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Existence of Solution for a New Class of Fractional Differential Equations

Hajar Sbai, Meryeme El Harrak and Ahmed Hajji∗

LabMIA-SI, Department of Mathematics, Faculty of Sciences, Mohammed V University in Rabat, Rabat, Morocco

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Abstract: In this paper, we investigate the global existence and uniqueness of a solution to a specific class of fractional differential equations, noted by Φ-fractional differential equation with nonlocal condition, which Atangana–Baleanu Caputo fractional derivative operators and compare the two studies. At first, we need to introduce a new topology in $C([0,+\infty[,E)$ with E is a Banach space. Then, we provide the necessary hypothesis on Φ and $\|\Phi\|_\infty$ for each problem applying Banach's fixed point theorem. Moreover, we give some illustrative examples which exhibited the applicability of the founded hypotheses.

Keywords: Fractional differential equation, Caputo fractional derivative, Atangana–Baleanu Caputo fractional derivative, Banach space.

1 Introduction, motivation and preliminaries

Fractional differential equations applied and used in many fields such as physics, economics, engineering, chemistry and biology [1,2,3,4,5,6,7,8]. There exist many papers on the existence and uniqueness of solution of fractional differential equations within finite intervals, see for instance [9,10,11,12,13,14,15]. For example, A. Keten, M. Yavuz and D. Baleanu [12] established the existence and uniqueness conditions for solutions for a nonlinear differential equation containing the Caputo–Fabrizio operator in Banach spaces

$$\begin{cases} {}^{CF}D_t^\alpha w(t) = Tw(t) + h(t,w(t)), \quad 0 \le t \le 1, \\ w(0) = \int_0^1 g(\xi)w(\xi)d\xi. \end{cases} \tag{1}$$

In [9], Bai studied the existence and uniqueness of a positive solution for a nonlocal boundary value problem of fractional differential equation

$$\begin{cases} D_{0+}^\alpha u(t) + f(t,u(t)) = 0, \quad 0 \le t \le 1, \\ u(0) = 0, \quad u(1) = \beta u(\eta),\ 0 < \eta < 1. \end{cases} \tag{2}$$

Jarad et al. [16] and Syam [17] investigated the local existence and uniqueness of the general equation

$$\begin{cases} {}^{ABC}D_{0,t}^\alpha y(t) = f(t,y(t)), \quad 0 \le \alpha \le 1, \\ y(0) = y_0, \end{cases} \tag{3}$$

with ${}^{ABC}D_{0,t}^\alpha$ is the Atangana–Baleanu Caputo fractional derivative.

In the literature, few results have been treated the problem in unbounded domains. Arara et al. [18] have concerned with the existence of a bounded solution of a boundary value problem on an unbounded domain for fractional order differential inclusions involving the Caputo fractional derivative

$$\begin{cases} {}^{c}D_{0+}^\alpha u(t) = f(t,u(t)), \quad t \in J := [0,+\infty), \\ u(0) = u_0. \end{cases} \tag{4}$$

∗ Corresponding author e-mail: a-hajji@um5r.ac.ma

Hassouna, El Kinani and Ouhadan [19] have provided conditions of the global existence and uniqueness of the Atangana–Baleanu Caputo fractional differential equation

$$\begin{cases} {}^{ABC}D^{\alpha}_{0,t}y(t)+N(y(t))=g(t,y(t)),\ t\in[0,+\infty),\ (4.1) \\ y(0)=y_0, \qquad\qquad\qquad\qquad (4.2), \end{cases} \tag{5}$$

In the present paper, we investigate the global existence of a solution to the following class of fractional differential equations, noted by $\Phi-$fractional differential equations, with nonlocal condition on an infinite interval through using Banach's fixed point theorem.

$$\begin{cases} {}^{c}D^{\alpha}(y(t)+\sigma(t,y(t)))=A_{\Phi}(t,y(t))y(t)+f_{\Phi}(t,y(t)),\ t\in[0,+\infty),\ (4.1) \\ y(0)+g(y)=y_0, \qquad\qquad\qquad\qquad (4.2), \end{cases} \tag{6}$$

and

$$\begin{cases} {}^{ABC}D^{\alpha}(y(t)+\sigma(t,y(t)))=A_{\Phi}(t,y(t))y(t)+f_{\Phi}(t,y(t)),\ t\in[0,+\infty),\ (4.1) \\ y(0)+g(y)=y_0, \qquad\qquad\qquad\qquad (4.2), \end{cases} \tag{7}$$

where

(1) ${}^{ABC}D$ is the Atangana–Baleanu Caputo fractional derivative.
(2) ${}^{c}D^{\alpha}$ is the Caputo fractional derivative of order $0<\alpha<1$.
(3) $\Phi:\mathbb{R}^{+}\to(0,a]$ is a bounded, continuous and decreasing function, where $a>0$.
(4) $\sigma, f_{\Phi}:[0,+\infty)\times E\to E$ are continuous functions.
(5) $A_{\Phi}(t,x)\colon [0,+\infty)\times E\to\mathscr{B}(E)$ is a bounded operator, where $\mathscr{B}(E)$ denote the space of all bounded linear operators on a Banach space E.
(6) $g:C([0,+\infty),E)\to E$ is a continuous function defined by

$$g(x)=\sum_{i=1}^{p}C_i x(t_i),$$

where C_i are given constants, for $i\in\{1,2,.....,p\}$.

To prove the existence of the solution of (6), we introduce a new norm depended on Φ, noted by $\|.\|_{C_{\Phi}}$, in the space $(C[0,+\infty),E)$ and we show the existence of the real number $\rho>0$ such that the solution u of (6) satisfies $\|u\|_{C_{\Phi}}<\rho$. i.e., u is an element of the ball $B_{\rho}=\{u\in(C[0,+\infty),E):\|u\|_{C_{\Phi}}<\rho\}$. To do this, we give some necessary conditions on Φ in Lemma 4 and since we noted that the existence of the solution of Problem (6) is depended on $\|\Phi\|_{\infty}$, we construct the following set:

$$H_{\Phi}=\{\varphi:\mathbb{R}^{+}\to(0,a] \text{ is a bounded, continuous and decreasing function such that } \|\Phi\|_{\infty}=\|\varphi\|_{\infty}\}.$$

Then if we change Φ by any $\varphi\in H_{\Phi}$ with $\|\Phi\|_{\infty}=\|\varphi\|_{\infty}$, the problem (6) has a solution in B_{ρ}. For the problem (7), we proceed in a similar way making use of Lemma 5. Finally, some illustrative examples are given to confirm the applicability and the effectiveness of the obtained results.
In this section, we introduce some basic preliminaries that will be used in this paper.

Definition 1([20]). *The Riemann-Liouville fractional integral of order α for function f can be written as*

$$I_0^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds,$$

where Γ is the gamma function.

Definition 2([20]). *The Riemann-Liouville fractional derivative of order α for function f is defined as*

$$\begin{aligned} D_0^{\alpha}f(x)&=\frac{d^n}{dx^n}[I^{n-\alpha}f(x)] \\ &=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x(x-t)^{n-\alpha-1}f(t)\,dt, \end{aligned}$$

where $n=[\alpha]+1$, with $[\alpha]$ denotes integer part of α.

Definition 3([20]). *The Caputo fractional derivative of order α of a function f is defined by*

$$
\begin{aligned}
{}^{c}D_0^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds \\
&= I_0^{n-\alpha}\left(\frac{d^n}{dt^n} f(t)\right),
\end{aligned}
$$

where $f^{(n)}(s) = \frac{d^n}{dt^n} f(s)$.

