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Application on the Validity of Weak Compactness in Variable Exponent Spaces

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Abstract: This research aimed to show the validity following Francisco L.Hernández, César Ruiz and Mauro Sanchiz [1], of the necessary and sufficient conditions on subsets of variable exponent spaces $L^{p(\cdot)}(\Omega)$ in order to be weakly compact. Useful criteria are given extending Andô results for Orlicz spaces. This research aimed to show that all separable variable exponent spaces are weakly Banach-Saks. Also, L-weakly compact and weakly compact inclusions between variable exponent spaces are studied.

Keywords: Phishing attacks, Advanced phishing tools, Cyberattack, Internet security, Machine learning, Anti-phishing.

1 Introduction

The Riesz-Kolmogorov compactness theorem in $L_{1+\epsilon}$-spaces $(0 \leq \epsilon < \infty)$ has been extended to the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ (or Nakano spaces) by Górka and Macios [2], Górka and Bandaliyev [3] and Dong et al. [4].

They give useful versions of the theorem according with the underlying measure space considered (Ω, μ) (f.i. Euclidean spaces, metric measure spaces or locally compact groups). [4] study the compactness of Riemann-Liouville fractional integral operators in the variable exponent $L^{p(\cdot)}(\Omega)$ setting. The variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ (and their corresponding Sobolev spaces) are being used successfully in several areas of harmonic analysis and related differential equations and applications (cf. [5-7]).

Variable exponent Lebesgue spaces belong to the general class of non-symmetric Musielak-Orlicz spaces [8, 9] . Francisco L.Hernández, César Ruiz and Mauro Sanchiz [1] are describing the weakly compact sets in non-reflexive variable exponent spaces $L^{p(\cdot)}(\Omega)$. We follow and show an application on [1] this topic has been widely studied for symmetric (or rearrangement invariant) function spaces. Recall the classical Dunford and Pettis result for $L_1(\Omega)$ describing the relative weakly compact subsets as the equi-integrable sets. For Orlicz spaces $L^{\varphi}(\Omega)$ with the Δ_2-condition, useful weak compactness criteria were given by Andô in [10] (see [11] chapter 4). Later on, many extensions have been given for general symmetric function spaces (see f.i. [12] and references within) and also for the vectorial case of Orlicz-Bochner spaces in [13].

They extend Andô weak compactness characterizations in Orlicz spaces to the variable exponent $L^{p(\cdot)}(\Omega)$ setting. Also, equi-integrable subsets in $L^{p(\cdot)}(\Omega)$ spaces are studied, obtaining a De la Vallée Poussin type theorem [14] in $L^{p(\cdot)}(\Omega)$ spaces. Recall that De la Vallée Poussin's classical result characterizes equiintegrable sets in $L_1(\Omega)$ by their boundness in certain Orlicz spaces. As an application, [1] obtain criteria for when the inclusions between two variable exponent spaces $L^{p(\cdot)+\epsilon(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ are weakly compact or L-weakly compact operators (this means that the unit ball $B_{L^{p(\cdot)+\epsilon(\cdot)}}$ is equi-integrable in $L^{p(\cdot)}(\Omega)$). It turns out that, even for "closed" exponent functions $p(\cdot)$ and $p(\cdot)+\epsilon(\cdot)$ (i.e. ess $\inf(\epsilon(\cdot)) = 0$), the inclusion $L^{p(\cdot)+\epsilon(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ can be L-weakly compact.

The obtained weak compactness criteria are used later to study the weak Banach-Saks property in $L^{p(\cdot)}(\Omega)$ spaces (i.e. when every weakly convergent sequence in $L^{p(\cdot)}(\Omega)$ contain a subsequence which is Cesàro convergent).

We point out that no extra conditions on the regularity of the exponent functions (like the log-Hölder continuous conditions) will be assumed along the paper.

We give in section 3 a characterization for $L^{p(\cdot)}(\Omega)$-equi-integrable subsets obtaining a De la Vallée Poussin type result in $L^{p(\cdot)}(\Omega)$ spaces (Theorem 3.2). In section 4, we obtain the Andô type criteria for a subset S of $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$ and $\mu((1+\epsilon)^{-1}\{1\}) = 0$ to be relatively weakly compact (Theorem 4.3), namely

$$\lim_{\lambda \to 0} \sup_{f \in S} \frac{1}{\lambda} \int_{\Omega} \sum_{m_0} |\lambda f(t^{m_0})|^{p(t^{m_0})} d\mu = 0.$$

In particular, weakly convergent sequences in $L^{p(\cdot)}(\Omega)$-spaces are characterized (see Propositions 4.5 and 4.6). In section 5, we apply previous results to study the weak Banach-Saks property in $L^{p(\cdot)}(\Omega)$

spaces, showing that all separable $L^{p(\cdot)}(\Omega)$ spaces are weakly Banack-Saks (Theorem 5.1). In the last section 6 , we obtain another Andô type characterization of weak

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compactness of a set S in a $L^{p(\cdot)}(\Omega)$ space in terms of the existence of a Musielak-Orlicz function $\Psi(t^{m_0},x^2)$ increasing uniformly more rapidly than $x^{2p(t^{m_0})}$ such that S is bounded in the Musielak-Orlicz space $L^{\Psi}(\Omega)$ (see [1]).

2 Preliminaries

Throughout the paper (Ω,Σ,μ) is a finite separable non-atomic measurable space and $L_0(\Omega)$ is the space of all real measurable function classes. Given a μ-measurable function $(1+\epsilon):\Omega\to[1,\infty)$, the Variable Exponent Lebesgue space (or Nakano space) $L^{p(\cdot)}(\Omega)$ is defined by the set of all measurable scalar function classes $f\in L_0(\Omega)$ such that the modular $\rho_{p(\cdot)}(\frac{f}{1+\epsilon})$ is finite for some $\epsilon\geq 0$, where

$$\rho_{p(\cdot)}(f):=\int_\Omega\sum_{m_0}|f(t^{m_0})|^{p(t^{m_0})}d\mu(t^{m_0})<\infty.$$

The associated Luxemburg norm is defined as

$$\| f \|_{p(\cdot)}:=\inf\left\{\epsilon\geq 0:\rho_{p(\cdot)}(\frac{f}{1+\epsilon})\leq 1\right\}.$$

With the usual pointwise order, $\left(L^{p(\cdot)}(\Omega),\|\cdot\|_{p(\cdot)}\right)$ is a Banach lattice.

We write $1+2\epsilon:=\operatorname{essinf}\sum_{m_0}\{p(t^{m_0}):t^{m_0}\in\Omega\}$ and $p^+:=\operatorname{esssup}\sum_{m_0}\{p(t^{m_0}):t^{m_0}\in\Omega\}$. Equally, $p^+_{|A^{m_0}}$ and $p^-_{|A^{m_0}}$ will denote the essential supremum and infimum of the function $p(\cdot)$ over a measurable subset $A^{m_0}\subset\Omega$. The conjugate function $p^*(\cdot)$ of $p(\cdot)$ is defined by the equation $\frac{1}{p(t^{m_0})}+\frac{1}{p^*(t^{m_0})}=1$ almost everywhere $t^{m_0}\in\Omega$. Thus, the topological dual of the space $L^{p(\cdot)}(\Omega)$, for $p^+<\infty$, is the variable exponent space $L^{p^*(\cdot)}(\Omega)$.

A $L^{p(\cdot)}(\Omega)$ space is separable if and only if $p^+<\infty$ or, equivalently, if and only if $L^{p(\cdot)}(\Omega)$ contains no isomorphic copy of ℓ_∞. In the sequel, only separable variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ will be considered. An space $L^{p(\cdot)}(\Omega)$ is reflexive if and only if $1<p^-\leq p^+<\infty$. This is also equivalent to $L^{p(\cdot)}(\Omega)$ being uniformly convex ([15] Theorem 3.3).

Notice that, for $p^+<\infty, \| f \|_{p(\cdot)}=1$ if and only if the modular $\rho_{p(\cdot)}(f)=1$. Also, every sequence $(f_n)\subset L^{p(\cdot)}(\Omega)$ satisfies $\lim_{n\to\infty}\|f_n\|_{p(\cdot)}=0$ if and only if $\lim_{n\to\infty}\rho_{p(\cdot)}(f_n)=0$ ([6]). By $B_{L^{p(-)}}$ we denote the closed unit ball of $L^{p(\cdot)}(\Omega)$. The essential range of the exponent function $p(\cdot)$ is defined as

$$R_{p(\cdot)}:=\{p+\epsilon\in[1,\infty):\ \forall\varepsilon>0\ \mu((1+\epsilon)^{-1}(p,p+2\epsilon))>0\}.$$

It is a closed subset of $[1,\infty)$ and it is compact when $p(\cdot)$ is essentially bounded. The values $p^-$and $p^+$are always in the set $R_{p(\cdot)}$. It holds for $p^+<\infty$ that a $L^{p(\cdot)}(\Omega)$ space has a lattice isomorphic copy of $\ell_{p+\epsilon}$ if and only if $p+\epsilon\in R_{p(\cdot)}$ ([16] Theorem 3.5). Indeed, for every $p+\epsilon\in R_{p(\cdot)}$ there exists a suitable sequence of disjoint measurable subsets $\left(A_k^{m_0}\right)$ such that the normalized sequence

$$g_k^{m_0}:=\sum_{m_0}\frac{\chi_{A_k^{m_0}}}{\left(\mu\left(A_k^{m_0}\right)\right)^{\frac{1}{p(\cdot)}}}$$

is equivalent to the canonical basis of $\ell_{p+\epsilon}$. Even more, we can choose suitable sets $\left(A_k^{m_0}\right)$ in order to get that the orthogonal projection

$$P(f)=\sum_{k=1}^{\infty}\sum_{m_0}\left(\int_{A_k^{m_0}}\frac{f(s)}{\mu\left(A_k^{m_0}\right)^{\frac{1}{p^*(s)}}}d\mu(s)\right)\frac{\chi_{A_k^{m_0}}}{\mu\left(A_k^{m_0}\right)^{\frac{1}{p(\cdot)}}}$$

is bounded ([16] Proposition 4.4).

