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# Sine F-Loss Family of Distributions: Properties and Insurance Applications

*John Abonongo*[1,*], *Ivivi J. Mwaniki*[2] *and Jane A. Aduda*[3]

[1]Institute for Basic Sciences, Technology and Innovation. Pan African University, Nairobi, Kenya
[2]School of Mathematics, Statistics and Actuarial Science Division, University of Nairobi, Kenya
[3]Department of Statistics and Actuarial Science, Jomo Kenyatta University of Agriculture and Technology, Nairobi, Kenya

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**Abstract:** Heavy-tailed distributions are vital in actuarial and financial modeling. In this article, a new family of heavy-tailed distributions known as sine F-Loss is proposed. The mathematical properties are derived and maximum likelihood estimators of the model parameters are obtained. For illustrative purposes, three special distributions; sine Weibull loss, sine Fréchet loss and sine Burr XII loss are proposed. Monte Carlo simulations are carried out to assess the behavior of the estimators. The densities and hazard rate functions exhibit increasing, decreasing, increasing-constant-decreasing, symmetric, right skewed, reversed J-shaped, bathtub and upside down bathtub shapes. The application of the proposed distributions is presented with two insurance loss datasets.

**Keywords:** Heavy-tailed distributions, Insurance losses, Monte Carlo simulation, F-Loss, Weibull distribution, Fréchet distribution, Maximum likelihood estimation.

## 1 Introduction

Statistical distributions are essential in modeling data in applied fields, primarily in actuarial, financial sciences, and risk management. Heavy-tailed distributions have proven to be the best choice for modeling financial datasets. Actuaries often look for such distributions to better describe actuarial or financial datasets [1]. Insurance data usually are positive, and their distribution is typically unimodal humped shaped with extreme values yielding tails heavier than some conventional distributions [2]. Therefore, traditional distributions may not be flexible in modeling these heavy-tailed datasets [3].
Some classical distributions like the Pareto [4], Lomax [5], beta [6], Burr [7], and Weibull [8] still have some drawbacks as they cannot model heavy-tailed datasets adequately. Also, some of the methods employed in developing heavy tail distributions in the literature have been the compounding of distributions, addition of variables, transformation of variables, composition method, and the finite mixture of distributions; most of the heavy-tailed distributions from these methods are mostly not flexible enough in modeling insurance loss data and at times lead to over-parameterization.
Since there is an increased interest in data analysis, further research is needed to have more choices regarding distributions that dwell on trigonometric functions [9]. The merits of these distributions are; to allow a good understanding of the mathematical properties, limits over parameterization, and give better applicability in modeling different datasets. These points come from proper use of the trigonometric functions. Also, the trigonometric transformation provides flexibility because of the periodic function, which controls how the distribution curve behaves, and parameter(s) oscillate with value changes [10] .
Most of the statistical distributions proposed in the literature have many parameters to make them flexible. According to [11] , these estimates may be challenging to obtain numerically. Therefore, it is essential to develop distributions with a few parameters but a significant degree of flexibility for modeling data. Few researchers decided to look for other distributions using trigonometric functions to achieve this. Among them are; [12, 13, 14, 15, 16, 17, 18].
More recently, [19] introduced the F-loss family of distributions by adding a shape parameter to the baseline distribution. They proposed the Weibull-Loss (W-Loss)

* Corresponding author e-mail: abonongojohn@gmail.com

distribution as a sub-model. The distributions from this family are not flexible enough in handling varying shape of the density and hazard rate functions. Hence, we propose an extension of this family without adding any extra parameters by using trigonometric functions.
Also, to address the drawbacks of some of the existing heavy-tailed distributions and methods of developing probability distributions, we propose the sine F-Loss Family of distributions. We show the flexibility of the sine F-Loss family of distributions as a better alternative to the F-Loss family of distributions and other heavy tailed distributions in real-world phenomenon. In this regard, we use the F-Loss family of distributions introduced by [19]. A random variable $X$ is said to follow the F-Loss family, if its cumulative distribution (CDF) is given by

$$H(x;\sigma,\boldsymbol{\omega}) = 1 - \frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma - \log\left(\bar{F}(x;\boldsymbol{\omega})\right)}, \qquad \sigma > 0, x \in \mathbb{R}, \tag{1}$$

where $\bar{F}(x;\boldsymbol{\omega}) = 1 - F(x;\boldsymbol{\omega})$ is the survival function (sf) of the baseline distribution, $\boldsymbol{\omega}$ is a $p \times 1$ vector of parameters and $\sigma$ is a shape parameter.
[20] proposed the sine-G family of distributions. A random variable $X$ is said to follow the sine-G family of distributions, if its CDF is given by

$$G(x;\sigma,\boldsymbol{\omega}) = \sin\left(\frac{\pi}{2}H(x;\sigma,\boldsymbol{\omega})\right), \; x \in \mathbb{R}. \tag{2}$$

We propose a new heavy-tailed family of distributions based on the F-Loss family and sine-G family of distributions called the sine F-Loss (SFL) family of distributions without adding an extra parameter. A random variable $X$ is said to follow the SFL family of distributions if its CDF is designated as

$$G(x;\sigma,\boldsymbol{\omega}) = \sin\left[\frac{\pi}{2}\left(1 - \frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))}\right)\right], \; \sigma > 0, x \in \mathbb{R}, \tag{3}$$

where $\bar{F}(x;\boldsymbol{\omega}) = 1 - F(x;\boldsymbol{\omega})$ is the survival function (sf) of the baseline distribution which may depend on the vector parameter $\boldsymbol{\omega}$ and $\sigma$ is a shape parameter.
It is constructed from the insertion of the CDF, $H(x;\sigma,\boldsymbol{\omega})$ in equation (1) into the CDF; $G(x;\sigma,\boldsymbol{\omega})$ given in equation (2).
The PDF is given by

$$g(x;\sigma,\boldsymbol{\omega}) = \frac{\pi}{2}\left[\frac{\sigma f(x;\boldsymbol{\omega})\left[1+\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))\right]}{\left[\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))\right]^2}\right] \times \cos\left[\frac{\pi}{2}\left(1 - \frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma - \log\left(\bar{F}(x;\boldsymbol{\omega})\right)}\right)\right], x \in \mathbb{R}. \tag{4}$$

The hazard rate function of the SFL family of distributions is given by

$$h(x;\sigma,\boldsymbol{\omega}) = \frac{\pi\left[\sigma f(x;\boldsymbol{\omega})[1+\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))]\right]\cos\left[\frac{\pi}{2}\left(1-\frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))}\right)\right]}{2\left[\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))\right]^2\left[1-\sin\left[\frac{\pi}{2}\left(1-\frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))}\right)\right]\right]}, x \in \mathbb{R}. \tag{5}$$

Our motivation for proposing an extension of the F-Loss family of distributions are; to improve the flexibility of the F-Loss family of distributions without introducing any additional parameter(s); to produce heavy tailed distributions with fewer parameters that gives better parametric fit to a given data sets than some existing distributions; to generate distributions which are approximately symmetric, right-skewed and reversed-J shaped and distributions capable of modeling monotonic and non-monotonic hazard rates. Therefore, there is the need to extend the F-Loss family of distributions so as to capture all these variations.
The rest of the paper is organized as follows: Section 2, gives the mixture representation of the SFL family of distributions, Section 3 presents the mathematical properties of the SFL family. In Section 4, we present the parameter estimation. Section 5, presents three special distributions of the family of distributions. The behavoir of the parameters of the special distributions are ascertain in Section 6 using Monte Carlo simulations. In Section 7, the applications of the special distributions are illustrated using insurance loss dataset. The conclusion is presented in Section 8.

## 2 Mixture Representation

This section presents the mixture representation of the PDF of the SFL family of distributions.
**Lemma 1.** The PDF of the SFL family of distributions have a mixture representation of the form

$$g(x;\sigma,\boldsymbol{\omega}) = \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^{*}_{ijk}(1+\sigma)L_k f(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+t+m} + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^{*}_{ijk}L_{k+1}f(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+1+t+m}. \tag{6}$$

where $\varpi^{*}_{ijk} = \frac{\varpi_{ijk}(-1)^t}{\sigma^{k+1}}\binom{j}{t}$, $L_k = k\sum_{m=0}^{\infty}\binom{m-k}{m}\sum_{r=0}^{m}\frac{(-1)^{m+r}}{k-r}\binom{m}{r}P_{r,m}$ and $L_{k+1} = (k+1)\sum_{m=0}^{\infty}\binom{m-k-1}{m}\sum_{r=0}^{m}\frac{(-1)^{m+r}}{k+1-r}\binom{m}{r}P_{r,m}$.
**Proof.** Using the power series expansion of cosine function,

$$\cos(x) = \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i)!}x^{2i}. \tag{7}$$

That is, from equation (4),

$$\cos\left[\frac{\pi}{2}\left(1 - \frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma - \log\left(\bar{F}(x;\boldsymbol{\omega})\right)}\right)\right] = \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i)!} \times \left[\frac{\pi}{2}\left(1 - \frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma - \log\bar{F}(x;\boldsymbol{\omega})}\right)\right]^{2i}. \tag{8}$$

From the generalized binomial expansion given by

$$(1-z)^n = \sum_{j=0}^{\infty}(-1)^j\binom{n}{j}z^j, \quad |z|\leq 1, \tag{9}$$

and

$$(1+x)^{-s} = \sum_{k=0}^{\infty}(-1)^k\binom{s+k-1}{k}x^k, \quad |x|\leq 1. \tag{10}$$