Lemma 1([20]). *Let $\alpha > 0$. Then we have*

$$
{}^{c}D_0^{\alpha}(I_0^{\alpha} f(t)) = f(t).
$$

Lemma 2([20]). *Let $\alpha > 0$ and $n = [\alpha] + 1$. Then*

$$
I_0^{\alpha}({}^{c}D_0^{\alpha} f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k.
$$

Definition 4([7]). *The Mittag-leffler in term of the power series is as follow:*

$$
E_a(w) = \sum_{m=0}^{\infty} \frac{w^m}{\Gamma(am+1)}, \quad a > 0,
$$

Definition 5([21]). *Let $p \in [1, \infty[$ and Ω be an open subset of $\mathbb{R}$ the Sobolev space $H^p(\Omega)$is defined by*

$$
H^p(\Omega) = \{f \in L^2(\Omega) : \; D^{\beta} f \in L^2(\Omega) \; for \; all \quad |\beta| \leq p\}
$$

Definition 6([3]). *Let $f \in H^1(0,1)$ and $0 < \alpha < 1$, the left Atangana-Baleanu fractional derivative of Caputo sense is defined by*

$$
{}^{ABC}D_0^{\alpha} f(t) = \frac{B(\alpha)}{(1-\alpha)} \int_0^t f'(t) E_{\alpha}\left[\frac{-\alpha}{1-\alpha}(t-s)^{\alpha}\right] ds,
$$

where $B(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)} > 0$ is a normalization function satisfying $B(0) = B(1) = 1$, and
E_{α} is the well-known Mittag-Leffer function of one variable.
The associated fractional integral is defined by

$$
{}^{AB}I_0^{\alpha} f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t h(s)(t-s)^{\alpha-1} ds.
$$

Definition 7. *Let (X,d) be a complete metric space. A mapping $T : X \to X$ is said to be a contraction mapping, or contraction, if there exists a constant $k \in [0;1)$ such that*

$$
d\big(T(x), T(y)\big) \leqslant k d(x,y) \tag{8}
$$

for all $x, y \in X$.

Theorem 1. *(Banach's Fixed Point Theorem) If $T : X \to X$ is a contraction mapping on a complete metric space (X,d), then T has a unique fixed point $x \in X$.*

2 Main results

To prove the existence of the solution for our $\Phi-$fractional differential equations (6) and (7), at first we need to introduce a new topology of $C([0,+\infty),E)$. Then, we provide the necessary hypotheses.

Now, we defined the norm $\|.\|_{C_\Phi}$ depending of the function Φ in $C([0,+\infty),E)$ by

$$
\|y\|_{C_\Phi} = \sup_{t \geq 0} \Phi(t)\|y(t)\|, \quad \text{for all } y \in C([0,+\infty),E)
$$

Lemma 3. *The space* $(C([0,+\infty),E),\|.\|_{C_\Phi})$ *is a Banach space.*

Proof. Firstly, we show that $\|.\|_{C_\Phi}$ is a norm on $C([0,+\infty),E)$.
For all $y_1,y_2 \in C([0,+\infty),E)$ and $\lambda \in \mathbb{R}$, we have

(i) Positivity:

$$\begin{aligned}\|y_1\|_{C_\Phi}=0 &\Leftrightarrow \sup_{t\in\mathbb{R}^+} \Phi(t)\|y_1(t)\|=0\\ &\Leftrightarrow \Phi(t)\|y_1(t)\|=0, \quad \text{for all } t\in\mathbb{R}^+\\ &\Leftrightarrow \|y_1(t)\|=0, \quad \text{for all } t\in\mathbb{R}^+\\ &\Leftrightarrow y_1=0.\end{aligned}$$

(ii) Homogeneity:

$$\begin{aligned}\|\lambda y_1\|_{C_\Phi} &= \sup_{t\in\mathbb{R}^+} \Phi(t)\|\lambda y_1(t)\|\\ &= |\lambda| \sup_{t\in\mathbb{R}^+} \Phi(t)\|y_1(t)\|\\ &= |\lambda|\|y_1\|_{C_\Phi}.\end{aligned}$$

(iii) Subadditivity:

$$\begin{aligned}\|y_1+y_2\|_{C_\Phi} = \sup_{t\in\mathbb{R}^+} \Phi(t)\|y_1(t)+y_2(t)\| &\le \sup_{t\in\mathbb{R}^+} \Phi(t)\big(\|y_1(t)\|+\|y_2(t)\|\big)\\ &\le \|y_1\|_{C_\Phi}+\|y_2\|_{C_\Phi}.\end{aligned}$$

Secondly, we will verify that $(C([0,+\infty),E),\|.\|_{C_\Phi})$ is complete.
Since the function Φ is bounded, then there exist two real numbers m and M such that

$$m\le\Phi(t)\le M, \quad \text{for all } t\in[0,+\infty),$$

which implies

$$m\sup_{t\ge0}\|y(t)\|\le\sup_{t\ge0}\Phi(t)\|y(t)\|\le M\sup_{t\ge0}\|y(t)\|,$$

it means that

$$m\|y\|_\infty\le\|y\|_{C_\Phi}\le M\|y\|_\infty,$$

where $\|y\|_\infty=\sup_{t\ge0}\|y(t)\|$. Hence $\|.\|_\infty$ and $\|.\|_{C_\Phi}$ are equivalent, and therefore $(C([0,+\infty),E),\ \|.\|_{C_\Phi})$ is a Banach space.

Concerning Problems (6) and (7), we give the following assumptions:

(H_1) There exist two continuous and bounded functions $L:[0,+\infty)\to\mathbb{R}^+$ and $\Phi:\mathbb{R}^+\to(0,a]$, and a positive constant λ such that

$$\beta=\sup_{t\in\mathbb{R}+} t^\alpha\Phi(t)<+\infty,$$

$$\|A_\Phi(t,u)-A_\Phi(t,v)\|\le L(t)\Phi(t)^2\|u-v\|, \text{ for any } u,v\in E \textit{ and } t\ge0,$$

and

$$A_\Phi(t,0)<\lambda\Phi(t).$$

(H_2) There exist a continuous and bounded function $p:[0,+\infty)\to\mathbb{R}^+$, and a constant η such that

$$\|f_\Phi(t,u)-f_\Phi(t,v)\|\le p(t)\Phi(t)\|u-v\|,\ \ u,v\in E,\ t\ge0,$$

and

$$\eta=\frac{1}{\Gamma(\alpha+1)}\sup_{t\ge0}\|f(t,0)\|<+\infty.$$

(H_2') There exist a continuous and bounded function $p : [0,+\infty) \to \mathbb{R}^+$, and a constant η such that

$$\|f_\Phi(t,u) - f_\Phi(t,v)\| \leq p(t)\Phi(t)\|u-v\|, \;\; u,v \in E, \; t \geq 0,$$

$$\eta_1 = \frac{\alpha}{B(\alpha)}\eta,$$

$$\varepsilon = \frac{(1-\alpha)\Gamma(\alpha+1)}{B(\alpha)},$$

and,

$$\eta_2 = \varepsilon\eta.$$

(H_3) There exist two constants $\zeta > 0$ and δ such that

$$\|\sigma(t,u) - \sigma(t,v)\| \leq \zeta\|u-v\|, \;\; u,v \in E,$$

and

$$\delta = \sup_{t\geq 0}\|\sigma(t,0)\| < +\infty.$$

(H_4) There exists a constant $G > 0$ such that

$$\|g(u) - g(v)\| \leq G\|u-v\|, \;\; u,v \in C([0,+\infty),E).$$

We consider the following notations

$$M = \frac{\|L\|_\infty}{\Gamma(\alpha+1)}, \; N = \frac{\|p\|_\infty}{\Gamma(\alpha+1)}, \; \gamma = \frac{\lambda}{\Gamma(\alpha+1)},$$

$$M_1 = \frac{\alpha}{B(\alpha)}M, \; N_1 = \frac{\alpha}{B(\alpha)}N, \; \gamma_1 = \frac{\alpha}{B(\alpha)}\gamma, \quad M_2 = \varepsilon M, \; N_2 = \varepsilon N, \; \gamma_2 = \varepsilon\gamma,$$

and for $\rho > 0$, we set

$$B_\rho = \{y \in C([0,+\infty),\mathbb{R}) : \; \|y\|_{C_\Phi} \leq \rho\}.$$

2.1 Global Existence and Uniqueness of a solution of problem (6) using Caputo fractional derivative

Now, we study the global existence and uniqueness of our problem by using Caputo fractional derivative operator (6). The following lemma is nedeed to show Theorem 2 below.

Lemma 4. *If* $\beta\gamma + G + \zeta + \beta N - 1 < 0$ *and* $\|\Phi\|_\infty < \frac{(\beta\gamma+G+\zeta+\beta N-1)^2 - 4\beta^2 M\eta}{4\beta M(\|y_0\|+\|\sigma_0\|+\|g_0\|+\delta)}$, *then the following equation*

$$E(\rho) = \beta M\rho^2 + (\beta\gamma + G + \zeta + \beta N - 1)\rho + \|\Phi\|_\infty(\|y_0\| + \|\sigma_0\| + \|g(0)\| + \delta) + \beta\eta = 0$$

has two distinct real positive roots ρ_1 *and* ρ_2 *such*

$$\rho_1 = \frac{-(\beta\gamma + G + \zeta + \beta N - 1) + \sqrt{\Delta}}{2\beta M},$$

and

$$\rho_2 = \frac{-(\beta\gamma + G + \zeta + \beta N - 1) - \sqrt{\Delta}}{2\beta M},$$

where

$$\Delta = (\beta\gamma + G + \zeta + \beta N - 1)^2 - 4\beta M(\|\Phi\|_\infty(\|y_0\| + \|\sigma_0\| + \|g(0)\| + \delta) + \beta\eta).$$

Hence the set of solutions of $E(\rho) \leq 0$ *is* $[\rho_1, \rho_2]$.

