\gamma+\gamma\left(\frac{u}{s}\right)^{\gamma}\right)\mathscr{N}\left\{\varphi_\mu-\psi-\varphi\right\}\right] \end{aligned} \tag{40}$$

Now, we represent solution as an infinite series given below

$$\varphi(\mu,\tau)=\sum_{\mathrm{n}=0}^{\infty}\sqrt{}^{\mathrm{n}}\varphi_{\mathrm{n}}(\mu,\tau),\ \psi(\mu,\tau)=\sum_{\mathrm{n}=0}^{\infty}\sqrt{}^{\mathrm{n}}\psi_{\mathrm{n}}(\mu,\tau) \tag{41}$$

Substituting (42) in (41) and by applying HPM on Eq.(41),

$$\begin{aligned}\sum_{n=0}^{\infty}p^n\varphi_n(\mu,\tau)&=\sinh(\mu)-p\mathscr{N}^{-1}\left[\left(1-\delta+\delta\left(\frac{u}{S}\right)^{\delta}\right)\mathscr{N}\left\{\psi_{\mu n}-\psi_n-\varphi_n\right\}\right]\\ \sum_{n=0}^{\infty}p^n\psi_{\mathrm{n}}(\mu,\tau)&=\cosh(\mu)-p\mathscr{N}^{-1}\left[\left(1-\gamma+\gamma\left(\frac{u}{s}\right)^{\gamma}\right)\mathscr{N}\left\{\varphi_{\mu n}-\psi_n-\varphi_n\right\}\right]\end{aligned} \tag{42}$$

On comparing both sides of the (38),

$$\begin{aligned}&p^0:\varphi_0=v(x,0),p^0:\psi_0=\psi(x,0)\\ &p^1:\varphi_1=-\mathscr{N}^{-1}\left\{\left(1-\delta+\delta\left(\frac{u}{S}\right)^{\delta}\right)\mathscr{N}\left\{\psi_{\mu 0}-\psi_0-\varphi_0\right\}\right\}\\ &p^1:\psi_1=-\mathscr{N}^{-1}\left\{\left(1-\gamma+\gamma\left(\frac{u}{S}\right)^{\gamma}\right)\mathscr{N}\left\{\varphi_{\mu 0}-\psi_0-\varphi_0\right\}\right\}\\ &p^2:\varphi_2=-\mathscr{N}^{-1}\left\{\left(1-\delta+\delta\left(\frac{u}{S}\right)^{\delta}\right)\mathscr{N}\left\{\psi_{\mu 1}-\psi_1-\varphi_1\right\}\right\}\\ &p^2:\psi_2=-\mathscr{N}^{-1}\left\{\left(1-\gamma+\gamma\left(\frac{u}{S}\right)^{\gamma}\right)\mathscr{N}\left\{\varphi_{\mu 1}-\psi_1-\varphi_1\right\}\right\}\end{aligned}$$

$$\varphi_0=\sinh(\mu),\ \psi_0=\cosh(\mu)$$

$$\varphi_1=-\cosh(\mu)\left(1-\delta+\delta\frac{\tau^{\delta}}{\Gamma(\delta+1)}\right)$$

$$\psi_1=-\sinh(\mu)\left(1-\gamma+\gamma\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}\right)$$

$$\begin{aligned}\varphi_2=&\left[(1-\delta)(1-\gamma)+\gamma(1-\delta)\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}+\delta(1-\gamma)\frac{\tau^{\delta}}{\Gamma(\delta+1)}+\delta\gamma\frac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}\right](\sinh(\mu)\\ &-\cosh(\mu))+\left[(1-\delta)^2+2\delta(1-\delta)\frac{\tau^{\delta}}{\Gamma(\delta+1)}+\delta^2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}\right]\cosh(\mu)\\ \psi_2=&\left[(1-\delta)(1-\gamma)+\delta(1-\gamma)\frac{\tau^{\delta}}{\Gamma(\delta+1)}+\gamma(1-\delta)\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}+\delta\gamma\frac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}\right](\cosh(\mu)\\ &-\sinh(\mu))+\left[(1-\gamma)^2+2\gamma(1-\gamma)\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}+\delta^2\frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}\right]\sinh(\mu)\end{aligned}$$

Therefore, the approximate solution of (38) is given by

$$\begin{aligned}\varphi=&\sinh(\mu)-\cosh(\mu)\left(1-\delta+\delta\frac{\tau^{\delta}}{\Gamma(\delta+1)}\right)\\ &+\left[(1-\delta)(1-\gamma)+\gamma(1-\delta)\frac{\tau^{\gamma}}{\Gamma(\lambda+1)}+\delta(1-\gamma)\frac{\tau^{\delta}}{\Gamma(\delta+1)}\right.\\ &\left.+\delta\gamma\frac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)}\right](\sinh(\mu)-\cosh(\mu))\\ &+\left[(1-\delta)^2+2\delta(1-\delta)\frac{\tau^{\delta}}{\Gamma(\delta+1)}+\delta^2\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}\right]\cosh(\mu)\end{aligned}$$

$$\begin{aligned}\psi &= \cosh(\mu) - \sinh(\mu)\left(1-\gamma+\gamma\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}\right) \\ &+ \left[(1-\delta)(1-\gamma)+\delta(1-\gamma)\frac{\tau^{\delta}}{\Gamma(\delta+1)}+\gamma(1-\delta)\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}\right](\cosh(\mu)-\sinh(\mu)) \\ &+ \delta\lambda\frac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} \\ &+ \left[(1-\gamma)^2+2\gamma(1-\gamma)\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}+\delta^2\frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}\right]\sinh(\mu)\end{aligned}$$

If we put $\delta \to 1$ and $\gamma \to 1$, we reproduce the solution of the problem as follows

$$\varphi(\mu,\tau) = \sinh(\mu)\left(1+\frac{\tau^2}{2!}+\cdots\right) - \cosh(\mu)\left(\tau+\frac{\tau^3}{3!}+\cdots\right) \tag{43}$$

$$\psi(\mu,\tau) = \cosh(\mu)\left(\tau+\frac{\tau^3}{3!}+\cdots\right) - \sinh(\mu)\left(1+\frac{\tau^2}{2!}+\cdots\right) \tag{44}$$

This solution is equivalent to the exact solution in closed form:

$$\begin{aligned}\varphi(\mu,\tau) &= \sinh(\mu)\cosh(\tau) - \cosh(\mu)\sinh(\tau) \\ \psi(\mu,\tau) &= \cosh(\mu)\sinh(\tau) - \sinh(\mu)\cosh(\tau)\end{aligned} \tag{45}$$

6 Conclusion

In this article, the natural homotopy permutation method was presented and the following results were obtained:

– The method is effective and efficient in solving fractional differential equations with the AtanganaBaleanu-Caputo operator.
– The approximate solutions obtained by this method approximates the exact solution when $\delta, \gamma = 1$.
– The method can solve linear and nonlinear equations.

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