Substituting equation (8) in equation (4) and making use of equation (9) and the fact that $0<\frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma-\log\bar{F}(x;\boldsymbol{\omega})}<1$, we have

$$g(x;\sigma,\boldsymbol{\omega}) = \frac{\pi}{2}\frac{\sigma f(x;\boldsymbol{\omega})\left[1+\sigma-\log\bar{F}(x;\boldsymbol{\omega})\right]}{\left[\sigma-\log\bar{F}(x;\boldsymbol{\omega})\right]^2}$$
$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\frac{(-1)^{i+j}(\frac{\pi}{2})^{2i}}{(2i)!}\binom{2i}{j}\left[\frac{\sigma^j\bar{F}(x;\boldsymbol{\omega})^j}{\left[\sigma-\log\bar{F}(x;\boldsymbol{\omega})\right]^j}\right]$$
$$=\frac{\pi}{2}\sigma f(x;\boldsymbol{\omega})\left[1+\sigma-\log\bar{F}(x;\boldsymbol{\omega})\right]$$
$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\frac{(-1)^{i+j}(\frac{\pi}{2})^{2i}}{(2i)!}\binom{2i}{j}\left[\frac{\sigma^j\bar{F}(x;\boldsymbol{\omega})^j}{\sigma^{2+j}\left[1-\frac{\log\bar{F}(x;\boldsymbol{\omega})}{\sigma}\right]^{2+j}}\right]$$
$$=f(x;\boldsymbol{\omega})\left[1+\sigma-\log\bar{F}(x;\boldsymbol{\omega})\right]$$
$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\frac{(-1)^{i+j}(\frac{\pi}{2})^{2i+1}}{(2i)!}\binom{2i}{j}\frac{\bar{F}(x;\boldsymbol{\omega})^j}{\sigma\left[1-\frac{\log\bar{F}(x;\boldsymbol{\omega})}{\sigma}\right]^{2+j}}.$$

Using the binomial expansion in equation (10) and letting $x=-\frac{\log\bar{F}(x;\boldsymbol{\omega})}{\sigma}$,

$$g(x;\sigma,\boldsymbol{\omega}) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}(\frac{\pi}{2})^{2i+1}}{(2i)!}\binom{2i}{j}$$
$$\times\binom{j+k+1}{k}$$
$$\times\frac{(1+\sigma)f(x;\boldsymbol{\omega}))\bar{F}(x;\boldsymbol{\omega})^j\left[-\log\bar{F}(x;\boldsymbol{\omega})\right]^k}{\sigma^{k+1}}$$
$$+\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}(\frac{\pi}{2})^{2i+1}}{(2i)!}\binom{2i}{j}$$
$$\times\binom{j+k+1}{k}$$
$$\times\frac{f(x;\boldsymbol{\omega})\bar{F}(x;\boldsymbol{\omega})^j\left[-\log\bar{F}(x;\boldsymbol{\omega})\right]^{k+1}}{\sigma^{k+1}}.$$

Again, using the expansion;

$$(-\log(1-z))^a = a\sum_{m=0}^{\infty}\binom{m-a}{m}\sum_{r=0}^{m}\frac{(-1)^{m+r}}{a-r}\binom{m}{r}P_{r,m},z^{a+m},$$

where $a>0$ is any real value. The constants $P_{r,m}$ can be calculated, recursively, via
$P_{r,m}=\frac{1}{m}\sum_{s=1}^{m}\frac{rs+s-m}{s+1}P_{r,m-s}$ for $m=1,2,3,\ldots$ and $P_{r,0}=1$. Let $\varpi_{ijk}=\frac{(-1)^{i+j+k}(\frac{\pi}{2})^{2i+1}}{(2i)!}\binom{2i}{j}\binom{j+k+1}{k}$,

$$g(x;\sigma,\boldsymbol{\omega}) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{k=0}^{\infty}\frac{\varpi_{ijk}(1+\sigma)}{\sigma^{k+1}}f(x;\boldsymbol{\omega})\bar{F}(x;\boldsymbol{\omega})^j$$
$$\left[-\log\bar{F}(x;\boldsymbol{\omega})\right]^k+\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{k=0}^{\infty}\frac{\varpi_{ijk}}{\sigma^{k+1}}f(x;\boldsymbol{\omega})$$
$$\bar{F}(x;\boldsymbol{\omega})^j\left[-\log\bar{F}(x;\boldsymbol{\omega})\right]^{k+1}.$$

This implies that,

$$g(x;\sigma,\boldsymbol{\omega}) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{k=0}^{\infty}\frac{\varpi_{ijk}(1+\sigma)}{\sigma^{k+1}}f(x;\boldsymbol{\omega})\bar{F}(x;\boldsymbol{\omega})^j$$
$$\times k\sum_{m=0}^{\infty}\binom{m-k}{m}\sum_{r=0}^{m}\frac{(-1)^{m+r}}{k-r}\binom{m}{r}P_{r,m}$$
$$F(x;\boldsymbol{\omega})^{k+m}+\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{k=0}^{\infty}\frac{\varpi_{ijk}}{\sigma^{k+1}}f(x;\boldsymbol{\omega})$$
$$\bar{F}(x;\boldsymbol{\omega})^j(k+1)\sum_{m=0}^{\infty}\binom{m-k-1}{m}$$
$$\times\sum_{r=0}^{m}\frac{(-1)^{m+r}}{k+1-r}\binom{m}{r}P_{r,m}F(x;\boldsymbol{\omega})^{k+1+m}.$$

Letting $L_k=k\sum_{m=0}^{\infty}\binom{m-k}{m}\sum_{r=0}^{m}\frac{(-1)^{m+r}}{k-r}\binom{m}{r}P_{r,m}$
and $L_{k+1}=(k+1)\sum_{m=0}^{\infty}\binom{m-k-1}{m}\sum_{r=0}^{m}\frac{(-1)^{m+r}}{k+1-r}\binom{m}{r}P_{r,m}$
and the fact
that, $\bar{F}(x;\boldsymbol{\omega})^j=\left[1-F(x;\boldsymbol{\omega})\right]^j=\sum_{t=0}^{j}(-1)^t\binom{j}{t}F(x;\boldsymbol{\omega})^t$,
we get,

$$g(x;\sigma,\boldsymbol{\omega}) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{k=0}^{\infty}\sum_{t=0}^{j}\frac{\varpi_{ijk}(1+\sigma)(-1)^tL_k}{\sigma^{k+1}}\binom{j}{t}$$
$$f(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+t+m}$$
$$+\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\sum_{k=0}^{\infty}\sum_{t=0}^{j}\frac{\varpi_{ijk}(-1)^tL_{k+1}}{\sigma^{k+1}}\binom{j}{t}$$
$$f(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+1+t+m}.$$

## 3 Mathematical Properties of the SFL Family of Distributions

In this section, the mathematical properties of the SFL family of distributions including quantile function, moments, moment generation function, limited expected value, mean deviation, median deviation, mean excess loss function, value at risk, tail value at risk, tail variance and tail variance premium are derived.

### 3.1 Quantile Function

The quantile function is vital in describing the random variable of a distribution. It helps in simulating random samples which are useful in computing the median, kurtosis and skewness of a distribution amongst others.

**Lemma 2.** The quantile function of the SFL family of distributions for $u \in (0,1)$ is given by

$$\sigma(1-t)-(\sigma-\log(1-t))\left(1-\frac{2}{\pi}\arcsin(u)\right)=0. \tag{11}$$

**Proof.** By definition the quantile function is given by

$$x_u = G^{-1}(u).$$

Thus,

$$1-\frac{2}{\pi}\arcsin(u)=\frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma-\log\left(\bar{F}(x;\boldsymbol{\omega})\right)}. \tag{12}$$

Solving equation (12), we get

$$\sigma(1-t)-(\sigma-\log(1-t))\left(1-\frac{2}{\pi}\arcsin(u)\right)=0,$$

where $t$ is the solution of the equation $\sigma(1-t)-(\sigma-\log(1-t))\left(1-\frac{2}{\pi}\arcsin(u)\right)=0$, and $u$ has the uniform distribution on the interval $(0,1)$.

The quantile function can also be employed in estimating the skewness and kurtosis especially when the moments of the distribution does not exist. It can be obtained by using Galton's skewness and Moor's kurtosis measure which are given respectively by

$$\mathscr{G}.\mathscr{S}=\frac{Q\left(\frac{6}{8}\right)+Q\left(\frac{2}{8}\right)-2Q\left(\frac{4}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)} \tag{13}$$

and

$$\mathscr{M}.\mathscr{K}=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)+Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}. \tag{14}$$

### *3.2 Moments*

The moments of a distribution is important in estimating measures of variation like the variance, standard deviation, coefficient of variation, mean deviation, median deviation, kurtosis, skewness amongst others.

**Proposition 1.** The $n^{th}$ non-central moment of the SFL family of distributions is given by

$$\mu_n' = \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi_{ijk}^{*}(1+\sigma)L_k\vartheta_{ktm}(x;\omega) + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi_{ijk}^{*}L_{k+1}\phi_{ktm}(x;\omega). \tag{15}$$

where $\vartheta_{ktm}(x;\omega)=\int_0^\infty x^n f(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+t+m}dx$, $\phi_{ktm}(x;\omega) = \int_0^\infty x^n f(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+1+t+m}dx$ and $n=1,2,\ldots$.