*Proof.*We solve the equation $E(\rho)=0$. We have

$$\Delta = (\beta\gamma + G + \zeta + \beta N - 1)^2 - 4\beta M(\|\Phi\|_\infty(\|y_0\| + \|\sigma_0\| + \|g(0)\| + \delta) + \beta\eta).$$

Since

$$\|\Phi\|_\infty < \frac{(\beta\gamma + G + \zeta + \beta N - 1)^2 - 4\beta^2 M\eta}{4\beta M(\|y_0\| + \|\sigma_0\| + \|g_0\| + \delta)}.$$

(1) Then $\Delta > 0$ and we ha

$$\Delta < (\beta\gamma + G + \zeta + \beta N - 1)^2,$$

hence

$$\sqrt{\Delta} < \left|(\beta\gamma + G + \zeta + \beta N - 1)\right|.$$

So

(i) If $\beta\gamma + G + \zeta + \beta N - 1 > 0$, then we have two solutions

$$\rho_1 = \frac{-(\beta\gamma + G + \zeta + \beta N - 1) + \sqrt{\Delta}}{2\beta M} < 0,$$

and

$$\rho_2 = \frac{-(\beta\gamma + G + \zeta + \beta N - 1) - \sqrt{\Delta}}{2\beta M} < 0.$$

(ii) If $\beta\gamma + G + \zeta + \beta N - 1 < 0$, then we have two solutions

$$\rho_1 = \frac{-(\beta\gamma + G + \zeta + \beta N - 1) + \sqrt{\Delta}}{2\beta M} > 0,$$

and

$$\rho_2 = \frac{-(\beta\gamma + G + \zeta + \beta N - 1) - \sqrt{\Delta}}{2\beta M} > 0.$$

(2) If

$$\|\Phi\|_\infty \geq \frac{(\beta\gamma + G + \zeta + \beta N - 1)^2 - 4\beta^2 M\eta}{4\beta M(\|y_0\| + \|\sigma_0\| + \|g_0\| + \delta)},$$

then $\Delta \leq 0$, and in this case the set of solution of $E(\rho) \leq 0$ is empty.

We conclude that the set of solutions of $E(\rho) \leq 0$ is $S = [\rho_1, \rho_2]$.

The following result gives the unique solution of the problem (6).

Theorem 2.*Under the hypothesis* (H_1), (H_2), (H_3), (H_4) *and* (H_5) : $\frac{1-(\beta\gamma+G+\zeta+\beta N)}{\beta M} - 1 > \rho$, *for any* $\rho \in [\rho_1, \rho_2]$, *where* ρ_1, ρ_2 *are the solutions of* $E(\rho) = 0$. *Our problem* (6) *has a unique solution* $y \in B_\rho$.

*Proof.*By Lemma 1 and Lemma 2, the problem (6) becames

$$y(t) + \sigma(t, y(t)) = y(0) + \sigma_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (A_\Phi(s, y(s))y(s) + f_\Phi(s, y(s)))ds.$$

Thus

$$y(t) = y_0 + \sigma_0 - g(y) - \sigma(t, y(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (A_\Phi(s, y(s))y(s) + f_\Phi(s, y(s))))ds. \tag{9}$$

Set $\Psi_\Phi(t, y(t)) = A_\Phi(t, y(t))y(t) + f_\Phi(t, y(t))$. Now, we consider the operator T defined by

$$(Ty)(t) = y_0 + \sigma_0 - g(y) - \sigma(t, y(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\Psi_\Phi(s, y(s)))ds.$$

Then the integral equation (9) is reduced to $y = Ty$. In order to establish our existence result, we need to show that T has a unique fixed point y in B_ρ, for any $\rho \in [\rho_1, \rho_2]$.

First step, we prove that $T(B_\rho) \subset B_\rho$.
Let $y \in B_\rho$, we have

$$\|\Phi(t)Ty(t)\| = \|\Phi(t)(y_0+\sigma_0) - \Phi(t)(g(y)+\sigma(t,y(t))) + \frac{1}{\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\Psi_\Phi(s,y(s))ds\|,$$

which implies that

$$\begin{aligned}\|\Phi(t)Ty(t)\| &\leq \Phi(t)(\|y_0\|+\|\sigma_0\|) + \Phi(t)\|g(y)\| + \Phi(t)\|\sigma(t,y(t))\| + \frac{1}{\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\|\Psi_\Phi(s,y(s))\|ds\\ &\leq \Phi(t)\big(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\|\sigma(t,0)\|+\|g(y)-g(0)\|+\|\sigma(t,y(t))-\sigma(t,0)\|\big)\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\big(\|A_\Phi(s,y(s))-A_\Phi(s,0)\|+\|A_\Phi(s,0)\|\big)\|y(s)\|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\big(\|f_\Phi(s,y(s))-f_\Phi(s,0)\|+\|f_\Phi(s,0)\|\big)ds.\end{aligned}$$

By using $(H_1),\ (H_2),\ (H_3)\ and\ (H_4)$, we have

$$\begin{aligned}\|\Phi(t)Ty(t)\| &\leq \Phi(t)\big(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta\big)+G\Phi(t)\|y(t)\|+\zeta\Phi(t)\|y(t)\|\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\big(L(s)\Phi(s)\|y(s)\|+\frac{1}{\Phi(s)}\|A_\Phi(s,0)\|\big)\Phi(s)\|y(s)\|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\big(p(s)\Phi(s)\|y(s)\|+\|f_\Phi(s,0)\|\big)ds.\end{aligned}$$

So we get

$$\sup_{t>0}\|\Phi(t)Ty(t)\| \leq \|\Phi\|_\infty(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta)+G\rho+\zeta\rho+(M\rho+\gamma)\beta\rho+\beta(N\rho+\eta),$$

and by Lemma 4, we have

$$\begin{aligned}\sup_{t>0}\|\Phi(t)Ty(t)\| &\leq \beta M\rho^2+(\beta\gamma+G+\zeta+\beta N)\rho+\|\Phi\|_\infty(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta)+\beta\eta\\ &\leq \rho.\end{aligned}$$

Then $\|Ty\|_{C_\Phi} \leq \rho$. Hence $T(B_\rho) \subset B_\rho$.
Second step, We prove that T is contraction.
For each $u,v \in B_\rho$, we have

$$\begin{aligned}\|\Phi(t)(Tu(t)-Tv(t))\| &= \|\Phi(t)(g(u)-g(v))+\Phi(t)(\sigma(t,u(t))-\sigma(t,v(t)))\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}(A_\Phi(s,u)u(s)-A_\Phi(s,v)v(s))ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}(f_\Phi(s,u(s))-f_\Phi(s,v(s)))ds\|\\ &\leq \Phi(t)\|(g(u)-g(v))\|+\Phi(t)\|(\sigma(t,u(t))-\sigma(t,v(t)))\|\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\|A_\Phi(s,u)u(s)-A_\Phi(s,v)v(s)\|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\|f_\Phi(s,u(s))-f_\Phi(s,v(s))\|ds\\ &\leq \Phi(t)\|(g(u)-g(v))\|+\Phi(t)\|(\sigma(t,u(t))-\sigma(t,v(t)))\|\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\Big(\|A_\Phi(s,u)\|\|u(s)-v(s)\|\\ &+\|A_\Phi(s,u)-A_\Phi(s,v)\|\|v(s)\|\Big)ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\|f_\Phi(s,u(s))-f_\Phi(s,v(s))\|ds.\end{aligned}$$

Again, by using (H_1), (H_2), (H_3) *and* (H_4), we have

$$\|\Phi(t)(Tu(t)-Tv(t))\| \leq \Phi(t)G\|u-v\| + \Phi(t)\zeta\|u-v\| + \frac{1}{\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\Big((L(s)\Phi(s)\|u(s)\| + \frac{1}{\Phi(s)}\|A_\Phi(s,0)\|)\Phi(s)\|u(s)-v(s)\| + L(s)\Phi(s)\|u(s)\|\Phi(s)\|v(s)\|\Big)ds + \frac{1}{\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}(p(s)\Phi(s)\|u(s)-v(s)\|)ds.$$