Variable exponent spaces are a special class of Musielak-Orlicz spaces. Recall that an Orlicz function $\varphi:[0,\infty)\to[0,\infty]$ is a convex increasing function that satisfies $\varphi(0)=0$, $\lim_{x^2\to 0^+}\varphi(x^2)=0$ and $\lim_{x^2\to\infty}\varphi(x^2)=\infty$. We say that a function $\Phi:\Omega\times[0,\infty)\to[0,\infty]$ is a Musielak-Orlicz function if $\Phi(t^{m_0},\cdot)$ is an Orlicz function for every $t^{m_0}\in\Omega$ and $t^{m_0}\mapsto\Phi(t^{m_0},x^2)$ is measurable for every $x^2\geq 0$. Given a Musielak-Orlicz function $\Phi(t^{m_0},x^2)$, the MusielakOrlicz space $L^{\Phi}(\Omega)$ is defined by the set of all measurable

3 $L^{p(\cdot)}$ Equi-Integrability

Recall that, given a Banach function space $E(\Omega)$, a bounded subset $S\subset E(\Omega)$ is equi-integrable if

$$\lim_{\mu(A^{m_0})\to 0}\sum_{m_0}\sup_{f\in S}\|f\chi_{A^{m_0}}\|_E=0.$$

As in classical $L_{1+\epsilon}$ spaces, equi-integrability plays an important role in the study of $L^{p(\cdot)}(\Omega)$ spaces. Let us mention, for example, Riesz-Kolmogorov compactness type theorems in $L^{p(\cdot)}(\Omega)$ spaces (see [4] Theorem 2.1, [2]).

The classical De la Vallée Poussin's result ([14]) characterizes the equi-integrable subsets in $L_1(\Omega)$ by their boundedness in some suitable Orlicz space $L^{\varphi}(\Omega)$ (cf. [11] Theorem 1.2). Here we will present an extension of this result to $L^{p(\cdot)}(\Omega)$ spaces. First, we give an equivalent statement of $L^{p(\cdot)}$-equi-integrability (see [1]):

Proposition 3.1. Let $L^{p(\cdot)}(\Omega)$ with $p^+<\infty$ and $S\subset L^{p(\cdot)}(\Omega)$ bounded. Then S is equi-integrable if and only if

$$\lim_{x^2\to\infty}\sup_{f\in S}\int_{\{|f|>x^2\}}\sum_{m_0}|f(t^{m_0})|^{p(t^{m_0})}d\mu=0. \quad (1)$$

Proof. Suppose that S is equi-integrable. Let us show that

$$\lim_{x^2\to\infty}\sup_{f\in S}\left\|f\chi_{\{|f|>x^2\}}\right\|_{p(\cdot)}=0,$$

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which is equivalent to (1) since $p^+ < \infty$. Let $\sup_{f\in S} \| f \|_{p(\cdot)} \leq C < \infty$. Define the sets $(A^{m_0})_f^{x^2} := \{t^{m_0} \in \Omega : |f(t^{m_0})| > x^2\}$. By the hypothesis, we just need to show that $\lim_{x^2\to\infty} \sup_{f\in S} \sum_{m_0} \mu\left((A^{m_0})_f^{x^2}\right) = 0$, but this follows from $\left\|\sum_{m_0} f\chi_{(A^{m_0})_f^{x^2}}\right\|_1 \leq \sum_{m_0} (1 + \mu(\Omega)) \left\|f\chi_{(A^{m_0})_f^{x^2}}\right\|_{p(\cdot)}$ (cf.[5]Corollary 2.48), as

$$\sup_{f\in S} \sum_{m_0} \mu\left((A^{m_0})_f^{x^2}\right) \leq \sup_{f\in S} \sum_{m_0} \frac{1}{x^2}\left\|f\chi_{(A^{m_0})_f^{x^2}}\right\|_1$$
$$\leq \sup_{f\in S} \sum_{m_0} \frac{1}{x^2}\big(1+\mu(\Omega)\big)\left\|f\chi_{X_f^{x^2}}\right\|_{p(\cdot)}$$
$$\leq \frac{C}{x^2}\big(1+\mu(\Omega)\big).$$

Conversely, given $\varepsilon > 0$, there exists $x^2 > 1$ such that $\sup_{f\in S} \left\|\sum_{m_0} f\chi_{(A^{m_0})_f^{x^2}}\right\|_{p(\cdot)} \leq \frac{\varepsilon}{2}$. Then, for every measurable subset A^{m_0} with $\sum_{m_0} (\mu(A^{m_0}))^{\frac{1}{p^+}} < \frac{\varepsilon}{2x^2}$, we have

$$\sup_{f\in S} \sum_{m_0} \|f\chi_{A^{m_0}}\|_{p(\cdot)} \leq \sup_{f\in S} \sum_{m_0} \begin{pmatrix} \left\|f\chi_{A^{m_0}\cap(A^{m_0})_f^{x^2}}\right\|_{p(\cdot)} + \\ \left\|f\chi_{A^{m_0}\cap\{|f|\leq x^2\}}\right\|_{p(\cdot)} \end{pmatrix}$$
$$\leq \sup_{f\in S} \sum_{m_0} \begin{pmatrix} \left\|f\chi_{(A^{m_0})_f^{x^2}}\right\|_{p(\cdot)} + \\ (\mu(A^{m_0}))^{\frac{1}{p^+}} x^2 \end{pmatrix} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Theorem 3.2. [1] Let $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$. A bounded subset $S \subset L^{p(\cdot)}(\Omega)$ is equi-integrable if and only if there exists an Orlicz function φ with $\lim_{x^2\to\infty} \frac{\varphi(x^2)}{x^2} = \infty$ such that

scalar functions on Ω such that $\rho_\Phi(\frac{f}{1+\epsilon})$ is finite for some $\epsilon \geq 0$, where $\rho_\Phi(\cdot)$ is the modular defined by

$$\rho_\Phi(f) = \int_\Omega \sum_{m_0} \Phi(t^{m_0}, |f(t^{m_0})|)d\mu(t^{m_0}) < \infty.$$

he associated Luxemburg norm is defined as

$$\| f \|_\Phi := \inf\left\{\epsilon \geq 0 : \rho_\Phi\left(\frac{f}{1+\epsilon}\right) \leq 1\right\}.$$

With the usual pointwise order, $(L^\Phi(\Omega), \|\cdot\|_\Phi)$ is a Banach lattice. In the special cases of (i) $\Phi(t^{m_0}, x^2) = x^{2p(t^{m_0})}$ we get $L^\Phi(\Omega) = L^{p(\cdot)}(\Omega)$; (ii) $\Phi(t^{m_0}, x^2) = \varphi(x^2)$ for every $t^{m_0} \in \Omega$ we get the Orlicz space $L^\varphi(\Omega)$.

See [5, 6, 17] for other definitions and basic facts regarding variable exponent spaces, Musielak-Orlicz spaces and Banach lattices.

$$\sup_{f\in S} \| \varphi(f) \|_{p(\cdot)} < \infty.$$

Proof. Assume S is equi-integrable. Using the above equivalence, consider a sequence (x_n^2) such that

$$\sup_{f\in S} \left\|f\chi_{\{|f|>x_n^2\}}\right\|_{p(\cdot)} \leq \frac{1}{n^2}$$

and $x_{n+1}^2 > 2x_n^2$ for each natural n. Define the function

$$\varphi(x^2) := \sum_{n=1}^{\infty} (x^2 - x_n^2)_+,$$

for $x^2 \geq 0$. Clearly, φ is an increasing convex function with $\varphi(0) = 0$. Moreover, $\lim_{x^2\to\infty} \frac{\varphi(x^2)}{x^2} = \infty$. Indeed, for $x^2 \in [x_n^2, x_{n+1}^2)$ we have

$$\varphi(x^2) = \sum_{k=1}^{n} (x^2 - x_k^2)_+ = nx^2 - \sum_{k=1}^{n} x_k^2 \geq nx^2 - 2x_n^2,$$

hence $\frac{\varphi(x^2)}{x^2} \geq n - 2\frac{x_n^2}{x^2} \geq n - 2$. Finally, for each $f \in S$, we have

$$\| \varphi(f) \|_{p(\cdot)} \leq \sum_{n=1}^{\infty} \left\|f\chi_{\{|f|>x_n^2\}}\right\|_{p(\cdot)} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Conversely, let us assume $\sup_{f\in S} \| \varphi(f) \|_{p(\cdot)} = C < \infty$. Given $\varepsilon > 0$, by hypothesis there exists $x_\varepsilon^2 > 0$ such that, for all $x^2 \geq x_\varepsilon^2$, we have $x^2 \leq \frac{\varepsilon}{C}\varphi(x^2)$. Then, for every $f \in S$, we have

$$\left\|f\chi_{\{|f|>x_\varepsilon^2\}}\right\|_{p(\cdot)} \leq \frac{\varepsilon}{C}\left\|\varphi(f)\chi_{\{|f|>x_\varepsilon^2\}}\right\|_{p(\cdot)} \leq \frac{\varepsilon}{C}\sup_{f\in S} \| \varphi(f) \|_{p(\cdot)} \leq \varepsilon,$$

and so the previous proposition ends the proof.

Note that the above result can be reformulated saying that a bounded subset $S \subset L^{p(\cdot)}(\Omega)$ is equi-integrable if and only if S is norm bounded in the Musielak-Orlicz space $L^\Phi(\Omega)$, where $\Phi(t^{m_0}, x^2) = (\varphi(x^2))^{p(t^{m_0})}$ and φ is a certain Orlicz function with $\lim_{x^2\to\infty} \frac{\varphi(x^2)}{x^2} = \infty$. In Section 6 we will extend this statement to the family of relative weakly compact subsets in $L^{p(\cdot)}(\Omega)$.

If we consider now a pair of exponent functions $p(\cdot) \leq p(\cdot) + \epsilon(\cdot)$, we have the continuous inclusion $L^{p(\cdot)+\epsilon(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$. The inclusion $L^{p(\cdot)+\epsilon(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ is said to be L-weakly compact when the unit ball $B_{L^{p(\cdot)+\epsilon(\cdot)}}$ is an equi-integrable set in $L^{p(\cdot)}(\Omega)$. L-weakly compact inclusions for symmetric function spaces have been studied in [18]. For variable exponent spaces, taking the set S as the unit ball $B_{L^{p(\cdot)+\epsilon(\cdot)}}$ in the above theorem we get the following (see [1]):

roposition 3.3. Let $p(\cdot) \leq p(\cdot) + \epsilon(\cdot)$ be exponent functions. The inclusion $L^{p(\cdot)+\epsilon(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ is L-weakly compact if and only if there exists an Orlicz function φ with $\lim_{x^2\mapsto\infty} \frac{\varphi(x^2)}{x^2} = \infty$ such that

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$L^{p(\cdot)+\epsilon(\cdot)}(\Omega) \subset L^{\Phi}(\Omega)$ where Φ is the Musielak-Orlicz function $\Phi(t^{m_0}, x^2) = (\varphi(x^2))^{p(t^{m_0})}$.