**Proof.** By definition the $n^{th}$ non- central moment is given by

$$\mu_n' = \int_0^\infty x^n g(x)dx.$$

This implies that,

$$\mu_n' = \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi_{ijk}^{*}(1+\sigma)L_k\int_0^\infty x^n f(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+t+m}dx + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi_{ijk}^{*}L_{k+1}\int_0^\infty x^n f(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+t+1+m}dx.$$

### *3.3 Expressing the Moment in terms of the Quantile Function*

**Proposition 2.** The moment of the SFL family of distributions in terms of the quantile function is given by

$$\mu_n' = \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi_{ijk}^{*}(1+\sigma)L_kA + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi_{ijk}^{*}L_{k+1}A^{*}. \tag{16}$$

where $A = \int_0^1 Q_G^n(u)u^{k+t+m}du$ and $A^{*}=\int_0^1 Q_G^n(u)u^{k+1+t+m}du$.

**Proof.** Letting $u=G(x)$, implies that if, $x\to-\infty$ then, $u\to 0$ and if $x\to\infty$, then, $u\to 1$. Also, $du=g(x)dx$ and $x=G^{-1}(u)=Q_G(u)$.

Thus,

$$\mu_n' = \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi_{ijk}^{*}(1+\sigma)L_k\int_0^1 Q_G^n(u)u^{k+t+m}du + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi_{ijk}^{*}L_{k+1}\int_0^1 Q_G^n(u)u^{k+1+t+m}du.$$

### *3.4 Moment Generating Function*

The moment generating function (MGF) helps in determining the moments of a random variable.

**Proposition 3.** The MGF of the SFL Family of distributions is given by

$$M_X(z) = \sum_{i,k,n=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\frac{Z^n\varpi_{ijk}^{*}(1+\sigma)L_k}{n!}\vartheta_{ktm}(x;\omega) + \sum_{i,k,n=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\frac{Z^n\varpi_{ijk}^{*}L_{k+1}}{n!}\phi_{ktm}(x;\omega). \tag{17}$$

**Proof.** By definition the MGF is given as;

$$M_X(z)=\mathbb{E}(e^{zX})=\int_0^\infty e^{zx}g(x)dx.$$

Using series expansion,

$$M_X(z) = \sum_{n=0}^{\infty} \frac{Z^n}{n!} \mu_n^{'}.$$

This implies,

$$M_X(z) = \sum_{i,k,n=0}^{\infty} \sum_{j=0}^{2i} \sum_{t=0}^{j} \frac{Z^n \varpi_{ijk}^{*}(1+\sigma)L_k}{n!} \times \int_0^{\infty} x^n f(x;\boldsymbol{\omega}) F(x;\boldsymbol{\omega})^{k+t+m} dx + \sum_{i,k,n=0}^{\infty} \sum_{j=0}^{2i} \sum_{t=0}^{j} \frac{Z^n \varpi_{ijk}^{*} L_{k+1}}{n!} \times \int_0^{\infty} x^n f(x;\boldsymbol{\omega}) F(x;\boldsymbol{\omega})^{k+t+1+m} dx.$$

### 3.5 Mean Excess Loss Function

The mean excess loss function measures the expected payment per claim on a policy with a fixed amount deductible $x$, ignoring the claims with amounts less than or equal to $x$.

**Proposition 4.** The mean excess loss function of the SFL family of distributions is given by

$$e_X(d) = \frac{1}{1-F_{(x;\boldsymbol{\omega})}(d)} \left\{ \sum_{i,k=0}^{\infty} \sum_{j=0}^{2i} \sum_{t=0}^{j} \varpi_{ijk}^{*}(1+\sigma)L_k B_{ktm} + \sum_{i,k=0}^{\infty} \sum_{j=0}^{2i} \sum_{t=0}^{j} \varpi_{ijk}^{*} L_{k+1} B_{ktm}^{*} \right\}. \tag{18}$$

where $B_{ktm} = \int_d^{\infty} (x-d) f(x;\boldsymbol{\omega}) F(x;\boldsymbol{\omega})^{k+t+m} dx$ and $B_{ktm}^{*} = \int_d^{\infty} (x-d) f(x;\boldsymbol{\omega}) F(x;\boldsymbol{\omega})^{k+1+t+m} dx$.

**Proof.** By definition the mean excess loss function is given as;

$$e_X(d) = \mathbb{E}(X-d|X>d) = \frac{1}{1-F_x(d)} \int_d^{\infty} (x-d) g(x).$$

This implies that,

$$e_X(d) = \frac{1}{1-F_{(x;\boldsymbol{\omega})}(d)} \left[ \sum_{i,k=0}^{\infty} \sum_{j=0}^{2i} \sum_{t=0}^{j} \varpi_{ijk}^{*}(1+\sigma)L_k \times \int_d^{\infty} (x-d) f(x;\boldsymbol{\omega}) F(x;\boldsymbol{\omega})^{k+t+m} dx + \sum_{i,k=0}^{\infty} \sum_{j=0}^{2i} \sum_{t=0}^{j} \varpi_{ijk}^{*} L_{k+1} \times \int_d^{\infty} (x-d) f(x;\boldsymbol{\omega}) F(x;\boldsymbol{\omega})^{k+1+t+m} dx \right].$$

### 3.6 Limited Expected Value

The limited expected value represents the expected amount per claim retained by insured on a policy with a fixed amount deductible of $x$. Also, it shows how the different parts of the claim size CDF contributes to the premium.

**Proposition 5.** The limited expected value of the SFL family of distributions is given by

$$L(u) = \left[ \sum_{i,k=0}^{\infty} \sum_{j=0}^{2i} \sum_{t=0}^{j} \varpi_{ijk}^{*}(1+\sigma)L_k A_u + \sum_{i,k=0}^{\infty} \sum_{j=0}^{2i} \sum_{t=0}^{j} \varpi_{ijk}^{*} L_{k+1} A_u^{*} \right] + u(1-F(x)) \tag{19}$$

where $A_u = \int_0^u x f(x;\boldsymbol{\omega}) F(x;\boldsymbol{\omega})^{k+t+m} dx$ and $A_u^{*} = \int_0^u x f(x;\boldsymbol{\omega}) F(x;\boldsymbol{\omega})^{k+1+t+m} dx$

**Proof.** By definition the limited expected value function of a claim size variable $X$ is given as;

$$L(u) = \mathbb{E}(X \wedge u) = \int_0^u x g(x) dx + u(1-F(x)).$$

This implies that,

$$L(u) = \left[ \sum_{i,k=0}^{\infty} \sum_{j=0}^{2i} \sum_{t=0}^{j} \varpi_{ijk}^{*}(1+\sigma)L_k \int_0^u x f(x;\boldsymbol{\omega}) F(x;\boldsymbol{\omega})^{k+t+m} dx + \sum_{i,k=0}^{\infty} \sum_{j=0}^{2i} \sum_{t=0}^{j} \varpi_{ijk}^{*} L_{k+1} \int_0^u x f(x;\boldsymbol{\omega}) F(x;\boldsymbol{\omega})^{k+1+t+m} dx \right] + u(1-F(x)).$$

### 3.7 Value at Risk

Value at risk (VaR) is commonly used as a benchmark in measuring market risk. It is also called the quantile premium principle or quantile risk measure. It is usually expressed with a confidence level $q$ (usually 90%, 95% or 99%), and represent the percentage of loss in portfolio value that will be equaled or exceeded only $X$ percent of the time. The VaR of a random $X$ is the $q^{th}$ quantile of its [21].

**Proposition 6.** For $\sigma > 0$, the $VaR_q(X)$ of the SFL family of distributions is given by

$$(1-t) - (\sigma - \log(1-t)) \left(1 - \frac{2}{\pi} \arcsin(q)\right) = 0, \tag{20}$$

where $t$ is the solution of equation $(1-t) - (\sigma - \log(1-t)) \left(1 - \frac{2}{\pi} \arcsin(q)\right) = 0$.

**Proof.** By definition

$$x_q = F^{-1}(t).$$

Thus, the VaR of SFL distribution can be written as;

$$(1-t) - (\sigma - \log(1-t)) \left(1 - \frac{2}{\pi} \arcsin(q)\right) = 0.$$

### 3.8 Tail Value at Risk

The Tail value at risk is also called the tail conditional expectation (TCE) or conditional tail expectation (CTE) and is for determining the average loss beyond a given probability level.

**Proposition 7.** For $\sigma \geq 0$, the $TVaR_q(X)$ for the SFL family of distributions is given by

$$TVaR_q(X) = \frac{1}{1-q}\left[\sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}(1+\sigma)L_k J_q + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}L_{k+1}J^*_q\right]. \quad (21)$$

where $J_q = \int_{VaR_q}^{\infty} xf(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+t+m}dx$ and $J^*_q = \int_{VaR_q}^{\infty} xf(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+1+t+m}dx$.

**Proof.** By definition

$$TVaR_q(X) = \mathbb{E}(X|X > VaR_q) = \frac{1}{1-q}\int_{VaR_q}^{\infty} xg(x)dx.$$

$$TVaR_q(X) = \frac{1}{1-q}\left[\sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}(1+\sigma)L_k \times \int_{VaR_q}^{\infty} x\left(f(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+t+m}\right)dx + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}L_{k+1} \times \int_{VaR_q}^{\infty} x\left(f(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+1+t+m}\right)dx\right].$$

### 3.9 Tail Variance

TV is an important risk measure in insurance sciences. It is vital in determining the risk level at the tails.