By calculating the integrals, we get

$$\sup_{t>0}\|\Phi(t)(Tu(t)-Tv(t))\| \leq G\|u-v\|_{C_\Phi} + \zeta\|u-v\|_{C_\Phi} + \beta(M+\gamma)\|u-v\|_{C_\Phi} + \beta M\rho\|u-v\|_{C_\Phi} + \beta N\|u-v\|_{C_\Phi}.$$

Therefore

$$\|Tu-Tv\|_{C_\Phi} \leq ((1+\rho)\beta M + \beta\gamma + G + \zeta + \beta N)\|u-v\|_{C_\Phi}.$$

Based on (H_5), we conclude that T is a contraction from B_ρ into B_ρ and

$$(1+\rho)\beta M + \beta\gamma + G + \zeta + \beta N < 1.$$

Since

$$1-(\beta\gamma+G+\zeta+\beta N)\beta M - \beta M > \rho\beta M,$$

then

$$1 > \beta M\rho + \beta\gamma + G + \zeta + \beta N + \beta M = (1+\rho)\beta M + \beta\gamma + G + \zeta + \beta N.$$

Finally, the Banach fixed point theorem guarantees that T has a unique fixed point $y \in B_\rho$ which is a solution of the problem (6).

2.2 Global existence and uniqueness of a solution of problem (7) using Atangana–Baleanu Caputo fractional derivative

Next, we study the global existence and uniqueness of our problem (7) by using the Atangana–Baleanu Caputo fractional derivative operator.

We need the following assertion to prove Theorem 3 below.

Lemma 5. *If we have these two conditions*

(i) $\|\Phi\|_\infty < \frac{1}{\varepsilon}(\frac{1-G-\zeta}{N+\gamma} - \frac{\alpha}{B(\alpha)}\beta)$.

(ii) $\big(-4\varepsilon M(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta+\varepsilon\eta)+\varepsilon(N+\gamma)^2\big)\|\Phi\|_\infty^2 + \big(-4M\frac{\alpha}{B(\alpha)}\beta(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta+\varepsilon\eta)+2\frac{\alpha}{B(\alpha)}\beta\varepsilon(N+\gamma)^2+2(N+\gamma)(G+\zeta-1)\varepsilon\big)\|\Phi\|_\infty + (N+\gamma)^2(\frac{\alpha}{B(\alpha)}\beta)^2 + 2(N+\gamma)\frac{\alpha}{B(\alpha)}\beta(G+\zeta-1)+(G+\zeta-1)^2 - 4\frac{\alpha}{B(\alpha)}\beta\eta > 0$.

Then the following equation

$$E(\rho') = (\|\Phi\|_\infty\varepsilon + \frac{\alpha}{B(\alpha)}\beta)M\rho'^2 + ((\frac{\alpha}{B(\alpha)}\beta + \|\Phi\|_\infty\varepsilon)(N+\gamma)+G+\zeta-1)\rho' + \|\Phi\|_\infty(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta+\varepsilon\eta) + \frac{\alpha}{B(\alpha)}\beta\eta = 0$$

has two distinct real positive roots ρ_1' *and* ρ_2' *such that*

$$\rho_1' = \frac{-\big((\frac{\alpha}{B(\alpha)}\beta + \|\Phi\|_\infty\varepsilon)(N+\gamma)+G+\zeta-1\big)+\sqrt{\Delta}}{2(\|\Phi\|_\infty\varepsilon + \frac{\alpha}{B(\alpha)}\beta)M},$$

and

$$\rho_2' = \frac{-\big((\frac{\alpha}{B(\alpha)}\beta + \|\Phi\|_\infty\varepsilon)(N+\gamma)+G+\zeta-1\big)-\sqrt{\Delta}}{2(\|\Phi\|_\infty\varepsilon + \frac{\alpha}{B(\alpha)}\beta)M},$$

where

$$\Delta = \big((\frac{\alpha}{B(\alpha)}\beta + \|\Phi\|_\infty\varepsilon)(N+\gamma) + G + \zeta - 1\big)^2 - 4\big((\|\Phi\|_\infty\varepsilon + \frac{\alpha}{B(\alpha)}\beta)M\big)\big(\|\Phi\|_\infty(\|y_0\| + \|\sigma_0\| + \|g(0)\| + \delta + \varepsilon\eta) + \frac{\alpha}{B(\alpha)}\beta\eta\big).$$

Hence, the set of solution of $E(\rho') \leq 0$ *is* $[\rho'_1, \rho'_2]$.

Proof. We solve the equation $E(\rho') = 0$. We have

$$\Delta = \big((\frac{\alpha}{B(\alpha)}\beta + \|\Phi\|_\infty\varepsilon)(N+\gamma) + G + \zeta - 1\big)^2 - 4\big((\|\Phi\|_\infty\varepsilon + \frac{\alpha}{B(\alpha)}\beta)M\big)\big(\|\Phi\|_\infty(\|y_0\| + \|\sigma_0\| + \|g(0)\| + \delta + \varepsilon\eta) + \frac{\alpha}{B(\alpha)}\beta\eta\big).$$

Since $\|\Phi\|_\infty$ satisfies the condition (ii), then $\Delta > 0$ and we have

$$\Delta < \big((\frac{\alpha}{B(\alpha)}\beta + \|\Phi\|_\infty\varepsilon)(N+\gamma) + G + \zeta - 1\big)^2,$$

which implies that

$$\sqrt{\Delta} < \big|\big((\frac{\alpha}{B(\alpha)}\beta + \|\Phi\|_\infty\varepsilon)(N+\gamma) + G + \zeta - 1\big)\big|.$$

From condition (i), we get $\big((\frac{\alpha}{B(\alpha)}\beta + \|\Phi\|_\infty\varepsilon)(N+\gamma) + G + \zeta - 1\big) < 0$, so we obtain two positive solutions

$$\rho'_1 = \frac{-\big((\frac{\alpha}{B(\alpha)}\beta + \|\Phi\|_\infty\varepsilon)(N+\gamma) + G + \zeta - 1\big) + \sqrt{\Delta}}{2(\|\Phi\|_\infty\varepsilon + \frac{\alpha}{B(\alpha)}\beta)M} > 0,$$

and

$$\rho'_2 = \frac{-\big((\frac{\alpha}{B(\alpha)}\beta + \|\Phi\|_\infty\varepsilon)(N+\gamma) + G + \zeta - 1\big) - \sqrt{\Delta}}{2(\|\Phi\|_\infty\varepsilon + \frac{\alpha}{B(\alpha)}\beta)M} > 0.$$

We conclude that the set of solutions of $E(\rho') \leq 0$ is $S = [\rho'_1, \rho'_2]$.

*Remark.*If $\big((\frac{\alpha}{B(\alpha)}\beta + \|\Phi\|_\infty\varepsilon)(N+\gamma) + G + \zeta - 1\big) > 0$, we get two negative solutions

$$\rho'_1 = \frac{-\big((\frac{\alpha}{B(\alpha)}\beta + \|\Phi\|_\infty\varepsilon)(N+\gamma) + G + \zeta - 1\big) + \sqrt{\Delta}}{2(\|\Phi\|_\infty\varepsilon + \frac{\alpha}{B(\alpha)}\beta)M} < 0,$$

and

$$\rho'_2 = \frac{-\big((\frac{\alpha}{B(\alpha)}\beta + \|\Phi\|_\infty\varepsilon)(N+\gamma) + G + \zeta - 1\big) - \sqrt{\Delta}}{2(\|\Phi\|_\infty\varepsilon + \frac{\alpha}{B(\alpha)}\beta)M} < 0.$$

In the case where $\Delta \leq 0$, the set of solution of $E(\rho') \leq 0$ is empty.

The following result gives the unique solution of the problem (7).

Theorem 3.*If the hypothesis* (H_1), (H'_2), (H_3), (H_4) *and* (H'_5) : $\frac{1-((N_2+\gamma_2)\|\Phi\|_\infty+\beta\gamma_1+G+\zeta+\beta N_1)}{(\|\Phi\|_\infty M_2+\beta M_1)} - 1 > \rho$, *for any* $\rho \in [\rho'_1, \rho'_2]$ *with* ρ'_1, ρ'_2 *are the solutions of* $E(\rho') = 0$ *hold, then our problem* (7) *has a unique solution* $y \in B_\rho$.