We give now an easy sufficient condition to use for when the inclusion $L^{p(\cdot)+\epsilon(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ is L-weakly compact (see [1]).

Proposition 3.4. Let $p(\cdot) \leq p(\cdot) + \epsilon(\cdot)$ be exponent functions. If ess $\inf(\epsilon(x^2)) = \delta > 0$, then the inclusion $L^{p(\cdot)+\epsilon(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ is L-weakly compact.

Proof. It is enough to show that

$$\lim_{\mu(A^{m_0})\to 0} \sup_{\|f\|_{p(\cdot)+\epsilon(\cdot)}\leq 1} \sum_{m_0} \rho_{p(\cdot)}(f\chi_{A^{m_0}}) = 0.$$

Let us denote by $r(x^2) = \frac{p(x^2)+\epsilon(x^2)}{p(x^2)} \geq 1$ the exponent function with conjugate function $r^*(x^2) = \frac{p(x^2)+\epsilon(x^2)}{\epsilon(x^2)}$ for $x^2 \in \Omega$. It holds that $(r^*)^+ \leq \frac{p^+ + \epsilon}{\delta} < \infty$. Using Hölder's inequality ([5] Theorem 2.26, Remark 2.27), we have

$$\rho_{p(\cdot)}(f\chi_{A^{m_0}}) = \int_\Omega \sum_{m_0} |f|^{p(t^{m_0})}\chi_{A^{m_0}} d\mu \leq 4\|f^{p(\cdot)}\|_{r(\cdot)}\|\chi_{A^{m_0}}\|_{r^*(\cdot)}.$$

Now, as

$$\rho_{r(\cdot)}(f^{p(\cdot)}) = \int_\Omega \sum_{m_0} |f|^{p(t^{m_0})+\epsilon(t^{m_0})} d\mu \leq \| f \|_{p(\cdot)+\epsilon(\cdot)}^{p^- + \epsilon} \leq 1,$$

we have $\|f^{p(\cdot)}\|_{r(\cdot)} \leq 1$. Hence, since $\|\cdot\|_{r^*(\cdot)}$ is order continuous, we conclude that

$$\lim_{\mu(A^{m_0})\to 0} \sup_{\|f\|_{p(\cdot)+\epsilon(\cdot)}\leq 1} \sum_{m_0} \rho_{p(\cdot)}(f\chi_{A^{m_0}}) \leq \lim_{\mu(A^{m_0})\to 0} \sum_{m_0} 4\|\chi_{A^{m_0}}\|_{r^*(\cdot)} = 0.$$

The above condition ess $\inf(\epsilon(x^2)) = \delta > 0$ is far from be necessary for the L weak compactness of the inclusion $L^{p(\cdot)+\epsilon(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$. Here we give a weaker condition (see also [19] and [1]):

Proposition 3.5. Let $p(\cdot) \leq p(\cdot) + \epsilon(\cdot)$ be exponent functions in $\Omega = [0,1]$ with $p^+ + \epsilon < \infty$ and $\epsilon(\cdot)$ decreasing. Suppose that

(i) $\lim_{x^2\to 1}(1 - x^2)^{\epsilon(x^2)} = 0$, and

(ii) There exists a sequence (x_n^2) defined by $x_n^2 = \frac{x_{n-1}^2+1}{2}$ for $n \geq 1$, and $0 \leq x_0^2 < 1$ satisfying that

$$\sum_{n=0}^{\infty} \frac{1}{x_{n+1}^2 - x_n^2} \int_{x_n^2}^{x_{n+1}^2} (x_{n+1}^2 - x_n^2)^{\frac{\epsilon(t^{m_0})}{p(t^{m_0})+\epsilon(t^{m_0})}} dt^{m_0} < \infty.$$

Then, the inclusion $L^{p(\cdot)+\epsilon(\cdot)}[0,1] \subset L^{p(\cdot)}[0,1]$ is L-weakly compact.

Proof. Let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} \frac{1}{x_{n+1}^2 - x_n^2} \int_{x_n^2}^{x_{n+1}^2} (x_{n+1}^2 - x_n^2)^{\frac{\epsilon(t^{m_0})}{p(t^{m_0})+\epsilon(t^{m_0})}} dt^{m_0} < \frac{\varepsilon}{3} \quad (*)$$

and

$$(x_{n+1}^2 - x_n^2)^{\frac{\epsilon(x^2)}{p(x^2)+\epsilon(x^2)}} \leq (1 - x_{n+1}^2)^{\frac{\epsilon(x_{n+1}^2)}{M}} < \frac{\varepsilon}{3} \qquad (**)$$

for every $x^2 \in [x_n^2, x_{n+1}^2), n \geq n_0$ and $M = p^+ + \epsilon$.

Let $1 + \epsilon = \epsilon(x_{n_0}^2) > 0$. Take an arbitrary function $f \in B_{L^{p(\cdot)+\epsilon(\cdot)}}$ and any measurable set E with $\mu(E) \leq \left(\frac{\varepsilon}{6}\right)^{\frac{M}{1+\epsilon}+1}$. We define the two sets

$$E_1 := \left\{x^2 \in [0, x_{n_0}^2) \cap E: |f(x^2)| \leq \left(\frac{6}{\varepsilon}\right)^{\frac{1}{1+\epsilon}}\right\}, E_2: = \left\{x^2 \in [0, x_{n_0}^2) \cap E: |f(x^2)| > \left(\frac{6}{\varepsilon}\right)^{\frac{1}{1+\epsilon}}\right\}.$$

This way, using that $f \in B_{Lp(\cdot)+\epsilon(\cdot)}$ and $\mu(E) \leq \left(\frac{\varepsilon}{6}\right)^{\frac{M}{1+\epsilon}+1}$, we get that

$$\int_{[0,x_{n_0}^2)\cap E} \sum_{m_0} |f|^{p(t^{m_0})} dt^{m_0} = \int_{E_1} \sum_{m_0} |f|^{p(t^{m_0})} dt^{m_0} + \int_{E_2} \sum_{m_0} |f|^{p(t^{m_0})} dt^{m_0}$$

$$\leq \left(\frac{6}{\varepsilon}\right)^{\frac{M}{1+\epsilon}} \mu(E) + \int_{E_2} \sum_{m_0} |f|^{p(t^{m_0})+\epsilon(t^{m_0})}|f|^{-\epsilon(t^{m_0})} dt^{m_0}$$

$$\leq \frac{\varepsilon}{6} + \int_{E_2} \sum_{m_0} |f|^{p(t^{m_0})+\epsilon(t^{m_0})} \frac{\varepsilon}{6} dt^{m_0} \leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.$$

On the other hand,

$$\int_{x_{n_0}^2}^1 \sum_{m_0} |f|^{p(t^{m_0})}\chi_E dt^{m_0} = \sum_{n=n_0}^{\infty} \int_{x_n^2}^{x_{n+1}^2} \sum_{m_0} |f|^{p(t^{m_0})}\chi_E dt^{m_0}$$

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$$= \sum_{n=n_0}^{\infty} \int_{E_{n,1}} \sum_{m_0} |f|^{p(t^{m_0})} \chi_E dt^{m_0} + \int_{E_{n,2}} \sum_{m_0} |f|^{p(t^{m_0})} \chi_E dt^{m_0},$$

where

$$E_{n,1} := E \bigcap \left\{ x^2 \in [x_n^2, x_{n+1}^2) : |f(x^2)| \leq \frac{1}{(x_{n+1}^2 - x_n^2)^{\frac{1}{p(x^2)+\epsilon(x^2)}}} \right\}$$

and

$$E_{n,2} := E \bigcap \left\{ x^2 \in [x_n^2, x_{n+1}^2) : |f(x^2)| > \frac{1}{(x_{n+1}^2 - x_n^2)^{\frac{1}{p(x^2)+\epsilon(x^2)}}} \right\}$$

Then, using $(*)$, we have

$$\sum_{n=n_0}^{\infty} \int_{E_{n,1}} \sum_{m_0} |f|^{p(t^{m_0})} dt^{m_0} \leq \sum_{n=n_0}^{\infty} \int_{x_n^2}^{x_{n+1}^2} \sum_{m_0} \frac{1}{(x_{n+1}^2 - x_n^2)^{\frac{p(t^{m_0})}{p(t^{m_0})+\epsilon(t^{m_0})}}} dt^{m_0}$$
$$\leq \sum_{n=n_0}^{\infty} \int_{x_n^2}^{x_{n+1}^2} \sum_{m_0} \frac{1}{x_{n+1}^2 - x_n^2} (x_{n+1}^2 - x_n^2)^{\frac{\epsilon(t^{m_0})}{p(t^{m_0})+\epsilon(t^{m_0})}} dt^{m_0} < \frac{\varepsilon}{3},$$

and, using $(**)$ and $f \in B_{L^{p(\cdot)+\epsilon(\cdot)}}$,

$$\sum_{n=n_0 E_{n,2}}^{\infty} \int_n \sum_{m_0} |f|^{p(t^{m_0})} dt^{m_0} \leq \sum_{n=n_0}^{\infty} \int_{x_n^2}^{x_{n+1}^2} \sum_{m_0} |f|^{p(t^{m_0})+\epsilon(t^{m_0})} |f|^{-\epsilon(t^{m_0})} dt^{m_0}$$
$$\leq \sum_{n=n_0}^{\infty} \int_{x_n^2}^{x_{n+1}^2} \sum_{m_0} |f|^{p(t^{m_0})+\epsilon(t^{m_0})} (x_{n+1}^2 - x_n^2)^{\frac{\epsilon(t^{m_0})}{p(t^{m_0})+\epsilon(t^{m_0})}} dt^{m_0}$$
$$\leq \sum_{n=n_0}^{\infty} \int_{x_n^2}^{x_{n+1}^2} \sum_{m_0} |f|^{p(t^{m_0})+\epsilon(t^{m_0})} \frac{\varepsilon}{3} dt^{m_0} \leq \frac{\varepsilon}{3}$$

which ends the proof.