**Proposition 8.** For $\sigma \geq 0$, the $TV_q(X)$ for the SFL family of distributions is given by

$$TV_q(X) = \frac{1}{1-q}\left[\sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}(1+\sigma)L_k B_q + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}L_{k+1}B^*_q\right] - \left[\frac{1}{1-q}\left\{\sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}(1+\sigma)L_k J_q \times \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}L_{k+1}J^*_q\right\}\right]^2, \quad (22)$$

where $B_q = \int_{VaR_q}^{\infty} x^2 f(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+t+m}dx$ and $B^*_q = \int_{VaR_q}^{\infty} x^2 f(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+1+t+m}dx$.

**Proof.** From definition

$$TV_q(X) = \mathbb{E}(X^2|X > x_q) - (TVaR_q)^2. \quad (23)$$

Using conditional moments, $\mathbb{E}(X^n|X > x) = \frac{1}{s(x)}\tau_n(x)$, where $\tau_n(x) = \int_x^{\infty} y^n g(y)dy$ and $s(x) = 1 - F(x)$.
This implies that,

$$\mathbb{E}(X^2|X > x_q) = \frac{1}{1-q}\left\{\sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}(1+\sigma)L_k \times \int_{VaR_q}^{\infty} x^2 f(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+t+m}dx + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}L_{k+1} \times \int_{VaR_q}^{\infty} x^2 f(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+1+t+m}dx\right\}. \quad (24)$$

Substituting equation (24) into equation (23) completes the proof.

### 3.10 Tail Variance Premium

The TVP is another important measure that plays an important role in insurance sciences. It is vital in determining the premium for a risk.

**Proposition 9.** For $0 < \delta < 1$ , the $TVPq(X)$ for the SFL distribution is given by

$$TVP_q = \frac{1}{1-q}\left[\sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}(1+\sigma)L_k J_q + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}L_{k+1}J^*_q\right] + \delta\left[\frac{1}{1-q}\left[\sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}(1+\sigma)L_k B_q + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}L_{k+1}B^*_q\right] - \left[\frac{1}{1-q}\left\{\sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}(1+\sigma)L_k J_q \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}L_{k+1}J^*_q\right\}\right]^2\right]. \quad (25)$$

**Proof.** From definition,

$$TVP_q(X) = TVaR_q + \delta TV_q. \quad (26)$$

Substituting equation (21) and equation (22) in equation (26) completes the proof.

### 3.11 Mean Deviation

**Proposition 10.** The mean deviation of the SFL family of distribution is given by

$$\delta_\mu = 2\mu F(\mu) - 2\mu + 2\left[\left\{\sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}(1+\sigma)L_k D_\mu + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi^*_{ijk}L_{k+1}U_\mu\right\}\right], \quad (27)$$

where $D_\mu = \int_\mu^\infty xf(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+t+m}dx$ and $U_\mu = \int_\mu^\infty xf(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+1+t+m}dx$.

**Proof.** By definition

$$\delta_\mu = \int_0^\infty |x-\mu| f(x)dx = 2\mu F(\mu) - 2\mu + 2\tau_1(\mu).$$

This implies that,

$$\delta_\mu = 2\mu F(\mu) - 2\mu + 2\left[\left\{\sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi_{ijk}^{*}(1+\sigma)L_k\int_\mu^\infty xf(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+t+m}dx + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi_{ijk}^{*}L_{k+1}\int_\mu^\infty xf(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+1+t+m}dx\right\}\right].$$

### 3.12 Median Deviation

**Proposition 11.** The median deviation of the SFL family of distribution is given by

$$\delta_M = 2\left[\left\{\sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi_{ijk}^{*}(1+\sigma)L_kA_M + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi_{ijk}^{*}L_{k+1}A_M^{*}\right\}\right] - \mu, \quad (28)$$

where $A_M = \int_M^\infty xf(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+t+m}dx$ and $A_M^{*} = \int_M^\infty xf(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+1+t+m}dx$. **Proof.** By definition

$$\delta_M = \int_0^\infty |x-M| f(x)dx = 2\tau_1(M) - \mu.$$

That is,

$$\delta_M = 2\left[\left\{\sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi_{ijk}^{*}(1+\sigma)L_k \times \int_M^\infty xf(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+t+m}dx + \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\sum_{t=0}^{j}\varpi_{ijk}^{*}L_{k+1} \times \int_M^\infty xf(x;\boldsymbol{\omega})F(x;\boldsymbol{\omega})^{k+1+t+m}dx\right\}\right] - \mu.$$

### 3.13 Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a sample of size $n$ from the SFL family of distributions and $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the order statistics of the sample. The PDF of the $i^{th}$ order statistics $g_{i:n}(x)$ is defined as

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}[G(x)]^{i-1}[1-G(x)]^{n-1}g(x). \quad (29)$$

Using binomial series expansion, we have

$$[1-G(x)]^{n-1} = \sum_{r=0}^{n-i}(-1)^r\binom{n-i}{r}[G(x)]^r. \quad (30)$$

That is, equation (29) becomes

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}g(x)\sum_{r=0}^{n-i}(-1)^r\binom{n-i}{r}[G(x)]^{r+i-1}. \quad (31)$$

Substituting equation (3) and equation (4) in equation (31), we get the $i^{th}$ order statistics as

$$g_{i:n}(x) = \frac{\pi\sigma f(x;\boldsymbol{\omega})[1+\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))]\cos\left[\frac{\pi}{2}\left(1-\frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))}\right)\right]n!}{2[\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))]^2(i-1)!(n-i)!} \times \sum_{r=0}^{n-i}(-1)^r\binom{n-i}{r}\left[\sin\left[\frac{\pi}{2}\left(1-\frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))}\right)\right]\right]^{r+i-1}. \quad (32)$$

The PDF of the first order statistics is defined as

$$g_{1:n}(x) = n[1-G(x)]^{n-1}g(x). \quad (33)$$

Substituting equation (3) and equation (4) in equation (33), we get the PDF of the first order statistics as

$$g_{1:n}(x) = n\left[1-\sin\left[\frac{\pi}{2}\left(1-\frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))}\right)\right]\right]^{n-1} \times \frac{\pi}{2}\left[\frac{\sigma f(x;\boldsymbol{\omega})[1+\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))]}{[\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))]^2}\right] \times \cos\left[\frac{\pi}{2}\left(1-\frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))}\right)\right]. \quad (34)$$

Also, the PDF of the $n^{th}$ order statistics is defined as

$$g_{n:n}(x) = n[G(x)]^{n-1}g(x). \quad (35)$$

Substituting equation (3) and equation (4) in equation (35), we get the PDF of the $n^{th}$ order statistics as

$$g_{n:n}(x) = n\left[\sin\left[\frac{\pi}{2}\left(1-\frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))}\right)\right]\right]^{n-1} \times \frac{\pi}{2}\left[\frac{\sigma f(x;\boldsymbol{\omega})[1+\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))]}{[\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))]^2}\right] \times \cos\left[\frac{\pi}{2}\left(1-\frac{\sigma\bar{F}(x;\boldsymbol{\omega})}{\sigma-\log(\bar{F}(x;\boldsymbol{\omega}))}\right)\right]. \quad (36)$$

## 4 Parameter Estimation

In this section, the unknown parameters of the SFL family of distributions were estimated using the maximum likelihood estimation (MLE) technique.

### 4.1 Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ are $n$ random sample from the SFL family of distributions. Therefore, the log-likelihood function which is a $p \times 1$ parameter vector $\Lambda = (\sigma, \boldsymbol{\omega})^T$ is given by

$$\begin{aligned}\ell = n\left(\log\left(\frac{\pi}{2}\right)\right) + n\log(\sigma) + \sum_{i=1}^{n}\log f(x_i;\boldsymbol{\omega}) \\ + \sum_{i=1}^{n}\log\left[1+\sigma-\log\left(1-\bar{F}(x_i;\boldsymbol{\omega})\right)\right] \\ -2\sum_{i=1}^{n}\log\left[\sigma-\log\left(1-\bar{F}(x_i;\boldsymbol{\omega})\right)\right] \\ + \sum_{i=1}^{n}\log\cos\left[\frac{\pi}{2}\left(1-\frac{\sigma\bar{F}(x_i;\boldsymbol{\omega})}{\sigma-\log\left(\bar{F}(x_i;\boldsymbol{\omega})\right)}\right)\right]. \quad (37)\end{aligned}$$