*Proof.*The problem (7) becames

$$y(t) + \sigma(t, y(t)) = y(0) + \sigma_0 + \frac{1-\alpha}{B(\alpha)}(A_\Phi(t, y(t))y(t) + f_\Phi(t, y(t))) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-1}(A_\Phi(s, y(s))y(s) + f_\Phi(s, y(s)))ds\Big).$$

Set $\Psi_\Phi(t,y(t)) = A_\Phi(t,y(t))y(t) + f_\Phi(t,y(t))$. Thus

$$y(t) = y_0 + \sigma_0 - g(y) - \sigma(t,y(t)) + \sigma(t,y(t)) + \frac{1-\alpha}{B(\alpha)}(\Psi_\Phi(t,y(t))) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\Psi_\Phi(s,y(s)))ds.$$

Now, we consider the operator T defined by

$$(Ty)(t) = y_0 + \sigma_0 - g(y) - \sigma(t,y(t)) + \frac{1-\alpha}{B(\alpha)}(\Psi_\Phi(t,y(t))) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\Psi_\Phi(s,y(s)))ds.$$

Then the above integral equation is reduced to $y = Ty$. In order to establish our existence result, we need to show that T has a unique fixed point y in B_ρ, for any $\rho \in [\rho_1,\rho_2]$.

First step, we prove that $T(B_\rho) \subset B_\rho$.

Let $y \in B_\rho$, we have

$$\begin{aligned}\|\Phi(t)Ty(t)\| = \|\Phi(t)\big((y_0+\sigma_0) - (g(y)+\sigma(t,y(t)))\big) + \frac{\Phi(t)(1-\alpha)}{B(\alpha)}\Psi_\Phi(t,y(t))\\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\Psi_\Phi(s,y(s))ds\|,\end{aligned}$$

which implies that

$$\begin{aligned}\|\Phi(t)Ty(t)\| &\le \Phi(t)(\|y_0\|+\|\sigma_0\|) + \Phi(t)\|g(y)\| + \Phi(t)\|\sigma(t,y(t))\| + \frac{\Phi(t)(1-\alpha)}{B(\alpha)}\|\Psi_\Phi(t,y(t))\|\\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\|\Psi_\Phi(s,y(s))\|ds\\ &\le \Phi(t)\big(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\|\sigma(t,0)\|+\|g(y)-g(0)\|+\|\sigma(t,y(t))-\sigma(t,0)\|\big)\\ &+ \frac{\Phi(t)(1-\alpha)}{B(\alpha)}\big(\|A_\Phi(t,y(t))-A_\Phi(t,0)\|+\|A_\Phi(t,0)\|\big)\|y(t)\|\\ &+ \frac{\Phi(t)(1-\alpha)}{B(\alpha)}\big(\|f_\Phi(t,y(t))-f_\Phi(t,0)\|+\|f_\Phi(t,0)\|\big)\\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\big(\|A_\Phi(s,y(s))-A_\Phi(s,0)\|+\|A_\Phi(s,0)\|\big)\|y(s)\|ds\\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\big(\|f_\Phi(s,y(s))-f_\Phi(s,0)\|+\|f_\Phi(s,0)\|\big)ds.\end{aligned}$$

By using $(H_1),\ (H_2),\ (H_3)$ *and* (H_4), we have

$$\begin{aligned}\|\Phi(t)Ty(t)\| &\le \Phi(t)\big(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta\big) + G\Phi(t)\|y(t)\| + \zeta\Phi(t)\|y(t)\|\\ &+ \Phi(t)\big((M_2\Phi(t)\|y(t)\|+\gamma_2)\Phi(t)\|y(t)\| + N_2\Phi(t)\|y(t)\| + \eta_2\big)\\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\big(L(s)\Phi(s)\|y(s)\| + \frac{1}{\Phi(s)}\|A_\Phi(s,0)\|\big)\Phi(s)\|y(s)\|ds\\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\big(p(s)\Phi(s)\|y(s)\| + \|f_\Phi(s,0)\|\big)ds.\end{aligned}$$

So we get

$$\begin{aligned}\sup_{t>0}\|\Phi(t)Ty(t)\| &\le \|\Phi\|_\infty(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta) + G\rho + \zeta\rho + \|\Phi\|_\infty(M_2\rho+\gamma_2)\rho\\ &+ \|\Phi\|_\infty(N_2\rho+\eta_2) + (M_1\rho+\gamma_1)\beta\rho + \beta(N_1\rho+\eta_1),\end{aligned}$$

and by Lemma 5, we have

$$\begin{aligned}\sup_{t>0}\|\Phi(t)Ty(t)\| &\le (\|\Phi\|_\infty M_2 + \beta M_1)\rho^2 + (\beta\gamma_1 + G + \zeta + \beta N_1 + \|\Phi\|_\infty N_2 + \|\Phi\|_\infty\gamma_2)\rho\\ &+ \|\Phi\|_\infty(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta+\eta_2) + \beta\eta_1\\ &\le \rho.\end{aligned}$$

Then $\|Ty\|_{C_\Phi} \leq \rho$. Hence, $T(B_\rho) \subset B_\rho$.
Second step, We prove that T is contraction.
For each $u, v \in B_\rho$, we have

$$\begin{aligned}
\|\Phi(t)(Tu(t) - Tv(t))\| &= \|\Phi(t)(g(u) - g(v)) + \Phi(t)(\sigma(t,u(t)) - \sigma(t,v(t))) \\
&+ \frac{\Phi(t)(1-\alpha)}{B(\alpha)}(A_\Phi(t,u)u(t) - A_\Phi(t,v)v(t)) + \frac{\Phi(t)(1-\alpha)}{B(\alpha)}(f_\Phi(t,u(t)) - f_\Phi(t,v(t))) \\
&+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}(A_\Phi(s,u)u(s) - A_\Phi(s,v)v(s))ds \\
&+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}(f_\Phi(s,u(s)) - f_\Phi(s,v(s)))ds\| \\
&\leq \Phi(t)\|(g(u) - g(v))\| + \Phi(t)\|(\sigma(t,u(t)) - \sigma(t,v(t)))\| \\
&+ \frac{\Phi(t)(1-\alpha)}{B(\alpha)}(\|A_\Phi(t,u)u(t) - A_\Phi(t,v)v(t)\|) + \frac{\Phi(t)(1-\alpha)}{B(\alpha)}(\|f_\Phi(t,u(t)) - f_\Phi(t,v(t))\|) \\
&+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\|A_\Phi(s,u)u(s) - A_\Phi(s,v)v(s)\|ds \\
&+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\|f_\Phi(s,u(s)) - f_\Phi(s,v(s))\|ds \\
&\leq \Phi(t)\|(g(u) - g(v))\| + \Phi(t)\|(\sigma(t,u(t)) - \sigma(t,v(t)))\| \\
&\frac{\Phi(t)(1-\alpha)}{B(\alpha)}\Big(\|A_\Phi(t,u)\|\|u(t) - v(t)\| + \|A_\Phi(t,u) - A_\Phi(t,v)\|\|v(t)\|\Big) \\
&+ \frac{\Phi(t)(1-\alpha)}{B(\alpha)}\|f_\Phi(s,u(s)) - f_\Phi(s,v(s))\| \\
&+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\Big(\|A_\Phi(s,u)\|\|u(s) - v(s)\| + \|A_\Phi(s,u) - A_\Phi(s,v)\|\|v(s)\|\Big)ds \\
&+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}\|f_\Phi(s,u(s)) - f_\Phi(s,v(s))\|ds.
\end{aligned}$$

Again, by using (H_1), (H_2), (H_3) *and* (H_4), we have

$$\begin{aligned}
\|\Phi(t)(Tu(t) - Tv(t))\| &\leq \Phi(t)G\|u - v\| + \Phi(t)\zeta\|u - v\| \\
&+ \frac{\Phi(t)(1-\alpha)}{B(\alpha)}\Big((L(t)\Phi(t)\|u(t)\| + \frac{1}{\Phi(t)}\|A_\Phi(t,0)\|)\Phi(t)\|u(t) - v(t)\| \\
&+ L(t)\Phi(t)\|u(t)\|\Phi(t)\|v(t)\|\Big) \\
&+ \frac{\Phi(t)(1-\alpha)}{B(\alpha)}(p(t)\Phi(t)\|u(t) - v(t)\|) \\
&+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)})\int_0^t \Phi(t)(t-s)^{\alpha-1}\Big((L(s)\Phi(s)\|u(s)\| \\
&+ \frac{1}{\Phi(s)}\|A_\Phi(s,0)\|)\Phi(s)\|u(s) - v(s)\| + L(s)\Phi(s)\|u(s)\|\Phi(s)\|v(s)\|\Big)ds \\
&+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t \Phi(t)(t-s)^{\alpha-1}(p(s)\Phi(s)\|u(s) - v(s)\|)ds.
\end{aligned}$$