We give an example applying the above result. Take any bounded exponent function $p(\cdot)$ and consider the function in (0,1)

$$r(x^2) = \frac{\ln([\log_2(1-x^2)]^{2j})}{-\log_2(1-x^2)},$$

for some natural $j > 0$. If we define $\epsilon(\cdot) = r(1-2^{-e})\chi_{[0,1-2^{-c})} + r(\cdot)\chi_{[1-2^{-\epsilon},1]}$, then

$$\text{ess inf } (\epsilon(x^2)) = \text{ess} \inf_{x^2 \in [1-2^{-e},1]} (r(x^2)) \leq \lim_{x^2 \to 1} r(x^2) = \lim_{y^2 \to \infty} \frac{\ln(y^{4j})}{y^2} = 0,$$

yet the inclusion $L^{p(\cdot)}[0,1] \subset L^{p(\cdot)+\epsilon(\cdot)}[0,1]$ is L-weakly compact for j large enough. Indeed, let us see that the conditions in the above proposition are satisfied:

(i) The limit

$$\lim_{x^2 \to 1} (1-x^2)^{\epsilon(x^2)} = \lim_{x^2 \to 1} (1-x^2)^{r(x^2)} = \lim_{y^2 \to 0} y^{2\frac{\ln(\log_2(y^2)^{2j})}{-\log_2(y^2)}} - \log_2(y^2) = 0.$$

$ii)$ Let $x_n^2 = 1 - \frac{1}{2^{n+1}}$, so $x_{n+1}^2 - x_n^2 = \frac{1}{2^{n+2}}$. Then,

$$\sum_{n=0}^{\infty} \frac{1}{x_{n+1}^2 - x_n^2} \int_{x_n^2}^{x_{n+1}^2} \sum_{m_0} (x_{n+1}^2 - x_n^2)^{\frac{\epsilon(t^{m_0})}{p(t^{m_0})+\epsilon(t^{m_0})}} dt^{m_0} \leq \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+2}}\right)^{\frac{\epsilon(x_{n+1}^2)}{p^+ + \epsilon}}.$$

Now, for n_0 and j large enough (for example $j \geq p^+$), using the Cauchy condensation test, we conclude

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{2^{n+2}}\right)^{\frac{\epsilon(x_{n+1}^2)}{p^+ + \epsilon}} = \sum_{n=n_0}^{\infty} \left(\frac{1}{2^{n+2}}\right)^{\frac{r(x_{n+1}^2)}{p^+ + \epsilon}}$$

4 Weakly Compact Subsets of $L^{p(\cdot)}(\Omega)$

In this section we are interested in finding criteria for when a subset of a non reflexive $L^{p(\cdot)}(\Omega)$ is relatively weakly compact.

First note that every equi-integrable subset in a $L^{p(\cdot)}(\Omega)$ space with $p^+ < \infty$ is relatively weakly compact. This follows from a general statement in Banach lattices (cf. [20] Proposition 3.6.5). The converse is not true in general. For example, any space $L^{p(\cdot)}(\Omega)$ with $1 < p^+ < \infty$ contains relative weakly compact subsets which are not equiintegrable. Indeed, let $\epsilon = 0$ and consider disjoint subsets $A^{m_0}{}_n \subset p^{-1}\left(p + \epsilon - \frac{1}{n+1}, p + \epsilon - \frac{1}{n}\right)$ of positive measure (or even $A_n^{m_0} \subset p^{-1}(\{p + \epsilon\})$ if possible) and the normalized disjoint functions

$$f_n := \sum_{m_0} \frac{\chi_{A_n^{m_0}}}{(\mu(A_n^{m_0}))^{\frac{1}{p(\cdot)}}}.$$

Then, the sequence (f_n) is equivalent to the canonical basis of $\ell_{p+\epsilon}$ (cf. [16] Proposition 3.2). Hence, (f_n) is weakly convergent to 0 and, as (f_n) is normalized and $\mu(\Omega) < \infty$, we have $\mu(A_n^{m_0}) \to 0$ and so it is a non-equi-integrable relatively weakly compact subset of $L^{p(\cdot)}(\Omega)$. On the other hand, when $p^+ = 1$, i.e. in a $L_1(\Omega)$ space, it is well known that a bounded set is equi-integrable if and only if it is relatively weakly compact (Dunford-Pettis theorem, cf. [21] Theorem 5.2.9).

Recall that, by the classical Eberlian-Smulian Theorem (cf. [21] Theorem 1.6.3), a subset is weakly compact if and only if it is sequentially weakly compact. The following proposition is a consequence of ([17] Theorem 1.c.4), since

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the space $L^{p(\cdot)}(\Omega)$ does not have any isomorphic copy of c_0 when $p^+ < \infty$:

Proposition 4.1. A $L^{p(\cdot)}(\Omega)$ space is weakly sequentially complete if and only if$p^+ < \infty$

We will give now weak compactness criteria in $L^{p(\cdot)}(\Omega)$ spaces. We adapt the technique developed by Andô ([10]) in the context of Orlicz spaces to the non-symmetric setting of $L^{p(\cdot)}(\Omega)$ spaces (see [1]).

Theorem 4.2. Let $L^{p(\cdot)}(\Omega)$ be with $p^+ < \infty$. A subset $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact if and only if S is norm bounded and, for every$g^{m_0} \in L^{p^*(\cdot)}(\Omega)$,

$$\lim_{\mu(E)\to 0} \sup_{f\in S} \int_E \sum_{m_0} |f g^{m_0}| d\mu = 0. \qquad (2)$$

Proof. (⇒): Clearly, S is weakly bounded and hence norm bounded. Suppose now that (2) does not hold, i.e. there exist $\varepsilon > 0$, a function $g_0^{m_0} \in L^{p^*(\cdot)}$, a sequence (E_n) with $\mu(E_n) \to 0$ and $(f_n) \subset S$ such that

$$\int_{E_n} \sum_{m_0} |f_n g_0^{m_0}| d\mu \geq \varepsilon.$$

Since S is relatively weakly compact, there exists a subsequence $(f_{n_k}) \to f \in L^{p(\cdot)}(\Omega)$ weakly. Thus, for every $A^{m_0} \in \Sigma$,

$$\int_\Omega \sum_{m_0} f_{n_k} g_0^{m_0} \chi_{A^{m_0}} d\mu \overset{k\to\infty}{\longrightarrow} \int_{A^{m_0}} \sum_{m_0} f g_0^{m_0} d\mu < \infty.$$

Considering now the measures $\nu_k(A^{m_0}) := \int_{A^{m_0}} \sum_{m_0} f_{n_k} g_0^{m_0} d\mu$, which are μ-absolutely continuous, we have, by the Vitali-Hahn-Saks Theorem ([22] page 89), that the sequence (ν_k) is uniformly absolutely μ-continuous, i.e. it holds that $\lim_{n\to\infty} \sup_k \nu_k(A_n^{m_0}) = 0$ for every$= \sum_{n=n_0}^{\infty} \left(\frac{1}{2^{n+2}}\right)^{\frac{\ln((n+2)^{2j})}{(p^+ + \epsilon)(n+2)}} < \infty$

sequence $(A_n^{m_0})$ such that $\mu(A_n^{m_0}) \to 0$. In particular, we get that $\nu_k(E_{n_k}) \overset{k\to\infty}{\longrightarrow} 0$, which is a contradiction with the election of $g_0^{m_0}$ and (E_n).

(⇐) : Let S be norm bounded and a sequence $(f_n) \subset S$ with $\|f_n\|_{p(\cdot)} \leq M < \infty$. In virtue of Proposition 4.1 we have to find a weakly Cauchy subsequence, i.e. a subsequence (f_{n_k}) such that, for every $g^{m_0} \in L^{p^*(\cdot)}(\Omega)$

$$\int_\Omega \sum_{m_0} (f_{n_k} - f_{n_l}) g^{m_0} d\mu \overset{k,l\to\infty}{\longrightarrow} 0.$$

As Σ is separable, we first take a sequence $(A_j^{m_0})_{j=1}^{\infty}$ of subsets of $\Omega\, that\ genarets \backslash$Sigma. Thus, $(\chi_{A_j^{m_0}})_j \subset L^{p^*(\cdot)}(\Omega)$ and, for every $A^{m_0} \in \{A_j^{m_0}\}$, the sequence $\left(\int_\Omega \sum_{m_0} f_n \chi_{A^{m_0}} d\mu\right)_n$ is a bounded scalar sequence. Then, by the Cantor diagonal process, we can take a subsequence (f_{n_k}) such that the sequence $\left(\int_\Omega \sum_{m_0} f_{n_k} \chi_{A^{m_0}} d\mu\right)_k$ converges for each $A^{m_0} \in \{A_j^{m_0}\}$. Thus, if we define the sequence of measures

$$\nu_k(A^{m_0}) := \int_{A^{m_0}} \sum_{m_0} f_{n_k} d\mu = \int_\Omega \sum_{m_0} f_{n_k} \chi_{A^{m_0}} d\mu$$

we get that the measure $\nu(A^{m_0}) := \lim_{k\to\infty} \sum_{m_0} \nu_k(A^{m_0})$ is well defined for every $A^{m_0} \in \{A_j^{m_0}\}$ and it can be extended to any measurable subset $E \in \Sigma$ (cf. [22] page 91). Therefore, given a simple function $g_s^{m_0} = \sum_{i=1}^{N} \sum_{m_0} a_i^{m_0} \chi_{E_i}$ where the sets (E_i) are disjoint, we have

$$\int_\Omega \sum_{m_0} f_{n_k} g_s^{m_0} d\mu = \sum_{i=1}^{N} \sum_{m_0} a_i^{m_0} \nu_k(E_i) \overset{k\to\infty}{\longrightarrow} \sum_{i=1}^{N} \sum_{m_0} a_i^{m_0} \nu(E_i),$$

so we get that

$$\int_\Omega \sum_{m_0} (f_{n_k} - f_{n_l}) g_s^{m_0} d\mu \overset{k,l\to\infty}{\longrightarrow} 0.$$