The log-likelihood function in equation (37) is differentiated with respect to each parameter to obtain the score function, $U(\Lambda) = \left(\frac{\partial \ell}{\partial \sigma}, \frac{\partial \ell}{\partial \boldsymbol{\omega}}\right)^T$. Hence,

$$\begin{aligned}\frac{\partial \ell}{\partial \sigma} = \frac{n}{\sigma} + \sum_{i=1}^{n}\frac{1}{\left[1+\sigma-\log\left(\bar{F}(x_i;\boldsymbol{\omega})\right)\right]} \\ -2\sum_{i=1}^{n}\frac{1}{\left[\sigma-\log\left(\bar{F}(x_i;\boldsymbol{\omega})\right)\right]} \\ -\frac{\pi}{2}\sum_{i=1}^{n}\left(\frac{\sigma\bar{F}(x_i;\boldsymbol{\omega})}{\sigma-\log\bar{F}(x_i;\boldsymbol{\omega})} - \frac{\bar{F}(x_i;\boldsymbol{\omega})}{\sigma-\log\bar{F}(x_i;\boldsymbol{\omega})}\right) \\ \times\tan\left[\frac{\pi}{2}\left(1-\frac{\sigma\bar{F}(x_i;\boldsymbol{\omega})}{\sigma-\log\left(\bar{F}(x_i;\boldsymbol{\omega})\right)}\right)\right]. \quad (38)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \ell}{\partial \boldsymbol{\omega}} = \sum_{i=1}^{n}\frac{f'(x_i;\boldsymbol{\omega})}{f(x_i;\boldsymbol{\omega})} + \sum_{i=1}^{n}\frac{F'(x_i;\boldsymbol{\omega})}{\left[1+\sigma-\log\left(\bar{F}(x_i;\boldsymbol{\omega})\right)\right]} \\ -\frac{\pi}{2}\sum_{i=1}^{n}\left(\frac{\sigma F'(x_i;\boldsymbol{\omega})}{\sigma-\log\bar{F}(x_i;\boldsymbol{\omega})} - \frac{\sigma F'(x_i;\boldsymbol{\omega})}{\sigma-\log\bar{F}(x_i;\boldsymbol{\omega})}\right) \\ \times\tan\left[\frac{\pi}{2}\left(1-\frac{\sigma\bar{F}(x_i;\boldsymbol{\omega})}{\sigma-\log\left(\bar{F}(x_i;\boldsymbol{\omega})\right)}\right)\right]. \quad (39)\end{aligned}$$

where $F'(x_i;\boldsymbol{\omega}) = \frac{\partial F(x_i;\boldsymbol{\omega})}{\partial \boldsymbol{\omega}}$ and $f'(x_i;\boldsymbol{\omega}) = \frac{\partial f(x_i;\boldsymbol{\omega})}{\partial \boldsymbol{\omega}}$.

## 5 Special Distributions

In this section, three special distributions are presented. These are the sine Weibull Loss (SWL) , sine Fréchet Loss (SFrL) and sine Burr-XII Loss (SBXIIL) distributions.

### 5.1 Sine Weibull Loss Distribution

If we consider the Weibull distribution as the baseline distribution with CDF and PDF defined as $F(x) = 1 - e^{-\alpha x^{\beta}}$ and $f(x) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}$ for $x > 0$ and $\alpha, \beta > 0$, respectively, we obtain the SWL distribution. From equation (3), the CDF of the SWL distribution is given by

$$G(x;\alpha,\beta,\sigma) = \sin\left[\frac{\pi}{2}\left(1-\frac{\sigma e^{-\alpha x^{\beta}}}{\sigma+\alpha x^{\beta}}\right)\right], \; x>0, \alpha,\beta,\sigma>0, \quad (40)$$

where $\beta$ and $\sigma$ are shape parameters and $\alpha$ is a scale parameter.
The related PDF is given by

$$\begin{aligned}g(x;\alpha,\beta,\sigma) = \frac{\pi\alpha\beta\sigma}{2}\left[\frac{x^{\beta-1}e^{-\alpha x^{\beta}}(1+\sigma+\alpha x^{\beta})}{(\sigma+\alpha x^{\beta})^2}\right] \\ \times\cos\left[\frac{\pi}{2}\left(1-\frac{\sigma e^{-\alpha x^{\beta}}}{\sigma+\alpha x^{\beta}}\right)\right], \quad x>0. \quad (41)\end{aligned}$$

The corresponding hazard rate function is given by

$$h(x;\alpha,\beta,\sigma) = \frac{\pi\alpha\beta\sigma x^{\beta-1}e^{-\alpha x^{\beta}}(1+\sigma+\alpha x^{\beta})\cos\left[\frac{\pi}{2}\left(1-\frac{\sigma e^{-\alpha x^{\beta}}}{\sigma+\alpha x^{\beta}}\right)\right]}{2(\sigma+\alpha x^{\beta})^2\left[1-\sin\left[\frac{\pi}{2}\left(1-\frac{\sigma e^{-\alpha x^{\beta}}}{\sigma+\alpha x^{\beta}}\right)\right]\right]}, \quad x>0. \quad (42)$$

The plots of the density function of the SWL distribution is shown in Figure . The density function exhibits decreasing, right skewed, approximately symmetric and symmetric shapes.

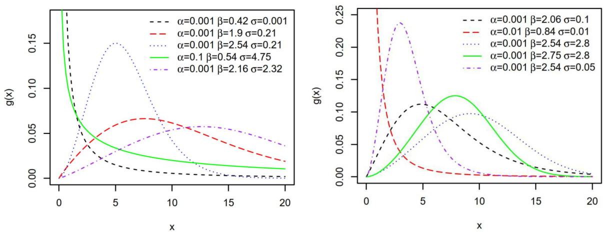


**Fig. 1: Different Plots for the density function of the SWL distribution**

From Figure 2, the hazard rate function of the SWL distribution exhibits increasing, decreasing, increasing-constant-increasing, reversed-J and bathtub shapes.

### 5.2 Sine Fréchet Loss Distribution

Consider the Fréchet distribution as the baseline distribution with CDF and PDF defined as $F(x) = e^{-\alpha x^{-\beta}}$ and $f(x) = \beta\alpha x^{-(\beta+1)}e^{-\alpha x^{-\beta}}$ for $x > 0$ and $\alpha, \beta > 0$,

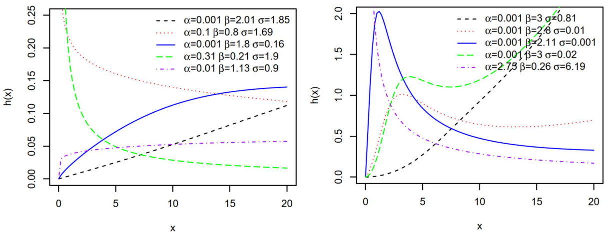


**Fig. 2: Different Plots for the hazard rate function of the SWL distribution**

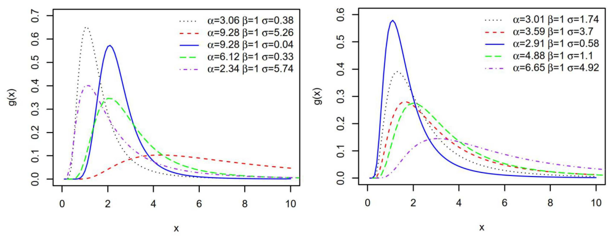


**Fig. 3: Different Plots for the density function of the SFrL distribution**

respectively, we obtain the SFrL distribution. Using equation (3), the CDF of the SFrL is given by

$$G(x;\alpha,\beta,\sigma) = \sin\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1-e^{-\alpha x^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)}\right)\right], \quad x>0,\alpha,\beta,\sigma>0, \tag{43}$$

where $\alpha$ is a scale parameter, $\beta$ and $\sigma>0$ are shape parameters.

The related PDF is given by

$$g(x;\alpha,\beta,\sigma) = \frac{\pi\alpha\beta\sigma}{2}\left[\frac{x^{-(\beta+1)}e^{-\alpha x^{-\beta}}\left[1+\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right]}{\left[\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right]^2}\right] \times\cos\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1-e^{-\alpha x^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)}\right)\right], \quad x>0. \tag{44}$$

The hazard rate function is given by

$$h(x;\alpha,\beta,\sigma) = \frac{\pi\alpha\beta\sigma\left[x^{-(\beta+1)}e^{-\alpha x^{-\beta}}\left[1+\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right]\right]\cos\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1-e^{-\alpha x^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)}\right)\right]}{2\left[\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)\right]^2\left[1-\sin\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1-e^{-\alpha x^{-\beta}}\right)}{\sigma-\log\left(1-e^{-\alpha x^{-\beta}}\right)}\right)\right]\right]}, \quad x>0. \tag{45}$$

The plots of the density function of the SFrL in Figure 3 exhibits right skewed and reversed-J shapes with varying degree of kurtosis.
The plots of the hazard rate function as shown in Figure 4, exhibits decreasing, reversed-J and upside down bathtub shapes.

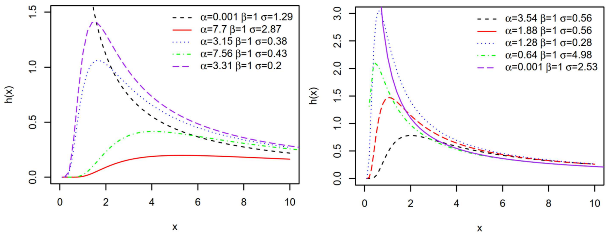


**Fig. 4: Different Plots for the hazard rate function of the SFrL distribution**

### *5.3 Sine Burr XII Loss Distribution*

Consider the Burr XII distribution as the baseline distribution with CDF and PDF defined as $F(x) = 1-\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)^{-\tau}$ and $f(x) = \tau\gamma\alpha^{-\gamma}x^{\gamma-1}\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)^{-(\tau+1)}$ for $x>0$ and $\alpha,\gamma,\tau>0$, respectively, we obtain the SBXIIL distribution. Using equation (3), the CDF of the SBXIIL is given by

$$G(x;\alpha,\gamma,\tau,\sigma) = \sin\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)^{-\tau}}{\sigma+\tau\log\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)}\right)\right], \quad x>0,\alpha,\gamma,\tau,\sigma>0, \tag{46}$$

where $\alpha$ is a scale parameter, $\gamma,\tau$ and $\sigma$ are shape parameters.
The PDF is given by

$$g(x;\alpha,\gamma,\tau,\sigma) = \frac{\pi}{2}\left[\frac{\sigma\tau\gamma\alpha^{-\gamma}x^{\gamma-1}\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)^{-(\tau+1)}\left[1+\sigma+\tau\log\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)\right]}{\left[\sigma+\tau\log\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)\right]^2}\right] \times\cos\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)^{-\tau}}{\sigma+\tau\log\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)}\right)\right], x>0. \tag{47}$$

The hazard rate function is given by

$$h(x;\alpha,\gamma,\tau,\sigma) = \frac{\pi\sigma\tau\gamma\alpha^{-\gamma}\left[x^{\gamma-1}\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)^{-(\tau+1)}\left[1+\sigma+\tau\log\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)\right]\right]\cos\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)^{-\tau}}{\sigma+\tau\log\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)}\right)\right]}{2\left[\sigma+\tau\log\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)\right]^2\left[1-\sin\left[\frac{\pi}{2}\left(1-\frac{\sigma\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)^{-\tau}}{\sigma+\tau\log\left(1+\left(\frac{x}{\alpha}\right)^{\gamma}\right)}\right)\right]\right]}, x>0. \tag{48}$$

The density plots of the SBXIIL distribution in Figure 5 exhibits right skewed, decreasing, reversed-J shapes with some level of peakedness.