By calculating the integrals, we get

$$\begin{aligned}
\sup_{t>0}\|\Phi(t)(Tu(t) - Tv(t))\| &\leq G\|u - v\|_{C_\Phi} + \zeta\|u - v\|_{C_\Phi} + \|\Phi\|_\infty(M_2 + \gamma_2)\|u - v\|_{C_\Phi} + M_2\|\Phi\|_\infty\rho\|u - v\|_{C_\Phi} \\
&+ \|\Phi\|_\infty N_2\|u - v\|_{C_\Phi} + \beta(M_1 + \gamma_1)\|u - v\|_{C_\Phi} + \beta M_1\rho\|u - v\|_{C_\Phi} + \beta N_1\|u - v\|_{C_\Phi}.
\end{aligned}$$

Therefore

$$\|Tu - Tv\|_{C_\Phi} \leq ((\|\Phi\|_\infty M_2 + \beta M_1)\rho + (M_2 + N_2 + \gamma_2)\|\Phi\|_\infty + \beta M_1 + \beta\gamma_1 + G + \zeta + \beta N_1)\|u - v\|_{C_\Phi}.$$

Based on (H_5'), we conclude that T is a contraction from B_ρ into B_ρ and

$$(\|\Phi\|_\infty M_2+\beta M_1)\rho+(M_2+N_2+\gamma_2)\|\Phi\|_\infty+\beta M_1+\beta\gamma_1+G+\zeta+\beta N_1<1.$$

Since

$$1-(M_2+N_2+\gamma_2)\|\Phi\|_\infty-(\beta M_1+\beta\gamma_1+G+\zeta+\beta N_1)>(\|\Phi\|_\infty M_2+\beta M_1)\rho,$$

then

$$1>(\|\Phi\|_\infty M_2+\beta M_1)\rho+(M_2+N_2+\gamma_2)\|\Phi\|_\infty+\beta M_1+\beta\gamma_1+G+\zeta+\beta N_1$$

Finally, the Banach fixed point theorem guarantees that T has a unique fixed point $y\in B_\rho$ which is a solution of the problem (7).

3 Examples

3.1 Examples associated to Caputo fractional derivative

We study the two following problems using Caputo fractional derivative operator:

$$\begin{cases} {}^cD^{\frac{1}{2}}_{0^+}(y(t)+\frac{|y(t)|e^{-t}}{2(|y(t)|+5)})=\frac{e^{-t}ln(6+|y(t)|)}{5}\Phi(t)^2y(t)+\frac{sin(t)\Phi(t)}{5(|y(t)|^2+1)}, & t>0,\\ y(0)+\frac{1}{8}cos(y(1))=0,001, \end{cases} \tag{10}$$

with $\Phi(t)=\frac{1}{1+t\sqrt{t}}$ or $\Phi(t)=e^{-t}$ or $\Phi(t)=\frac{e^{-t}}{1+t}$. And

$$\begin{cases} {}^cD^{\frac{1}{2}}_{0^+}(y(t)+\frac{1}{5}sin(t+y(t)))=\frac{cos(t^3+1)}{15(1+t\sqrt{t})^{\frac{1}{2}}}\Phi(t)^2|y|^{\frac{2}{3}}ln((\frac{|y|}{1+t\sqrt{t}})^{\frac{1}{3}}+1)y(t)+\frac{|y|\Phi(t)}{19+e^{2t}}+\frac{1}{10}, & t>0,\\ y(0)+\frac{1}{4}sin(y(1))=0,01, \end{cases} \tag{11}$$

with $\Phi(t)=\frac{1}{1+t\sqrt{t}}$.

Example 1. Consider the following $\Phi-Fractional$ functional differential equation.

$$\begin{cases} {}^cD^{\frac{1}{2}}_{0^+}(y(t)+\frac{|y(t)|e^{-t}}{2(|y(t)|+5)})=\frac{e^{-t}ln(6+|y(t)|)}{5}\Phi(t)^2y(t)+\frac{sin(t)\Phi(t)}{5(|y(t)|^2+1)}, & t>0,\\ y(0)+\frac{1}{8}cos(y(1))=0,001. \end{cases} \tag{12}$$

Let

$$\begin{aligned} \Phi(t)&=\frac{1}{1+t\sqrt{t}},\\ A_\Phi(t,y)&=\frac{e^{-t}ln(6+|y|)}{5(1+t\sqrt{t})^2},\\ \sigma(t,y)&=\frac{|y|e^{-t}}{2(|y|+5)},\\ f_\Phi(t,y)&=\frac{sin(t)}{5(1+t\sqrt{t})(|y|^2+1)},\\ g(y)&=\frac{1}{8}cos(y(1)). \end{aligned}$$

For all $x,y\in\mathbb{R}$ and $t\geq 0$, we have

$$\begin{aligned} |A_\Phi(t,x)-A_\Phi(t,y)|&\leq\frac{e^{-t}}{5(1+t\sqrt{t})^2}\big|ln(6+x)-ln(6+y)\big|\\ &\leq\frac{e^{-t}}{30(1+t\sqrt{t})^2}|x-y|. \end{aligned}$$

Thus the operator $A_\Phi(t,y)$ satisfies the hypothesis (H_1) with

$$\|L\|_\infty = \frac{1}{30},\ \lambda = \frac{ln(6)}{5},\ M = \frac{1}{15\sqrt{\pi}},\ \gamma = \frac{2ln(6)}{5\sqrt{\pi}},\ \beta \le \frac{\alpha^{\frac{\alpha}{\alpha+1}}}{1+\alpha} < 1.$$

On the other hand, we have

$$\begin{aligned} |f_\Phi(t,x) - f_\Phi(t,y)| &\le \frac{|sin(t)|}{5(1+t\sqrt{t})}\left|\frac{1}{(x^2+1)} - \frac{1}{(y^2+1)}\right| \\ &\le \frac{|sin(t)|}{10(1+t\sqrt{t})}|x-y|. \end{aligned}$$

So (H_2) is satisfied with

$$\|p\|_\infty = \frac{1}{10},\ N = \frac{1}{5\sqrt{\pi}},\ \eta = \frac{2}{5\sqrt{\pi}}.$$

For the hypothesis (H_3), we obtain

$$\begin{aligned} |\sigma(t,x) - \sigma(t,y)| &= \frac{e^{-t}}{2}\left|\frac{|x|}{(|x|+5)} - \frac{|y|}{(|y|+5)}\right| \\ &\le \frac{1}{10}|x-y|. \end{aligned}$$

Then (H_3) is satisfied with

$$\zeta = \frac{1}{10},\ \sigma_0 = 0\ et\ \delta = 0.$$

Moreover, we have

$$|g(x) - g(y)| \le \frac{1}{8}|x-y|.$$

Hence the assumption (H_4) is verified, with $G = \frac{1}{8}$.
Finally, we verify Hypothesis (H_5). We have

$$\frac{2ln(6)}{5\sqrt{\pi}} + \frac{1}{8} + \frac{1}{10} + \frac{1}{5\sqrt{\pi}} - 1 < 0.$$

Lemma 4 implies that the set of solution of the following inequality

$$\frac{1}{15\sqrt{\pi}}\rho^2 + (\frac{2ln(6)}{5\sqrt{\pi}} + \frac{1}{8} + \frac{1}{10} + \frac{1}{5\sqrt{\pi}} - 1)\rho + (0{,}001 + \frac{1}{8} + \frac{1}{5\sqrt{\pi}}) < 0$$

is $[\rho_1,\rho_2]$, with $\rho_1 = 1.1$ and $\rho_2 = 7.5$. Moreover, for any $\rho \in [\rho_1,\rho_2]$, we get

$$\frac{1-(\beta\gamma + G + \zeta + \beta N)}{\beta M} - 1 > \rho.$$

Consequently, the assumption (H_5) is obtained. We conclude that the problem has a unique solution in B_ρ for any $\rho \in [1.1, 7.5]$.