Our aim now is to get the same for every function $g^{m_0} \in L^{p^*(\cdot)}$. Thus, fixed g^{m_0} and $\varepsilon > 0$, by hypothesis there exist $\delta > 0$ such that, if $\mu(E) < \delta$ and $n \in \mathbb{N}$,

$$\int_E \sum_{m_0} |f_n g^{m_0}| d\mu < \frac{\varepsilon}{6}.$$

Let us denote $G_m := \{t^{m_0} \in \Omega : |g^{m_0}(t^{m_0})| \leq m\}$. Since $g^{m_0} \in L_1(\Omega)$, consider $m \in \mathbb{N}$ large enough so that $\mu(G_m^c) \leq \delta$. Then, given $g_m^{m_0} := g^{m_0} \cdot \chi_{G_m}$, using the dominated convergence Theorem, consider a simple function $g_s^{m_0}$ such that $\sum_{m_0} \|g_m^{m_0} - g_s^{m_0}\|_{p^*(\cdot)} \leq \frac{\varepsilon}{24M}$. ([5]Theorem 2.26). Thus, for k, l large enough so that $\int_\Omega \sum_{m_0} |(f_{n_k} - f_{n_l}) g_s^{m_0}| d\mu < \frac{\varepsilon}{3}$, we can use the Hölder **i**nequality ([5]Theorem 2.26) to get

$$\begin{aligned}
\left|\int_\Omega \sum_{m_0} (f_{n_k} - f_{n_l}) g^{m_0} d\mu\right| &\leq \int_{G_m} \sum_{m_0} |(f_{n_k} - f_{n_l}) g^{m_0}| d\mu + \int_{G_m^c} \sum_{m_0} |(f_{n_k} - f_{n_l}) g^{m_0}| d\mu \\
&\leq \int_\Omega \sum_{m_0} |(f_{n_k} - f_{n_l}) g_m^{m_0}| d\mu + \frac{\varepsilon}{3} \\
&\leq \int_\Omega \sum_{m_0} |(f_{n_k} - f_{n_l})(g_m^{m_0} - g_s^{m_0})| d\mu + \int_\Omega \sum_{m_0} |(f_{n_k} - f_{n_l}) g_s^{m_0}| d\mu + \frac{\varepsilon}{3} \\
&\leq 4 \sum_{m_0} \|f_{n_k} - f_{n_l}\|_{p(\cdot)} \|g_m^{m_0} - g_s^{m_0}\|_{p^*(\cdot)} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Thus, we conclude that (f_{n_k}) is a weakly Cauchy sequence so, by Proposition 4.1, (f_{n_k}) is weakly convergent to a function $f \in L^{p(\cdot)}(\Omega)$ and S is relatively weakly compact.

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Theorem 4.3. [1] Let $L^{p(\cdot)}(\Omega)$ with $p_+ < \infty$ and $\mu(\Omega_1) = 0$. A subset $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact if and only if it is norm bounded and

$$\lim_{\lambda\to 0}\sup_{f\in S}\frac{1}{\lambda}\int_\Omega \sum_{m_0} |\lambda f(t^{m_0})|^{p(t^{m_0})} d\mu = 0. \qquad (\diamond)$$

Proof. In the case $p_- > 1$ it is clear, since $L^{p(\cdot)}(\Omega)$ is reflexive so the relative weak compactness is equivalent to the norm boundless and, if that condition is met, the equation (\diamond) holds. Assume in the following that $p^- = 1$.

($\Rightarrow$) : Clearly S is norm bounded and we can suppose $S \subset B_{L^{p(\cdot)}}$. Thus, for every $f \in S$, we have $\int_\Omega \sum_{m_0} |f(t^{m_0})|^{p(t^{m_0})} d\mu \le 1$. Suppose that $(\diamond)$ does not hold, so there exist $\varepsilon > 0, (\lambda_n) \searrow 0$ and a sequence (f_n) in S such that, for every $n \in \mathbb{N}$,

$$\sum_{m_0} |\lambda_n f_n(t^{m_0})|^{p(t^{m_0})} d\mu \ge \lambda_n \varepsilon \qquad (3)$$

and let us find a contradiction.

Since $p_- = 1$ and $\mu(\Omega_1) = 0$, we can take a sequence $(\delta_n) \searrow 1$ such that the sets $A_n^{m_0} := \{t^{m_0} \in \Omega : p(t^{m_0}) \le \delta_n\}$ satisfy $0 < \mu(A_n^{m_0}) \le \frac{\varepsilon}{3n}$ and thus (up to subsequence) we can suppose that (λ_n) verifies the properties:

$$0 \le \lambda_n \le \frac{1}{2n}, \sum_n \lambda_n \le 1, \sup_{t^{m_0}\in(A^{m_0})_n^c} \frac{(n\lambda_n)^{p(t^{m_0})}}{\lambda_n} \le \frac{(n\lambda_n)^{\delta_n}}{\lambda_n} \le \frac{\varepsilon}{3}.$$

Now consider the function $g_n^{m_0}(t^{m_0}) := \sum_{m_0} |\lambda_n f_n(t^{m_0})|^{p(t^{m_0})-1}$. For a.e. $t^{m_0} \in \Omega$ we have

$$2\sum_{m_0} \left|\lambda_n f_n(t^{m_0}) g_n^{m_0}(t^{m_0})\right| = \sum_{m_0} |\lambda_n f_n(t^{m_0})|^{p(t^{m_0})} + \sum_{m_0} \left|g_n^{m_0}(t^{m_0})\right|^{p^*(t^{m_0})}.$$

Therefore, we conclude that

$$\begin{aligned}
\int_\Omega \sum_{m_0} |\lambda_n f_n(t^{m_0})|^{p(t^{m_0})} d\mu = & \int_{B_n^{m_0}} \sum_{m_0} |\lambda_n f_n(t^{m_0})|^{p(t^{m_0})} d\mu + \int_{B_n^{m_0 c}} \sum_{m_0} |\lambda_n f_n|^{p(t^{m_0})} d\mu \\
\le & \int_{B_n^{m_0}} \sum_{m_0} |\lambda_n f_n(t^{m_0})|^{p(t^{m_0})} d\mu + \sup_{t^{m_0}\in(A^{m_0})_n^c} \sum_{m_0} (n\lambda_n)^{p(t^{m_0})} \mu\left(B_n^{m_0 c} \cap (A^{m_0})_n^c\right) \\
& + \sup_{t^{m_0}\in A_n^{m_0}} \sum_{m_0} (n\lambda_n)^{p(t^{m_0})} \mu\left(B_n^{m_0 c} \cap A_n^{m_0}\right) \\
\le & \int_{B_n^{m_0}} \sum_{m_0} 2\left|\lambda_n f_n(t^{m_0}) g_n^{m_0}(t^{m_0})\right| d\mu + \lambda_n \frac{\varepsilon}{3} + n\lambda_n \frac{\varepsilon}{3n} \\
\le & \ 2\lambda_n \int_{B_n^{m_0}} \sum_{m_0} |f_n(t^{m_0}) g^{m_0}(t^{m_0})| d\mu + \lambda_n \frac{2\varepsilon}{3} \\
& < \lambda_n \varepsilon,
\end{aligned}$$

which is a contradiction with (3).

Given a variable exponent space $L^{p(\cdot)}(\Omega)$, let us denote $\Omega_1 := p^{-1}(\{1\})$. Indeed, since $2\left|\lambda_n f_n(t^{m_0}) g_n^{m_0}(t^{m_0})\right| = 2|\lambda_n f_n(t^{m_0})|^{p(t^{m_0})}$ and $p(t^{m_0}) = p(t^{m_0}) \cdot p^*(t^{m_0}) - p^*(t^{m_0})$, we have

$$\begin{aligned}
\sum_{m_0} \left|g_n^{m_0}(t^{m_0})\right|^{p^*(t^{m_0})} &= \sum_{m_0} \left(|\lambda_n f_n(t^{m_0})|^{p(t^{m_0})-1}\right)^{p^*(t^{m_0})} \\
&= \sum_{m_0} |\lambda_n f_n|^{p(t^{m_0})\cdot p^*(t^{m_0}) - p^*(t^{m_0})} \\
&= \sum_{m_0} |\lambda_n f_n(t^{m_0})|^{p(t^{m_0})}.
\end{aligned}$$

$$= \sum_{m_0} |\lambda_n f_n(t^{m_0})|^{p(t^{m_0})} + \sum_{m_0} \left|g_n^{m_0}(t^{m_0})\right|^{p^*(t^{m_0})}.$$

ndeed, since $2\left|\lambda_n f_n(t^{m_0}) g_n^{m_0}(t^{m_0})\right| = 2|\lambda_n f_n(t^{m_0})|^{p(t^{m_0})}$ and $p(t^{m_0}) = p(t^{m_0}) \cdot p^*(t^{m_0}) - p^*(t^{m_0})$, we have

$$\begin{aligned}
\sum_{m_0} \left|g_n^{m_0}(t^{m_0})\right|^{p^*(t^{m_0})} &= \sum_{m_0} \left(|\lambda_n f_n(t^{m_0})|^{p(t^{m_0})-1}\right)^{p^*(t^{m_0})} \\
&= \sum_{m_0} |\lambda_n f_n|^{p(t^{m_0})\cdot p^*(t^{m_0}) - p^*(t^{m_0})} \\
&= \sum_{m_0} |\lambda_n f_n(t^{m_0})|^{p(t^{m_0})}.
\end{aligned}$$

c

$$\begin{aligned}
\sup_{f\in S} \int_E \sum_{m_0} |f(t^{m_0}) g^{m_0}(t^{m_0})| d\mu
&\le \frac{1}{\lambda_0(1+\epsilon)} \sum_{m_0} \left[\sup_{f\in S} \int_E |\lambda_0 f(t^{m_0})|^{p(t^{m_0})} d\mu \right. \\
&\quad \left. + \int_E |(1+\epsilon) g^{m_0}(t^{m_0})|^{p^*(t^{m_0})} d\mu \right] \\
&< \frac{1}{1+\epsilon}\left(\frac{\varepsilon(1+\epsilon)}{2}\right) \\
&\quad + \frac{1}{\lambda_0(1+\epsilon)}\left(\frac{\varepsilon\lambda_0(1+\epsilon)}{2}\right) = \varepsilon.
\end{aligned}$$

Thus, applying Theorem 4.2, we conclude that S is relatively weakly compact.