From Figure 6, the hazard rate function plots of the SBXIIL distribution shows increasing-constant-decreasing, decreasing, bathtub, upside down bathtub and reversed-J shapes.

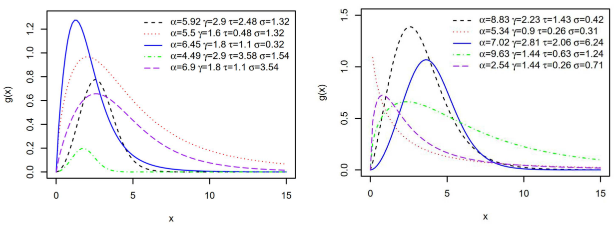


**Fig. 5: Different Plots for the density function of the SBXIIL distribution**

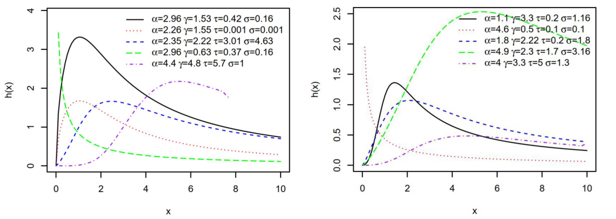


**Fig. 6: Different Plots for the hazard rate function of the SBXIIL distribution**

## 6 Monte Carlo Simulation

In this section, the simulation results are presented in examining the properties of the maximum likelihood estimators for the parameters of the SWL distribution. Five different combinations of the parameter values of these distributions are specified and their quantile functions used in generating four different random samples of size, $n = 50, 100, 150, 200$. The simulations are replicated for $N = 5000$ times. The properties of the estimators are investigated by computing average bias (AB) and root mean square error (RMSE) for each of the parameters. The simulation steps are as follows:

i. Specify the values of the parameters and the sample size $n$.
ii. Generate random samples of size $n = 50, 100, 150, 200$ from the SWL the quantiles.
iii. Find the maximum likelihood estimates for the parameters.
iv. Repeat steps ii-iii for 5000 times.
v. Calculate the average bias (AB) and root mean square error (RMSE) for the parameters of the distributions using the following formulas: $AB = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{\xi}_i - \xi\right)$ and $RMSE = \sqrt{\frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{\xi}_i - \xi\right)^2}$ for $\xi = (\alpha, \beta, \sigma)$, respectively.

Table 1 shows the simulation results for the SWL distribution using parameter values I: $\alpha = 0.5, \beta = 0.5, \sigma = 0.8$, II: $\alpha = 0.3, \beta = 0.3, \sigma = 0.3$, III: $\alpha = 0.4, \beta = 0.7, \sigma = 0.3$, IV: $\alpha = 0.6, \beta = 0.4, \sigma = 0.6$, and V: $\alpha = 0.1, \beta = 0.9, \sigma = 0.5$. It can be observed that, as the sample size increase, the AB and RMSE for the estimators of the parameters decreases. This shows that the estimators of the SWL distribution are consistent.

## 7 Applications

This section illustrates the usefulness and flexibility of the SFL distributions using insurance loss dataset. The performance of the SWL, SFrL, and SBXIIL distributions are compared with other loss distributions. The performance of the distributions about providing reasonable parametric fit to the dataset are compared using the Akaike information criterion (AIC), corrected Akaike information criterion (AICc), Bayesian information criterion (BIC), Cramér-Von Misses ($W^*$), Anderson-Darling ($A^*$) and Kolmogorove-Smirnov (K-S) statistics. The distribution with the least of these measures provides a reasonable fit to the dataset. The fit for the SWL, SFrL, and SBXIIL are compared with other heavy-tailed distributions, including the 2-parameters Weibull, 2-parameters Burr XII (B-XII), Weibull-Loss (W-Loss), 2-parameter Burr III (BIII), Frechét, Weibull-Lomax, sine inverse Weibull (SIW), Lomax, Dagum, exponentiated Weibull (EW) and power Lomax distributions. The distribution functions of the competitive models are:

1. Weibull

$$F(x) = 1 - e^{-\gamma x^{\alpha}}, \quad x \geq 0, \alpha, \gamma > 0.$$

2. B-XII

$$F(x) = 1 - (1 + x^{c})^{-k}, \quad x \geq 0, c, k > 0.$$

3. W-Loss

$$F(x) = 1 - \frac{\sigma e^{-\gamma x^{\alpha}}}{\sigma + \gamma x^{\gamma}}, \quad x \geq 0, \sigma, \alpha > 0.$$

4. BIII

$$F(x) = \left(1 + x^{-c}\right)^{-k}, \quad x \geq 0, c, k > 0.$$

5. Frechét

$$F(x) = e^{-\alpha x^{-\beta}}, \quad x \geq 0, \alpha, \beta > 0.$$

6. Weibull-Lomax

$$F(x) = 1 - e^{-a\left(\left(1+\frac{x}{\beta}\right)^{\alpha} - 1\right)^{b}}, \quad x \geq 0, a, b, \beta, \alpha > 0.$$

7. SIW

$$F(x) = \sin\left(\frac{\pi}{2} e^{-\alpha x^{-\theta}}\right), \quad x \geq 0, \alpha, \theta > 0.$$

**Table 1: Monte Carlo Simulation Results: AB and RMSE for the Parameters of the SWL distribution**

| n | Parameter value | | | AB | | | RMSE | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\alpha$ | $\beta$ | $\sigma$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\sigma}$ | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\sigma}$ |
| 50 | 0.5 | 0.5 | 0.8 | 0.225 | 0.062 | 0.843 | 0.067 | 0.007 | 0.852 |
| 100 | 0.5 | 0.5 | 0.8 | 0.201 | 0.047 | 0.788 | 0.052 | 0.004 | 0.773 |
| 150 | 0.5 | 0.5 | 0.8 | 0.183 | 0.039 | 0.750 | 0.043 | 0.003 | 0.729 |
| 200 | 0.5 | 0.5 | 0.8 | 0.176 | 0.036 | 0.722 | 0.040 | 0.002 | 0.693 |
| 50 | 0.3 | 0.3 | 0.2 | 0.316 | 0.035 | 0.681 | 0.162 | 0.003 | 1.033 |
| 100 | 0.3 | 0.3 | 0.2 | 0.253 | 0.029 | 0.504 | 0.105 | 0.002 | 0.676 |
| 150 | 0.3 | 0.3 | 0.2 | 0.222 | 0.025 | 0.417 | 0.083 | 0.001 | 0.518 |
| 200 | 0.3 | 0.3 | 0.2 | 0.190 | 0.021 | 0.331 | 0.064 | 0.001 | 0.366 |
| 50 | 0.4 | 0.7 | 0.3 | 0.332 | 0.083 | 0.782 | 0.158 | 0.011 | 1.129 |
| 100 | 0.4 | 0.7 | 0.3 | 0.273 | 0.067 | 0.616 | 0.108 | 0.007 | 0.817 |
| 150 | 0.4 | 0.7 | 0.3 | 0.247 | 0.058 | 0.540 | 0.090 | 0.005 | 0.672 |
| 200 | 0.4 | 0.7 | 0.3 | 0.222 | 0.051 | 0.478 | 0.076 | 0.004 | 0.564 |
| 50 | 0.6 | 0.4 | 0.6 | 0.316 | 0.050 | 0.833 | 0.130 | 0.004 | 0.961 |
| 100 | 0.6 | 0.4 | 0.6 | 0.283 | 0.038 | 0.761 | 0.103 | 0.002 | 0.853 |
| 150 | 0.6 | 0.4 | 0.6 | 0.257 | 0.033 | 0.701 | 0.085 | 0.002 | 0.760 |
| 200 | 0.6 | 0.4 | 0.6 | 0.235 | 0.029 | 0.663 | 0.073 | 0.001 | 0.709 |
| 50 | 0.1 | 0.9 | 0.5 | 0.068 | 0.109 | 0.867 | 0.006 | 0.020 | 1.103 |
| 100 | 0.1 | 0.9 | 0.5 | 0.062 | 0.084 | 0.756 | 0.005 | 0.012 | 0.916 |
| 150 | 0.1 | 0.9 | 0.5 | 0.056 | 0.073 | 0.657 | 0.004 | 0.008 | 0.752 |
| 200 | 0.1 | 0.9 | 0.5 | 0.053 | 0.067 | 0.615 | 0.004 | 0.007 | 0.684 |

8. Lomax

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}, \quad x \geq 0, \alpha, \lambda > 0.$$

9. Dagum

$$F(x) = (1 + \lambda x^{-\alpha})^{-\beta}, \quad x \geq 0, \alpha, \beta, \lambda > 0.$$

10. EW

$$F(x) = \left(1 - e^{-\gamma x^{\alpha}}\right)^{\lambda}, \quad x \geq 0, \lambda, \alpha, \gamma > 0.$$

11. Power Lomax

$$F(x) = 1 - \lambda^{\alpha}\left(x^{\beta} + \lambda\right)^{-\alpha}, \quad x \geq 0, \alpha, \beta, \lambda > 0.$$

## *7.1 Application 1: U.S Indemnity Losses*

The first dataset consists of 1,500 U.S indemnity losses; general liability claims indemnity payment in thousands of U.S dollars. This dataset is reported in CASdatasets package of R software.