Example 2. Consider the following $\Phi - Fractional$ functional differential equation.

$$\begin{cases} {}^cD^{\frac{1}{2}}_{0^+}(y(t) + \frac{|y(t)|e^{-t}}{2(|y(t)|+5)}) = \frac{e^{-t}ln(6+|y(t)|)}{5}\Phi(t)^2 y(t) + \frac{sin(t)\Phi(t)}{5(|y(t)|^2+1)}, & t > 0, \\ y(0) + \frac{1}{8}cos(y(1)) = 0{,}001. \end{cases} \tag{13}$$

Let

$$\begin{aligned}\Phi(t) &= e^{-t},\\ A_\Phi(t,y) &= \frac{e^{-3t}ln(6+|y(t)|)}{5}, \quad t\geq 0,\ y\in\mathbb{R},\\ \sigma(t,y) &= \frac{|y|e^{-t}}{2(|y|+5)},\\ f_\Phi(t,y) &= \frac{sin(t)e^{-}t}{5(|y(t)|^2+1)},\\ g(y) &= \frac{1}{8}cos(y(1)).\end{aligned}$$

For all $x,y\in\mathbb{R}$

$$\begin{aligned}|A_\Phi(t,x)-A_\Phi(t,y)| &\leq \frac{e^{-3t}}{5}|ln(6+x)-ln(6+y)|\\ &\leq \frac{e^{-3t}}{30}|x-y|.\end{aligned}$$

So the hypothesis (H_1) satisfies, with

$$\|L\|_\infty = \frac{1}{30},\ \lambda = \frac{ln(6)}{5},\ M = \frac{1}{15\sqrt{\pi}},\ \gamma = \frac{2ln(6)}{5\sqrt{\pi}},$$
$$\|\Phi\|_\infty \leq 1,\ \beta \leq \alpha^\alpha e^{-\alpha} < 1.$$

For the hypothesis (H_2), we have

$$\begin{aligned}|f_\Phi(t,x)-f_\Phi(t,y)| &\leq \frac{|sin(t)|e^{-t}}{5}\left|\frac{1}{(x^2+1)}-\frac{1}{(y^2+1)}\right|\\ &\leq \frac{|sin(t)|e^{-t}}{10}|x-y|.\end{aligned}$$

Then (H_2) is holds with

$$\|p\|_\infty = \frac{1}{10}, N = \frac{1}{5\sqrt{\pi}},\ \eta = \frac{2}{5\sqrt{\pi}}.$$

Concerning (H_3), we get

$$\begin{aligned}|\sigma(t,x)-\sigma(t,y)| &= \frac{e^{-t}}{2}\left|\frac{|x|}{(|x|+5)}-\frac{|y|}{(|y|+5)}\right|\\ &\leq \frac{1}{10}|x-y|.\end{aligned}$$

So (H_3) is satisfied with

$$\zeta = \frac{1}{10},\ \sigma_0 = 0\ et\ \delta = 0.$$

For the hypothesis (H_4), we have

$$|g(x)-g(y)| \leq \frac{1}{8}|x-y|,$$

and then

$$G = \frac{1}{8}.$$

In the end, since $\frac{2ln(6)}{5\sqrt{\pi}}+\frac{1}{8}+\frac{1}{10}+\frac{1}{5\sqrt{\pi}}-1<0$ and by Lemma 4, the set of solution of the following inequality

$$\frac{1}{15\sqrt{\pi}}\rho^2+(\frac{2ln(6)}{5\sqrt{\pi}}+\frac{1}{8}+\frac{1}{10}+\frac{1}{5\sqrt{\pi}}-1)\rho+(0,001+\frac{1}{8}+\frac{1}{5\sqrt{\pi}})<0$$

is $[\rho_1,\rho_2]$, with $\rho_1 = 1.1$ and $\rho_2 = 7.5$. Furthermore, we obtain $\frac{1-(\beta\gamma+G+\zeta+\beta N)}{\beta M}-1>\rho$, for all $\rho\in[\rho_1,\rho_2]$. So the hypotheses of Theorem 2 holds, and therefore the problem has a unique solution in B_ρ, for any $\rho\in[1.1,7.5]$.

Example 3. Consider the following $\Phi - Fractional$ functional differential equation.

$$\begin{cases} {}^{c}D^{\frac{1}{2}}_{0^+}\left(y(t)+\frac{|y(t)|e^{-t}}{2(|y(t)|+5)}\right)=\frac{e^{-t}ln(6+|y(t)|)}{5}\Phi(t)^2y(t)+\frac{sin(t)\Phi(t)}{5(|y(t)|^2+1)}, & t>0,\\ y(0)+\frac{1}{8}cos(y(1))=0,001. \end{cases}$$

Let

$$\begin{aligned} \Phi(t) &= \frac{e^{-t}}{1+t},\\ A_{\Phi}(t,y) &= \frac{e^{-3t}ln(6+|y(t)|)}{5(1+t)^2},\ \ t\geq 0,\ y\in\mathbb{R},\\ \sigma(t,y) &= \frac{|y|e^{-t}}{2(|y|+5)},\\ f_{\Phi}(t,y) &= \frac{sin(t)e^{-}t}{5(|y(t)|^2+1)(1+t)},\\ g(y) &= \frac{1}{8}cos(y(1)). \end{aligned}$$

For all $x,y\in\mathbb{R}$, we have

$$\begin{aligned} |A_{\Phi}(t,x)-A_{\Phi}(t,y)| &\leq \frac{e^{-3t}}{5(1+t)^2}|ln(6+x)-ln(6+y)|\\ &\leq \frac{e^{-3t}}{30(1+t)^2}|x-y|. \end{aligned}$$

Thus the operator $A(t,y)$ satisfies the hypothesis (H_1), with

$$\|L\|_\infty=\frac{1}{30},\ \lambda=\frac{ln(6)}{5},\ M=\frac{1}{15\sqrt{\pi}},\ \gamma=\frac{2ln(6)}{5\sqrt{\pi}},\ \|\Phi\|_\infty\leq 1,\ \beta<1.$$

Moreover, we can get

$$\begin{aligned} |f_{\Phi}(t,x)-f_{\Phi}(t,y)| &\leq \frac{|sin(t)|e^{-t}}{5(1+t)}\left|\frac{1}{(x^2+1)}-\frac{1}{(y^2+1)}\right|\\ &\leq \frac{|sin(t)|e^{-t}}{10(1+t)}|x-y|. \end{aligned}$$

So (H_2) is satisfied, with

$$\|p\|_\infty=\frac{1}{10},N=\frac{1}{5\sqrt{\pi}},\ \eta=\frac{2}{5\sqrt{\pi}}.$$

As for the hypothesis (H_3), we have

$$\begin{aligned} |\sigma(t,x)-\sigma(t,y)| &= \frac{e^{-t}}{2}\left|\frac{|x|}{(|x|+5)}-\frac{|y|}{(|y|+5)}\right|\\ &\leq \frac{1}{10}|x-y|. \end{aligned}$$

Then (H_3) is satisfied with

$$\zeta=\frac{1}{10},\ \sigma_0=0\ et\ \delta=0.$$

For the condition (H_4), we have

$$|g(x)-g(y)|\leq\frac{1}{8}|x-y|,$$

and then

$$G=\frac{1}{8}.$$

On the other hand, we have

$$\frac{2ln(6)}{5\sqrt{\pi}}+\frac{1}{8}+\frac{1}{10}+\frac{1}{5\sqrt{\pi}}-1<0.$$

From Lemma 4, the set of solution of the following inequality

$$\frac{1}{15\sqrt{\pi}}\rho^2+(\frac{2ln(6)}{5\sqrt{\pi}}+\frac{1}{8}+\frac{1}{10}+\frac{1}{5\sqrt{\pi}}-1)\rho+(0,001+\frac{1}{8}+\frac{1}{5\sqrt{\pi}})<0$$

is $[\rho_1,\rho_2]$, with

$$\rho_1=1.1,$$
$$\rho_2=7.5.$$

In addition, for any $\rho\in[\rho_1,\rho_2]$, we obtain

$$\frac{1-(\beta\gamma+G+\zeta+\beta N)}{\beta M}-1>\rho.$$

Consequently, Hypothesis (H_5) is verified. So the hypothesis of Theorem 2 hold, then the problem has a unique solution in B_ρ, for any $\rho\in[1.1,7.5]$.