A characterization of weakly compact subsets in general $L^{p(\cdot)}(\Omega)$ spaces (without the restriction $(\Omega_1) = 0$) follows now putting together the above criterion and the classical Dunford-Pettis theorem for $L_1(\Omega)$ (cf. [21] Theorem 5.2.9). Indeed, as $L^{p(\cdot)}(\Omega) = L_1(\Omega_1) \oplus L^{p(\cdot)}(\Omega \setminus \Omega_1)$, a sequence (f_n) is weakly convergent in $L^{p(\cdot)}(\Omega)$ if and only if the sequences $(f_n \chi \Omega_1)$ and $(f_n \chi_{\Omega\setminus\Omega_1})$ are weakly convergent in $L_1(\Omega_1)$ and $L^{p(\cdot)}(\Omega \setminus \Omega_1)$ respectively. Thus:

Theorem 4.4. Let $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty . A^{m_0}$ subset $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact if and only if it is norm bounded,

$$\lim_{\lambda\to 0}\sup_{f\in S}\frac{1}{\lambda}\int_{\Omega\setminus\Omega_1} \sum_{m_0} |\lambda f(t^{m_0})|^{p(t^{m_0})} d\mu = 0$$

and

$$\lim_{\mu(A^{m_0})\to 0}\sup_{f\in S}\int_{A^{m_0}\cap\Omega_1}\sum_{m_0}|f(t^{m_0})|d\mu=0.$$

Criteria to be a weakly convergent sequence in $L^{p(\cdot)}(\Omega)$ spaces follow now (see [1]):

Proposition 4.5. Let $L^{p(\cdot)}(\Omega)$ with $p^+<\infty$ and a sequence (f_n) and f in $L^{p(\cdot)}(\Omega)$. Then, $f_n\to f$ weakly if and only if

(i) $\lim_n\int_{A^{m_0}}\sum_{m_0}(f_n-f)d\mu=0$ for each $A^{m_0}\in\Sigma$, and

(ii) $\lim_{\mu(A^{m_0})\to 0}\sup_n\int_{A^{m_0}}\sum_{m_0}|(f_n-f)g^{m_0}|d\mu=0$ for each function $g^{m_0}\in L^{p^*(\cdot)}(\Omega)$.

Proof. $(\Rightarrow)$: Clearly (i) holds since $\chi_{A^{m_0}}\in L^{p^*(\cdot)}$ and condition (ii) follows from above Theorem 4.2.

$(\Leftarrow)$: We can assume w.l.o.g. $f=0$. If $g^{m_0}\in L^{p^*(\cdot)}(\Omega)$ is a simple function then it follows directly from (i) that $\lim_n\int_\Omega\sum_{m_0}f_n g^{m_0}d\mu=0$. Assume now that g^{m_0} is a bounded function. Given $\varepsilon>0$ there exists a simple function $g_s^{m_0}$ such that $\sum_{m_0}\|g^{m_0}-g_s^{m_0}\|_\infty<\frac{\varepsilon}{2}$, so

$$\begin{aligned}\int_\Omega\sum_{m_0}|f_n g^{m_0}|d\mu &\le \int_\Omega\sum_{m_0}|f_n(g^{m_0}-g_s^{m_0})|d\mu\\ &\quad+\int_\Omega\sum_{m_0}|f_n g_s^{m_0}|d\mu\\ &\le\frac{\varepsilon}{2}\int_\Omega|f_n|d\mu+\int_\Omega\sum_{m_0}|f_n g_s^{m_0}|d\mu\end{aligned}$$

and hence $\int_\Omega\sum_{m_0}|f_n g^{m_0}|d\mu\le\varepsilon$ from a big enough $n\in\mathbb{N}$.

Now, for an arbitrary $g^{m_0}\in L^{p^*(\cdot)}(\Omega)$, by condition (ii), there exists $\delta>0$ such that $\int_{A^{m_0}}|f_n g^{m_0}|d\mu<\frac{\varepsilon}{2}$ if $\mu(A^{m_0})<\delta$. Consider $G_m=\{t^{m_0}\in\Omega\colon|g^{m_0}(t^{m_0})|\le m\}$ with m large enough so that $\mu(G_m^c)<\delta$. Then,

$$\begin{aligned}\int_\Omega\sum_{m_0}|f_n g^{m_0}|d\mu &=\int_{G_m^c}\sum_{m_0}|f_n g^{m_0}|d\mu+\int_{G_m}\sum_{m_0}|f_n g^{m_0}|d\mu\\ &\le\frac{\varepsilon}{2}+\int_\Omega\sum_{m_0}|f_n g^{m_0}\chi_{G_m}|d\mu\end{aligned}$$

Hence, we have $\int_\Omega\sum_{m_0}|f_n g^{m_0}|d\mu\le\varepsilon$ from a big enough $n\in\mathbb{N}$ as $g^{m_0}\chi_{G_m}$ is bounded.

Proposition 4.6. [1] Let $L^{p(\cdot)}(\Omega)$ with $p^+<\infty$ and $\mu(\Omega_1)=0$. A sequence (f_n) in $L^{p(\cdot)}(\Omega)$ converges weakly to $f\in L^{p(\cdot)}(\Omega)$ if and only if

(i) $\lim_n\int_{A^{m_0}}\sum_{m_0}f_n d\mu=\int_{A^{m_0}}\sum_{m_0}f d\mu$ for each $A^{m_0}\in\Sigma$, and

(ii) $\lim_{\lambda\to 0}\sup_n\frac{1}{\lambda}\int_\Omega\sum_{m_0}|\lambda(f_n-f)|^{p(t^{m_0})}d\mu=0$.

Proof. Clearly, if $f_n\to f$ weakly the necessity condition (i) holds, and using Theorem 4.3 we get also condition (ii). Conversely, reasoning as in the proof of Theorem 4.3 (using Young inequality), we get easily that condition (ii) of the above Proposition 4.5 is satisfied. Thus, we conclude that (f_n) is weakly convergent to f.

In particular, it follows that in reflexive $L^{p(\cdot)}(\Omega)$ spaces, a sequence (f_n) is weakly convergent to $f\in L^{p(\cdot)}(\Omega)$ if and only if (f_n) is norm bounded and $\int_{A^{m_0}}\sum_{m_0}f_n d\mu\to\int_{A^{m_0}}\sum_{m_0}f d\mu$, for every measurable $A^{m_0}\in\Sigma$. Moreover, it holds that if (f_n) is weakly convergent to f and $\|f_n\|_{p(\cdot)}$ converges to $\|f\|_{p(\cdot)}$, then $f_n\to f$ in $L^{p(\cdot)}(\Omega)$, since all reflexive $L^{p(\cdot)}(\Omega)$ spaces are uniformly convex (cf. [15] Theorem 3.3).

Finally, a direct consequence of Theorem 4.4 is a characterization for the inclusion $L^{p(\cdot)+\epsilon(\cdot)}(\Omega)\subset L^{p(\cdot)}(\Omega)$ to be weakly compact (see [1]):

Proposition 4.7. Let $p(\cdot)\le p(\cdot)+\epsilon(\cdot)$ be exponent functions. The inclusion $L^{p(\cdot)+\epsilon(\cdot)}(\Omega)\subset L^{p(\cdot)}(\Omega)$ is weakly compact if and only if

$$\lim_{\lambda\to 0}\sup_{\|f\|_{p(\cdot)+\epsilon(\cdot)}\le 1}\frac{1}{\lambda}\int_{\Omega\setminus\Omega_1}\sum_{m_0}|\lambda f(t^{m_0})|^{p(t^{m_0})}d\mu=0$$

and

$$\lim_{\mu(A^{m_0})\to 0}\sup_{\|f\|_{p(\cdot)+\epsilon(\cdot)}\le 1}\int_{A^{m_0}\cap\Omega_1}\sum_{m_0}|f(t^{m_0})|d\mu=0,$$

where $\Omega_1=(1+\epsilon)^{-1}(\{1\})$.

5 Banach-Saks Property

Let us apply now the above criteria to show that all $L^{p(\cdot)}(\Omega)$ spaces with $p^+<\infty$ are weakly Banach-Saks. First, let us recall some definitions:

A Banach space X is said to be Banach-Saks if for every bounded square sequence (x_n^2) in X there exists a square subsequence $(x_{n_k}^2)$ which is Cesàro convergent, i.e. there exists $x^2\in X$ such that

$$\lim_{k\to\infty}\left\|\frac{x_{n_1}^2+\cdots+x_{n_k}^2}{k}-x^2\right\|_X=0.$$

A Banach space X is said to be weakly Banach-Saks if for every weakly convergent square sequence (x_n^2) in X there exists a square subsequence $(x_{n_k}^2)$ which is Cesàro convergent.

Obviously, every Banach-Saks space is also weakly Banach-Saks. The property of a Banach space being Banach-Saks (or weakly Banach-Saks) passes to closed subspaces. Uniformly convex spaces are Banach-Saks. In particular, every reflexive $L^{p(\cdot)}(\Omega)$ space is Banach-Saks because reflexives $L^{p(\cdot)}(\Omega)$ spaces are always uniformly convex ([15] Theorem 3.3). However, when $p^-=1$, spaces $L^{p(\cdot)}(\Omega)$ are never Banach-Saks. Indeed, there exist ℓ_1-subspaces generated by normalized sequences (f_n) in $L^{p(\cdot)}(\Omega)$ ([16]Proposition 3.2).

Theorem 5.1. [1] $A^{m_0}L^{p(\cdot)}(\Omega)$ space is weakly Banach-Saks if and only if $p^+<\infty$.

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Proof. (⇒) : If $p^+ = \infty$, then $L^{p(\cdot)}(\Omega)$ has an isomorphic copy of ℓ_∞ which is not weakly Banach-Saks, so neither is $L^{p(\cdot)}(\Omega)$.

(⇐) : Since $L^{p(\cdot)}(\Omega)$ is a p^+-concave lattice, we have that $L^{p(\cdot)}(\Omega)$ satisfies the subsequence splitting property ([23]). Thus, by ([24]Corollary 3.4), it is enough to prove the weak Banach-Saks property for disjoint sequences.