Table 2 shows the descriptive statistics and Figure 7 shows the histogram, boxplot and kernel density plot of the data. It can be seen that, the data is right-skewed and leptokurtic with the histogram, boxplot and kernel density plot depicting a typical feature of insurance loss data. That is, the histogram and kernel density plot shows that the data is right skewed and heavy tailed. The boxplot shows the presence of outliers in the U.S indemnity loss dataset.

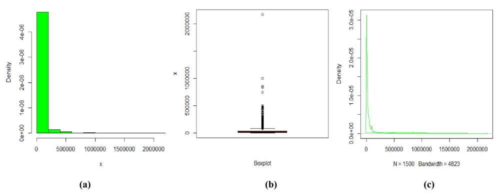

**Fig. 7: U.S Indemnity Losses: histogram (a), boxplot (b) and kernel density (c) plots**

To ascertain the nature of the hazard rate of a given

**Table 2: Descriptive Statistics of U.S Indemnity Losses**

| No. of Claims | Mean | Std. | Skewness | Kurtosis | Min. | Max. |
|---|---|---|---|---|---|---|
| 1,500 | 41208.000 | 102747.700 | 9.164 | 145.172 | 10.000 | 2,173,595.000 |

dataset, the TTT plot is employed. Figure 8 shows the TTT-transform plot for the U.S indemnity losses data. From the plot, there is evidence of a decreasing hazard rate function because the curve is convex below the 45 degree line.

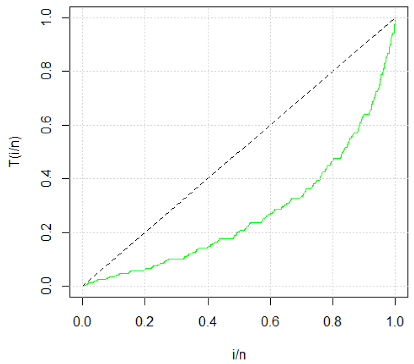


**Fig. 8: TTT-transform plot for U.S Indemnity Losses**

Table 3 shows the maximum likelihood estimates for the parameters of the fitted distributions with their corresponding errors in brackets. The parameters of the SWL, SFrL, W-Loss, Weibull, Lomax, Frechét, power Lomax, EW, SIW, BIII, Weibull Lomax and Dagum distributions are all significant at the 5% level. The SBXIIL distribution also had its parameters significant at the 5% level with the exception of $\sigma$ and $\gamma$ which are significant at 10%. All the parameters of the BXII distribution are significant at the 10% level.
Table 4 shows the goodness-of-fit and information criteria for the fitted distributions. The SWL was compared with other competing distributions. It can be seen that, the SWL provides a better fit to the dataset since it has smaller AIC, AICc, BIC, K-S, $A^*$, $W^*$ and $-2l$ values compared with the rest of the competitive distributions. Figure 9 shows the plots of the empirical density, the fitted density, the empirical CDF and the CDF of the fitted distributions. It is evident that, the SWL distribution also provide reasonable fit to the data among the other distributions.

## *7.2 Application 2: Automobile Insurance Claims*

The second dataset consists of 6,773 amounts (in U.S. dollars) paid, by a large midwestern (U.S) insurance company, to settle and close claims for private passengers

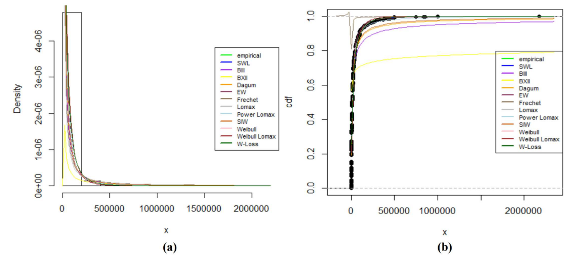


**Fig. 9: Empirical and fitted density (a) and CDF (b) plots of U.S Indemnity Losses data**

automobile policies. This dataset is available in CASdatasets package of R software.
Table 5 shows the descriptive statistics while Figure 10 shows the histogram, boxplot and kernel density plot of the automobile insurance claims data. It can be seen that, the losses are right skewed and leptokurtic, with a long right tail. Also, the histogram, boxplot and kernel density plot shows a typical characteristics of insurance loss data. That is, the histogram and kernel density plot shows that the data is right skewed and heavy tailed.
The boxplot is used to detect the presence of outliers which is a common feature of insurance loss data which is clearly exhibited by the boxplot in Figure 10.

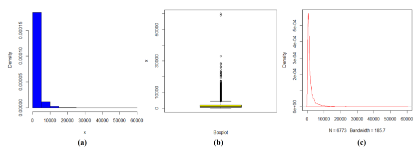

**Fig. 10: Automobile Insurance Claims: histogram (a), boxplot (b) and kernel density (c) plots**

Figure 11 shows the TTT-transform plot for the automobile insurance claims data so as to ascertain the nature of the failure rate. The data exhibits a unimodal (upside down bathtub ) hazard rate since it is first concave above the 45 degree line and then followed by a convex shape below.
Table 6 shows the maximum likelihood estimates for the parameters of the fitted distributions with their corresponding errors in brackets. The parameters of the SWL, SFrL, W-Loss, Weibull, Lomax, Frechét, power Lomax, EW, SIW, BIII, Weibull Lomax and Dagum distributions were all significant at the 5% level. The

**Table 3: Maximum likelihood estimates of the parameters and standard errors for U.S indemnity losses data**

| Model | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\lambda}$ | $\hat{\sigma}$ | $\hat{\gamma}$ | $\hat{\tau}$ | $\hat{\theta}$ |
|---|---|---|---|---|---|---|---|
| SWL | 0.003 | 0.524 | | 8.185 | | | |
| | (0.003) | (0.010) | | (0.005) | | | |
| SFrL | 4257.187 | 0.443 | | 406.466 | | | |
| | (0.002) | (0.009) | | (0.001) | | | |
| SBXIIL | 29.791 | | | 42.621 | 38.619 | 0.002 | |
| | (0.003) | | | (0.001) | (0.006) | (0.004) | |
| W-Loss | 0.556 | | | 41.265 | 0.004 | | |
| | (0.012) | | | (0.004) | (0.002) | | |
| Weibull | 0.568 | | | | 0.003 | | |
| | (0.012) | | | | (0.006) | | |
| Lomax | 1.522 | | 1406.423 | | | | |
| | (0.048) | | (6.389) | | | | |
| Frechét | 1476.849 | 0.370 | | | | | |
| | (6.414) | (0.010) | | | | | |
| Power Lomax | 1.839 | 0.754 | 2439.771 | | | | |
| | (0.189) | (0.014) | (0.002) | | | | |
| EW | 0.267 | | 8.565 | | 0.208 | | |
| | (0.020) | | (1.856) | | (0.056) | | |
| SIW | 68.989 | | | | | | 0.448 |
| | (4.725) | | | | | | (0.008) |
| | $\hat{c}$ | $\hat{k}$ | | | | | |
| BIII | 0.570 | 132.024 | | | | | |
| | (0.009) | (1.019) | | | | | |
| | $\hat{c}$ | $\hat{k}$ | | | | | |
| B-XII | 0.068 | 1.621 | | | | | |
| | (0.050) | (1.222) | | | | | |
| | | | $\hat{a}$ | $\hat{b}$ | | | |
| Weibull-Lomax | 6.070 | 39.038 | 0.016 | 0.104 | | | |
| | (0.002) | (0.005) | (0.001) | (0.002) | | | |
| Dagum | 0.737 | 3.031 | 240.971 | | | | |
| | (0.008) | (0.571) | (0.001) | | | | |

SBXIIL distribution also had its parameters significant at the 5% level with the exception of $\gamma$ which was significant at 10%. The BXII distribution had all its parameters significant at the 10% level.

Table 7 shows the goodness-of-fit and information criteria for the fitted distributions. The SFrL distribution was compared with other competing distributions. It can be seen that the SFrL distribution which is one of the proposed model provides a better fit to the dataset since it has the least AIC, AICc, BIC, K-S, $A^*$, $W^*$ and $-2l$ values compared with the other competitive distributions. Figure 12 shows the plots of the empirical density, the fitted density, the empirical CDF and the CDF of the fitted distributions. It is evident that, the SFrL distribution also provide reasonable fit to the data among the other distributions.

**Table 4: Goodness-of-fit and Information Criteria of U.S Indemnity Losses data**

| Model | $-2l$ | AIC | AICc | BIC | $W^*$ | $A^*$ | K-S |
|---|---|---|---|---|---|---|---|
| SWL | 33849.650 | 33855.650 | 33855.670 | 33871.590 | 0.144 | 1.033 | 0.065 |
| W-Loss | 34082.360 | 34088.360 | 34088.370 | 34104.300 | 11.921 | 12.068 | 0.110 |
| Weibull | 34069.280 | 34073.280 | 34073.280 | 34083.900 | 2.004 | 12.573 | 0.103 |
| Lomax | 34732.540 | 34736.540 | 34736.540 | 34747.160 | 4.001 | 15.666 | 0.656 |
| Frechét | 34128.540 | 34132.540 | 34132.550 | 34143.170 | 4.171 | 26.807 | 0.650 |
| Power-Lomax | 35946.790 | 35952.790 | 35952.810 | 35968.731 | 1.376 | 9.480 | 0.104 |
| EW | 34085.010 | 34151.020 | 34152.030 | 34196.960 | 2.001 | 12.499 | 0.100 |
| SIW | 34010.380 | 34014.380 | 34014.390 | 34025.010 | 1.210 | 6.497 | 0.067 |
| BIII | 34211.650 | 34215.650 | 34215.660 | 34226.280 | 2.617 | 16.682 | 0.086 |
| B-XII | 37834.100 | 37838.100 | 37848.110 | 37848.730 | 1.581 | 11.882 | 0.474 |
| Weibull-Lomax | 34056.000 | 34064.000 | 34064.030 | 34085.260 | 2.413 | 15.042 | 0.088 |
| Dagum | 33918.860 | 33955.860 | 33855.880 | 33881.810 | 0.824 | 6.601 | 0.069 |

**Table 5: Descriptive Statistics of the Automobile Insurance Claims**

| No. of Claims | Mean | Std. | Skewness | Kurtosis | Min. | Max. |
|---|---|---|---|---|---|---|
| 6,773 | 1853.000 | 2,646.909 | 6.236 | 87,278 | 9.500 | 60,000.000 |

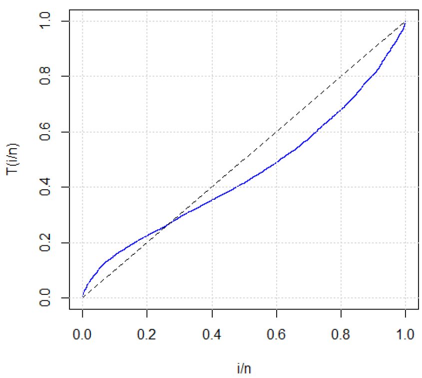


**Fig. 11: TTT-transform plot for Automobile Insurance Claims**

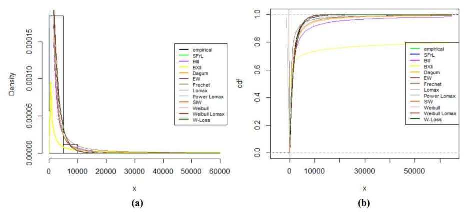


**Fig. 12: Empirical and fitted density (a) and CDF (b) plots of Automobile Insurance Claims data**

## 8 Conclusion

In this article, we have proposed a new heavy-tailed family of distributions known as sine F-Loss family of distributions, an extension of the F-Loss family of distributions. The mathematical properties and maximum likelihood estimators of the family are studied. Three special distributions, namely the sine Weibull loss, sine Fréchet loss, and sine B-XII loss distributions are proposed. Simulations are carried out to evaluate the behavior of the parameters of the proposed distributions. The densities exhibits different kinds of such as decreasing, right skewed, approximately symmetric, symmetric, and reversed-J shapes. The hazard rate functions show different kinds of increasing, decreasing, increasing-constant-decreasing, reversed-J, bathtub, and upside bathtub shapes. The usefulness of the proposed distributions is analyzed with two insurance loss datasets and compared with eleven other well-known loss distributions. From the applications, the sine Weibull loss distribution provides the best parametric fit for the U.S. indemnity losses dataset, whereas the sine Fréchet loss gives the best parametric fit for the automobile insurance claims dataset. The proposed models are reasonably good compared with the competitors. We hope the proposed models will attract broader application in the actuarial sciences and other related fields.

**Table 6: Maximum likelihood estimates of the parameters and standard errors for automobile insurance claims data**

| Model | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\lambda}$ | $\hat{\sigma}$ | $\hat{\gamma}$ | $\hat{\tau}$ | $\hat{\theta}$ |
|---|---|---|---|---|---|---|---|
| SWL | 0.001 | 0.718 | | 0.834 | | | |
| | (0.003) | (0.015) | | (0.073) | | | |
| SFrL | 1102.972 | 0.689 | | 282.505 | | | |
| | (1.619) | (0.006) | | (0.260) | | | |
| SBXIIL | 33.092 | | | 2.843 | 46.632 | 0.003 | |
| | (0.004) | | | (0.161) | (0.071) | (0.002) | |
| W-Loss | 0.858 | | | 1.038 | 0.002 | | |
| | (0.013) | | | (0.001) | (0.008) | | |
| Weibull | 0.806 | | | | 0.003 | | |
| | (0.007) | | | | (0.002) | | |
| Lomax | 1.957 | | 2137.220 | | | | |
| | (0.053) | | (73.326) | | | | |
| Frechét | 558.741 | 0.614 | | | | | |
| | (8.663) | (0.007) | | | | | |
| Power Lomax | 1.919 | 1.054 | 3194.496 | | | | |
| | (0.083) | (0.008) | (0.001) | | | | |
| EW | 0.378 | | 10.791 | | 0.200 | | |
| | (0.013) | | (1.155) | | (0.260) | | |
| SIW | 124.973 | | | | | | 0.690 |
| | (4.628) | | | | | | (0.006) |
| | $\hat{c}$ | $\hat{k}$ | | | | | |
| BIII | 0.875 | 276.011 | | | | | |
| | (0.007) | (1.175) | | | | | |
| | $\hat{c}$ | $\hat{k}$ | | | | | |
| B-XII | 0.068 | 0.870 | | | | | |
| | (0.028) | (0.870) | | | | | |
| | | | $\hat{a}$ | $\hat{b}$ | | | |
| Weibull-Lomax | 5.210 | 10.840 | 0.008 | 0.181 | | | |
| | (0.009) | (0.001) | (0.005) | (0.002) | | | |
| Dagum | 1.116 | 3.392 | 490.855 | | | | |
| | (0.005) | (0.090) | (0.006) | | | | |

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**Conflict of Interest**

The authors declare no conflict of interest.

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**Table 7: Goodness-of-fit and Information Criteria of Automobile Insurance Claims data**

| Model | $-2l$ | AIC | AICc | BIC | $W^*$ | $A^*$ | K-S |
|---|---|---|---|---|---|---|---|
| SFrL | 114368.800 | 114372.800 | 114372.800 | 114386.500 | 2.338 | 21.458 | 0.042 |
| W-Loss | 115520.900 | 115526.900 | 115526.900 | 115547.400 | 13.644 | 78.103 | 0.106 |
| Weibull | 115698.000 | 115702.000 | 115702.000 | 115715.600 | 12.324 | 70.464 | 0.123 |
| Lomax | 115378.900 | 115783.200 | 115783.200 | 115796.900 | 7.092 | 53.407 | 0.840 |
| Frechét | 115000.500 | 115006.500 | 115006.500 | 115026.900 | 12.596 | 84.836 | 0.274 |
| Power-Lomax | 115113.400 | 115119.400 | 115119.400 | 115139.800 | 6.848 | 35.836 | 0.106 |
| EW | 114845.900 | 114851.900 | 114851.900 | 114872.400 | 7.106 | 36.021 | 0.048 |
| SIW | 115003.000 | 115007.600 | 115007.600 | 115021.200 | 9.759 | 28.599 | 0.048 |
| BIII | 115937.700 | 115941.700 | 115941.700 | 115955.300 | 11.222 | 76.277 | 0.073 |
| B-XII | 134040.000 | 134044.000 | 134044.000 | 134057.600 | 8.564 | 29.948 | 0.482 |
| Weibull-Lomax | 115430.300 | 115438.300 | 115438.300 | 115465.600 | 15.684 | 89.878 | 0.079 |
| Dagum | 115198.200 | 115204.200 | 115204.200 | 115224.600 | 8.503 | 48.322 | 0.062 |


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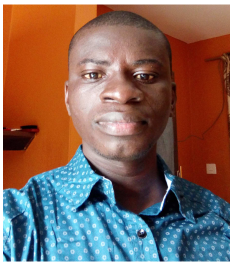

**John Abonongo** is a lecturer at the Statistics Department, School of Mathematical Sciences, C. K. Tedam University of Technology and Applied Sciences, Ghana. He has published several research articles in several reputable journals. His research areas: Financial Time Series, Distribution Theory, Stochastic Theory, Actuarial Mathematics and Applications.

**Joseph Ivivi Mwaniki** is an associate professor of Financial Mathematics and Statistics, University of Nairobi, Kenya. He is the Head of Financial and Actuarial Mathematics division-school of mathematics. He has several research articles published in very high reputable journals with over 10 years of experience in academic, research and supervision of postgraduate students. The research areas of professor Joseph Ivivi Mwaniki are probability modeling, option pricing, modeling returns, and computational methods.

**Jane Akinyi Aduda** is the program coordinator for Mathematics at the Pan African University, Institute of Basic Sciences, Technology and Innovation, Kenya. She has a Ph.D. in Financial Mathematics. Jane Akinyi Aduda is a lecturer at Jomo Kenyatta University of Agriculture and Technology, Kenya, and has several research articles in several reputable journals. Her research areas: Financial Mathematics, Statistics, Financial Econometrics, and Health Economics.