Assume that (f_n) is a pairwise disjoint weakly convergent sequence in $L^{p(\cdot)}(\Omega)$. Then, the sequences $\left(f_n\chi_{\Omega_1}\right)$ and $\left(f_n\chi_{\Omega\setminus\Omega_1}\right)$ are weakly convergent in $L_1(\Omega_1)$ and $L^{p(\cdot)}(\Omega\setminus\Omega_1)$ respectively. As $L_1(\Omega_1)$ is weakly Banach-Saks [25], there exists a subsequence $\left(f_{n_k}\chi_{\Omega_1}\right)$ which is Cesàro convergent. On the other hand, as

$$\left\|\frac{f_1+\cdots+f_n}{n}\right\|_{p(\cdot)} \le \left\|\frac{f_1\chi_{\Omega_1}+\cdots+f_n\chi_{\Omega_1}}{n}\right\|_{p(\cdot)} + \left\|\frac{f_1\chi_{\Omega\setminus\Omega_1}+\cdots+f_n\chi_{\Omega\setminus\Omega_1}}{n}\right\|_{p(\cdot)},$$

we just need to prove that $\left(f_{n_{k_l}}\chi_{\Omega\setminus\Omega_1}\right)$ is Cesàro convergent for some subsequence $\left(f_{n_{k_l}}\right)$. To simplify the notation, let's just suppose that (f_n) is in $L^{p(\cdot)}(\Omega\setminus\Omega_1)$. As it is a weakly convergent sequence, it is a relatively weakly compact set. So, by Theorem 4.3, we have

$$\limsup_{\lambda\to 0}\sup_{k\in\mathbb{N}}\frac{1}{\lambda}\int\sum_{m_0}|\lambda f_k(t^{m_0})|^{p(t^{m_0})}dt^{m_0} = \limsup_{\lambda\to 0}\sup_{k\in\mathbb{N}}\frac{\rho_{p(\cdot)}(\lambda f_k)}{\lambda} = 0.$$

Hence, we get

$$0 \le \lim_{n\to\infty}\rho_{p(\cdot)}\left(\frac{f_1+\cdots+f_n}{n}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\rho_{p(\cdot)}\left(\frac{f_k}{n}\right) \le \lim_{n\to\infty}\sum_{k=1}^{n}\sup_{m\in\mathbb{N}}\rho_{p(\cdot)}\left(\frac{f_m}{n}\right) = \lim_{n\to\infty}\sup_{m\in\mathbb{N}}\left(n\cdot\rho_{p(\cdot)}\left(\frac{f_m}{n}\right)\right) = 0.$$

This finishes the proof since, as $p^+ < \infty$, the modular convergence and the norm convergence are equivalent.

6 Weak Compactness and Musielak-Orlicz Spaces

We study the weak compactness of subsets of $L^{p(\cdot)}(\Omega)$ in relation with their norm boundedness in certain Musielak-Orlicz space $L^{\Psi}(\Omega) \subset L^{p(\cdot)}(\Omega)$.

The following definition generalizes the one given by Andô ([10]) for Orlicz functions.

Definition 6.1. A Musielak-Orlicz function $\Psi(t^{m_0}, x^2)$ increases uniformly more rapidly than another function $\Phi(t^{m_0}, x^2)$ if for each $\varepsilon > 0$ there exist some $\delta > 0$ and $x_0^2 > 0$ such that for all $x^2 \ge x_0^2$ and all $t^{m_0} \in \Omega$,

$$\varepsilon\Psi(t^{m_0}, x^2) \ge \frac{1}{\delta}\Phi(t^{m_0}, \delta x^2).$$

With this definition we characterize the relatively weak compact subsets of $L^{p(\cdot)}(\Omega)$ through their embedding in certain Musielak-Orlicz spaces. We follow a similar reasoning as the done for Orlicz spaces in [10].

Theorem 6.2. [1] Let $L^{p(\cdot)}(\Omega)$ with $p_+ < \infty$ and $\mu(\Omega_1) = 0. A^{m_0}$ subset $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact if and only if there exists a Musielak-Orlicz function $\Psi(t^{m_0}, x^2)$ increasing uniformly more rapidly than $\sum_{m_0}\Phi(t^{m_0}, x^2) = \sum_{m_0} x^{2p(t^{m_0})}$ such that S is norm bounded in $L^{\Psi}(\Omega)$.

Proof. Assume that S is norm bounded in the Musielak-Orlicz space $L^{\Psi}(\Omega)$ with $\Psi(t^{m_0}, x^2)$ increasing uniformly more rapidly than $x^{2p(t^{m_0})}$. Let us prove that S satisfies the conditions in Theorem 4.3, so it is a relatively weakly compact set in $L^{p(\cdot)}(\Omega)$. Suppose w.l.o.g. that $S \subset B_{L^{\Psi}}$. Given $\varepsilon > 0$, there exist $\delta > 0$ and $x_0^2 > 1$ such that, for all $x^2 \ge x_0^2$,

$$\frac{\varepsilon}{3}\sum_{m_0}\Psi(t^{m_0}, x^2) \ge \frac{1}{\delta}\sum_{m_0}(\delta x^2)^{p(t^{m_0})}.$$

Now, let $\gamma > 0$ be small enough so that the set $A^{m_0} = p^{-1}((1, 1+\gamma))$ has measure $\mu(A^{m_0}) < \frac{\varepsilon}{3x_0^{2p+}}$. Let $\delta_0 = \min\sum_{m_0}\left\{1, \delta, \left(\frac{\varepsilon}{3\mu(A^{m_0c})x_0^{2p+}}\right)^{\frac{1}{\gamma}}\right\}$. Then,

$$\frac{1}{\delta_0}\sum_{m_0}(\delta_0 x^2)^{p(t^{m_0})} \le \frac{1}{\delta}\sum_{m_0}(\delta x^2)^{p(t^{m_0})}$$

and, for every $t^{m_0} \in A^{m_0c}$ and $x^2 \le x_0^2$,

$$\frac{1}{\delta_0}\sum_{m_0}(\delta_0 x^2)^{p(t^{m_0})} \le \frac{\delta_0^{1+\gamma}}{\delta_0}\sum_{m_0}x^{2p(t^{m_0})} \le \delta_0^{\gamma}\sum_{m_0}x_0^{2p(t^{m_0})} \le \delta_0^{\gamma}x_0^{2p^+} \le \sum_{m_0}\frac{\varepsilon}{3\mu(A^{m_0c})}.$$

Now, for each $f \in \underline{S}$, define the set $B_f^{m_0} := \{t^{m_0} \in \Omega : |f(t^{m_0})| \ge x_0^2\}$. Then, for each $\lambda \le \delta_0$ we get

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$$\sup_{f\in S}\frac{1}{\lambda}\int_{\Omega}\sum_{m_0}\Phi\big(t^{m_0},\lambda f(t^{m_0})\big)d\mu$$
$$=\sup_{f\in S}\sum_{m_0}\left(\int_{B_f^{m_0}}\frac{1}{\lambda}|\lambda f(t^{m_0})|^{p(t^{m_0})}d\mu\right.$$
$$+\int_{A^{m_0}\cap B_f^{m_0c}}\frac{1}{\lambda}|\lambda f(t^{m_0})|^{p(t^{m_0})}d\mu$$
$$\left.+\int_{A^{m_0c}\cap B_f^{m_0c}}\frac{1}{\lambda}|\lambda f(t^{m_0})|^{p(t^{m_0})}d\mu\right)$$
$$\le\sup_{f\in S}\sum_{m_0}\left(\int_{B_f^{m_0}}\frac{1}{\delta}|\delta f(t^{m_0})|^{p(t^{m_0})}d\mu\right.$$
$$+\int_{A^{m_0}\cap B_f^{m_0}}|f(t^{m_0})|^{p(t^{m_0})}d\mu$$
$$\left.+\int_{A^{m_0c}\cap B_f^{m_0}}\frac{1}{\delta_0}|\delta_0 f(t^{m_0})|^{p(t^{m_0})}d\mu\right)$$
$$\le\sup_{f\in S}\sum_{m_0}\left(\int_{B_f^{m_0}}\frac{\varepsilon}{3}\Psi(t^{m_0},|f(t^{m_0})|)d\mu\right.$$
$$\left.+\int_{A^{m_0}}x_0^{2p^+}d\mu+\int_{A^{m_0c}}\frac{\varepsilon}{3\mu(A^{m_0c})}d\mu\right)$$
$$\le\sup_{f\in S}\frac{\varepsilon}{3}\int_{\Omega}\sum_{m_0}\Psi(t^{m_0},|f(t^{m_0})|)d\mu$$
$$+\sum_{m_0}x_0^{2p^+}\mu(A^{m_0})+\frac{\varepsilon}{3}\le\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}$$
$$=\varepsilon.$$

Conversely, let S be relatively weakly compact in $L^{p(\cdot)}(\Omega)$. Two cases are considered. First, assume that there exists $M>0$ so that $\parallel f\parallel_\infty< M$ for all $f\in S$. Then S is norm bounded in any $L^{p(\cdot)+\epsilon(\cdot)}(\Omega)$ with $\epsilon(\cdot)\ge 0$, in particular in $L^\infty(\Omega)$, which is defined by a function increasing uniformly more rapidly than $x^{2p(\cdot)}$.

Suppose now that $\sup_{f\in S}\parallel f\parallel_\infty=\infty$. Even more, assume that for each $t^{m_0}\in\Omega$ we have $\gamma(t^{m_0}):=\sup_{f\in S}\sum_{m_0}|f(t^{m_0})|=\infty$. If this were not the case, we can divide the space Ω in E and E^c (for $E=\{t^{m_0}:\gamma(t^{m_0})=\infty\}$) and we study the set E repeating the latter argument in E^c. Now, by Theorem 4.3, there exists a sequence $(\lambda_n)\searrow 0$ with

$$\sup_{f\in S}\frac{1}{\lambda_n}\int\sum_{m_0}|\lambda_n f(t^{m_0})|^{p(t^{m_0})}d\mu\le\frac{1}{2^{2n}}\qquad(4)$$

for every $n\in\mathbb{N}$. Let us define now the Musielak-Orlicz function

$$\sum_{m_0}\Psi(t^{m_0},x^2)=\sum_n\sum_{m_0}\frac{2^n}{\lambda_n}\Phi(t^{m_0},\lambda_n x^2)$$
$$=\sum_n\sum_{m_0}\frac{2^n}{\lambda_n}|\lambda_n x^2|^{p(t^{m_0})}.$$

Then, by Beppo-Levi Theorem, we get

$$\int_\Omega\sum_{m_0}\Psi(t^{m_0},f(t^{m_0}))d\mu$$
$$\le\sum_n\frac{2^n}{\lambda_n}\int_\Omega\sum_{m_0}|k_n f(t^{m_0})|^{p(t^{m_0})}d\mu$$
$$\le\sum_n\frac{1}{2^n}\le 1.$$

It is clear that $\Psi(t^{m_0},f(t^{m_0}))<\infty$ for every $f\in S$ a.e. $t^{m_0}\in\Omega$, so $\Psi(t^{m_0},x^2)<\infty$ for every $x^2<f(t^{m_0})$. As $\gamma(t^{m_0})=\infty$ for every t^{m_0}, we get that $\sum_{m_0}\Psi(t^{m_0},x^2)<\infty$ for all t^{m_0} and x^2, thus $\Psi(t^{m_0},x^2)$ is a Musielak-Orlicz function increasing uniformly more rapidly than the function $x^{2p(t^{m_0})}$. Indeed, since $\sum_{m_0}\Psi(t^{m_0},x^2)\ge\frac{2^i}{\lambda_i}\sum_{m_0}|\lambda_n x^2|^{p(t^{m_0})}$ for each natural i, we take n so that $\delta=2^n\ge\frac{1}{\varepsilon}$ getting $\varepsilon\Psi(t^{m_0},x^2)\ge\frac{1}{\delta}|\delta x^2|^{p(t^{m_0})}$. Furthermore, S is clearly norm bounded in $L^\Psi(\Omega)$.

We can get rid of the condition $\mu(\Omega_1)=0$ in above result. Indeed, assume that $\mu(\Omega_1)>0$, then a subset $S\subset L^1(\Omega_1)$ is weakly compact if and only if there is an Orlicz space $L^\varphi(\Omega_1)$ with $\frac{\varphi(x^2)}{x^2}\overset{x^2\to\infty}{\longrightarrow}\infty$ such that S is norm bounded in $L^\varphi(\Omega_1)$ (by Dunford-Pettis and De la Vallée Poussin theorems). Hence, considering the Musielak-Orlicz sum function

$$\bar\Psi(t^{m_0},x^2)=\varphi(x^2)\chi_{\Omega_1}+\Psi(t^{m_0},x^2)\chi_{\Omega_1^c}$$

we get:

Corollary 6.3. Let $L^{p(\cdot)}(\Omega)$ with $p^+<\infty$. A subset $S\subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact if and only if there exists a Musielak-Orlicz function $\bar\Psi(x^2,t^{m_0})$ increasing uniformly more rapidly than $x^{2p(t^{m_0})}$ such that S is notm bounded in $L^{\bar\Psi}(\Omega)$.

Note that the Musielak-Orlicz function $\Psi(x^2,t^{m_0})=(\varphi(x^2))^{p(t^{m_0})}$ associated to the Orlicz function φ defined in Theorem 3.2 increases uniformly more rapid than the function $x^{2p(t^{m_0})}$. Indeed, given $\varepsilon>0$, take $x_0^2>0$ and $0<\delta<1$ such that $\frac{\varphi(x_0^2)}{x_0^2}>\frac{1}{\varepsilon}$ and $\delta^{p-1}\le 1$.

Then, for every $x^2\ge x_0^2$ and $t^{m_0}\in\Omega$, it holds

$$\varepsilon\left(\frac{\varphi(x^2)}{x^2}\right)^{p(t^{m_0})}\ge\delta^{p-1}\ge\delta^{p(t^{m_0})-1}.$$

In the case of comparing exponent functions $p(\cdot)$ and $p(\cdot)+\epsilon(\cdot)$, the meaning of increasing more rapidly is easily characterized:

Proposition 6.4. [1] Let $p(\cdot)\le p(\cdot)+\epsilon(\cdot)$ exponent functions. Then, $\Psi(t^{m_0},x^2)=x^{2(p(t^{m_0})+\epsilon(t^{m_0}))}$ increases uniformly more rapidly than $\Phi(t^{m_0},x^2)=x^{2p(t^{m_0})}$ if and only if $p^-+\epsilon>1$.

Proof. First note that in variable exponent spaces the inequality relation is simplified to

$$\varepsilon x^{2(\epsilon(t^{m_0}))}\ge\delta^{p(t^{m_0})-1}.$$

Suppose that $p^-+\epsilon=1$. Let $(t_n^{m_0})$ be a sequence such that $p(t_n^{m_0})+\epsilon(t_n^{m_0})\to 1$ (and hence $p(t_n^{m_0})\to 1$). Let

$\varepsilon=\frac{1}{2}$. For any positives δ and x_0^2 there exists $t_{n_0}^{m_0}$ such that $x_0^{2\epsilon\left(t_{n_0}^{m_0}\right)}$ and $\delta^{p\left(t_{n_0}^{m_0}\right)-1}$ are sufficiently close to 1 and

$$\frac{1}{2}x_0^{2\epsilon\left(t_{n_0}^{m_0}\right)}<\delta^{p\left(t_{n_0}^{m_0}\right)-1},$$

showing that $x^{2p(\cdot)+\epsilon(\cdot)}$ does not increase uniformly more rapidly than $x^{2p(\cdot)}$.

Conversely, suppose $p^-+\epsilon>1$. Given $\varepsilon>0$, consider the set $A^{m_0}=\left\{t^{m_0}: p(t^{m_0})\geq\frac{1+p^-+\epsilon}{2}\right\}$. On one hand, taking $x_1^2\geq 1$ and $\delta_1<1$ small enough with $\varepsilon\geq\delta_1^{\left(\frac{p^-+\epsilon-1}{2}\right)}$, we get that, for all $t^{m_0}\in A^{m_0}$ and for all $x^2\geq x_1^2$,

$$\varepsilon x^{2\epsilon(t^{m_0})}\geq\varepsilon\geq\delta_1^{\left(\frac{p^-+\epsilon-1}{2}\right)}\geq\delta_1^{p(t^{m_0})-1}.$$

On the other hand, taking $\delta_2\leq 1$ and $x_2^2>1$ large enough to $\varepsilon x_2^{2\left(\frac{p^-+\epsilon-1}{2}\right)}\geq 1$, we get that, for all $t^{m_0}\in A^{m_0c}$ and for all $x^2\geq x_2^2$,

$$\varepsilon x^{2(\epsilon(t^{m_0}))}\geq\varepsilon x_2^{2\left(\frac{p^-+\epsilon-1}{2}\right)}\geq 1\geq\delta_2^{p(t^{m_0})-1}$$

Thus, taking $x_0^2=x_2^2$ and $\delta=\delta_1$, we get the desired inequality for all $t^{m_0}\in\Omega$.

Corollary 6.5. Let $S\subset L^{p(\cdot)}(\Omega)$ with $p^+<\infty$. If S is bounded in some $L^{p(\cdot)+\epsilon(\cdot)}(\Omega)$ with $\epsilon(\cdot)\geq 0$ and $p^-+\epsilon>1$, then S is relatively weakly compact in $L^{p(\cdot)}(\Omega)$.

The converse, however, is not true:

Proposition 6.6. [1] Let $L^{p(\cdot)}[0,1]$ with $1=p^-<p^+<\infty$. There exists a null sequence (f_n) in $L^{p(\cdot)}[0,1]$ such that (f_n) is not norm bounded in $L^{p(\cdot)+\epsilon(\cdot)}[0,1]$ for any exponent function $p(\cdot)+\epsilon(\cdot)\geq p(\cdot)$ with $p^-+\epsilon>1$.

Proof. Let $(p_n+\epsilon)\searrow 1$ be a sequence in the interval $[1,p^+]$. We can take a disjoint sequence of subsets $\left(A_n^{m_0}\right)$ of positive measure satisfying

$$A_n^{m_0}\subset p^{-1}\left(1,\frac{1+p_n+\epsilon}{2}\right)$$

and thus $p^+_{|A_n^{m_0}}\leq\frac{1+p_n+\epsilon}{2}<p_n+\epsilon$.

By Proposition 3.4 we know that, for every $n\in\mathbb{N}$, the inclusion $L_{p_n+\epsilon}\left(A_n^{m_0}\right)\subset L^{p(\cdot)}\left(A_n^{m_0}\right)$ is L-weakly compact. Let $\left(B_{n,k}^{m_0}\right)_k$ be a disjoint partition of each $A_n^{m_0}$ for $n\in\mathbb{N}$ and define the functions

$$s_{n,k}:=\sum_{m_0}\frac{\chi_{B_{n,k}^{m_0}}}{\mu\left(B_{n,k}^{m_0}\right)^{\frac{1}{p_n+\epsilon}}},$$

which are normalized in $L_{p_n+\epsilon}[0,1]$. For every $n\in\mathbb{N}$, $\mu\left(B_{n,k}^{m_0}\right)\overset{k\to\infty}{\longrightarrow}0$, so there exists some k_n such that, for every $k\geq k_n$,

$$\|s_{n,k}\|_{p(\cdot)}\leq\frac{1}{n}.$$

Then, the sequence $\left(s_{n,k_n}\right)_n$ converges to 0 in $L^{p(\cdot)}[0,1]$. So, let us see that $\left(s_{n,k_n}\right)$ is not norm bounded in any $L^{p(\cdot)+\epsilon(\cdot)}[0,1]$ with $p(\cdot)+\epsilon(\cdot)\geq p(\cdot)$ and $p^-+\epsilon>1$. Given such an exponent function $p(\cdot)+\epsilon(\cdot)$, there exist $n_0\in\mathbb{N}$ and $\delta>0$ such that $\frac{p^-+\epsilon}{p_n+\epsilon}>1+\delta$ for all $n\geq n_0$. Thus,

$$\rho_{p(\cdot)+\epsilon(\cdot)}\left(s_{n,k_n}\right)=\int_{B_{n,k_n}^{m_0}}\sum_{m_0}\left(\frac{1}{\mu\left(B_{n,k_n}^{m_0}\right)^{\frac{1}{p_n+\epsilon}}}\right)^{p(t^{m_0})+\epsilon(t^{m_0})}dt^{m_0}$$

$$\geq\int_{B_{n,k_n}^{m_0}}\sum_{m_0}\frac{1}{\mu\left(B_{n,k_n}^{m_0}\right)^{\frac{p^-+\epsilon}{p_n+\epsilon}}}dt^{m_0}\geq\sum_{m_0}\frac{1}{\mu\left(B_{n,k_n}^{m_0}\right)^{\frac{p^-+\epsilon}{p_n+\epsilon}-1}}$$

and $\frac{p^-+\epsilon}{p_n+\epsilon}-1>\delta$ for $n\geq n_0$, so

$$\lim_{n\to\infty}\rho_{p(\cdot)+\epsilon(\cdot)}\left(s_{n,k_n}\right)\geq\lim_{n\to\infty}\sum_{m_0}\frac{1}{\mu\left(B_{n,k_n}^{m_0}\right)^{\delta}}=\infty$$

and $\left(s_{n,k_n}\right)$ is not norm bounded in $L^{p(\cdot)+\epsilon(\cdot)}[0,1]$.

A tentative characterization of a weakly compact subset of $L^{p(\cdot)}(\Omega)$ in terms of norm boundness in some smaller $L^{p(\cdot)+\epsilon(\cdot)}(\Omega)$ space for some exponent functions $p(\cdot)+\epsilon(\cdot)$ with $p^-+\epsilon=1$ is left as an open question.

Conflict of interest

The authors declare that there is no conflict regarding the publication of this paper.

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