Example 4. Consider the following $\Phi-Fractional$ functional differential equation.

$$\begin{cases} {}^cD^{\frac{1}{2}}_{0^+}(y(t)+\frac{1}{5}sin(t+y(t)))=\frac{cos(t^3+1)}{15(1+t\sqrt{t})^{\frac{1}{2}}}\Phi(t)^2|y|^{\frac{2}{3}}ln((\frac{|y|}{1+t\sqrt{t}})^{\frac{1}{3}}+1)y(t)+\frac{|y|\Phi(t)}{19+e^{2t}}+\frac{1}{10}, & t>0,\\ y(0)+\frac{1}{4}sin(y(1))=0,01. \end{cases} \tag{14}$$

Let

$$\begin{aligned}
\Phi(t)&=\frac{1}{1+t\sqrt{t}},\\
A_\Phi(t,y)&=\frac{cos(t^3+1)}{15(1+t\sqrt{t})^{\frac{1}{2}}}|y|^{\frac{2}{3}}ln((\frac{|y|}{1+t\sqrt{t}})^{\frac{1}{3}}+1)\Phi(t)^2,\ t\geq 0,\ y\in\mathbb{R},\\
\sigma(t,y)&=\frac{1}{5}sin(t+y(t)),\\
f_\Phi(t,y)&=\frac{|y|\Phi(t)}{19+e^{2t}}+\frac{1}{10},\\
g(y)&=\frac{1}{4}sin(y(1)).
\end{aligned}$$

For all $x,y\in\mathbb{R}$, we have

$$\begin{aligned}
|A_\Phi(t,x)-A_\Phi(t,y)|&\leq\frac{cos(t^3+1)}{15(1+t\sqrt{t})^{\frac{1}{2}}}\Phi(t)^2\left||x|^{\frac{2}{3}}ln((\frac{|x|}{1+t\sqrt{t}})^{\frac{1}{3}}+1)-|y|^{\frac{2}{3}}ln((\frac{|y|}{1+t\sqrt{t}})^{\frac{1}{3}}+1)\right|\\
&\leq\frac{cos(t^3+1)}{15(1+t\sqrt{t})^{\frac{1}{2}}}\Phi(t)^2|x-y|.
\end{aligned}$$

Hence the condition (H_1) holds, with

$$\|L\|_\infty=\frac{1}{15},\ \lambda=0,\ M=\frac{2}{15\sqrt{\pi}},\ \gamma=0,$$

$$\|\Phi\|_\infty\leq 1,\ \beta\leq\frac{\alpha^{\frac{\alpha}{\alpha+1}}}{1+\alpha}<1.$$

On the other hand, we get

$$|f_\Phi(t,x)-f_\Phi(t,y)| \leq \frac{1}{19+e^{2t}}\Phi(t)\Big||x|-|y|\Big| \leq \frac{1}{19+e^{2t}}\Phi(t)|x-y|.$$

So (H_2) is satisfied with

$$\|p\|_\infty = \frac{1}{20}, N = \frac{1}{10\sqrt{\pi}},\ \eta = \frac{2}{5\sqrt{\pi}}.$$

Concerning the hypothesis (H_3), we have

$$|\sigma(t,x)-\sigma(t,y)| = \frac{1}{5}\big|sin(t+x)-sin(t+y)\big| \leq \frac{1}{5}|x-y|.$$

Then (H_3) is satisfied with

$$\zeta = \frac{1}{5},\ \sigma_0 = 0\ and\ \delta = \frac{1}{5}.$$

And for assumption (H_4), we obtain

$$|g(x)-g(y)| \leq \frac{1}{4}|x-y|,$$

it is obvious that

$$G = \frac{1}{4}.$$

Finally, since $\frac{1}{10\sqrt{\pi}}+\frac{1}{4}+\frac{1}{5}-1<0$. Then, by using Lemma 4, the set of solution of the following inequality

$$\frac{2}{15\sqrt{\pi}}\rho^2 + (\frac{1}{10\sqrt{\pi}}+\frac{1}{4}+\frac{1}{5}-1)\rho + (0,01+\frac{1}{5}+\frac{2}{5\sqrt{\pi}}) < 0$$

is $[\rho_1,\rho_2]$, with $\rho_1 = 1.02$ and $\rho_2 = 5.9$. Additionally, we get $\frac{1-(\beta\gamma+G+\zeta+\beta N)}{\beta M}-1>\rho$, for any $\rho\in[\rho_1,\rho_2]$, which implies that assumption (H_5) is satisfied. So the hypotheses of Theorem 2 hold, and therefore the problem has a unique solution in B_ρ for any $\rho\in[1.0,5.9]$.

3.2 Example associated to Atangana-Baleanu Caputo fractional derivative

Now, we consider the following problems using Atangana-Baleanu Caputo fractional derivative operator:

$$\begin{cases} {}^{ABC}D^{\frac{1}{2}}_{0^+}(y(t)+\frac{1}{5}sin(t+y(t))) = \frac{cos(t^3+1)}{15(1+t\sqrt{t})^{\frac{1}{2}}}\Phi(t)^2|y|^{\frac{2}{3}}ln((\frac{|y|}{1+t\sqrt{t}})^{\frac{1}{3}}+1)y(t)+\frac{|y|\Phi(t)}{19+e^{2t}}+\frac{1}{10}, & t>0, \\ y(0)+\frac{1}{4}sin(y(1)) = 0,01, \end{cases} \tag{15}$$

with $\Phi(t) = \frac{1}{1+t\sqrt{t}}$.
And

$$\begin{cases} {}^{ABC}D^{\frac{1}{2}}_{0^+}(y(t)+\frac{|y(t)|e^{-t}}{2(|y(t)|+5)}) = \frac{e^{-t}ln(6+|y(t)|)}{5}\Phi(t)^2y(t)+\frac{sin(t)\Phi(t)}{5(|y(t)|^2+1)}, & t>0, \\ y(0)+\frac{1}{8}cos(y(1)) = 0,001, \end{cases} \tag{16}$$

with $\Phi(t) = \frac{e^{-t}}{1+t}$.

Example 5. Consider the following $\Phi-Fractional$ functional differential equation.

$$\begin{cases} {}^{ABC}D^{\frac{1}{2}}_{0^+}(y(t)+\frac{1}{5}sin(t+y(t))) = \frac{cos(t^3+1)}{15(1+t\sqrt{t})^{\frac{1}{2}}}\Phi(t)^2|y|^{\frac{2}{3}}ln((\frac{|y|}{1+t\sqrt{t}})^{\frac{1}{3}}+1)y(t)+\frac{|y|\Phi(t)}{19+e^{2t}}+\frac{1}{10}, & t>0, \\ y(0)+\frac{1}{4}sin(y(1)) = 0,01. \end{cases} \tag{17}$$

Let

$$\begin{aligned}
\Phi(t) &= \frac{1}{1+t\sqrt{t}}, \quad \|\Phi\|_\infty = 1 \\
A_\Phi(t,y) &= \frac{cos(t^3+1)}{15(1+t\sqrt{t})^{\frac{1}{2}}}|y|^{\frac{2}{3}}ln((\frac{|y|}{1+t\sqrt{t}})^{\frac{1}{3}}+1)\Phi(t)^2, \;\; t\geq 0,\; y\in\mathbb{R}, \\
\sigma(t,y) &= \frac{1}{5}sin(t+y(t)), \\
f_\Phi(t,y) &= \frac{|y|\Phi(t)}{19+e^{2t}}+\frac{1}{10}, \\
g(y) &= \frac{1}{4}sin(y(1)).
\end{aligned}$$

For all $x,y\in\mathbb{R}$, we have

$$\begin{aligned}
|A_\Phi(t,x)-A_\Phi(t,y)| &\leq \frac{cos(t^3+1)}{15(1+t\sqrt{t})^{\frac{1}{2}}}\Phi(t)^2\left||x|^{\frac{2}{3}}ln((\frac{|x|}{1+t\sqrt{t}})^{\frac{1}{3}}+1)-|y|^{\frac{2}{3}}ln((\frac{|y|}{1+t\sqrt{t}})^{\frac{1}{3}}+1)\right| \\
&\leq \frac{cos(t^3+1)}{15(1+t\sqrt{t})^{\frac{1}{2}}}\Phi(t)^2|x-y|.
\end{aligned}$$

Then the condition (H_1) holds, with

$$\|L\|_\infty = \frac{1}{15},\; \lambda=0,\; M=\frac{2}{15\sqrt{\pi}},\; \gamma=0,$$

$$\|\Phi\|_\infty \leq 1,\; \beta \leq \frac{\alpha^{\frac{\alpha}{\alpha+1}}}{1+\alpha} < 1.$$

In addition, we have

$$\begin{aligned}
|f_\Phi(t,x)-f_\Phi(t,y)| &\leq \frac{1}{19+e^{2t}}\Phi(t)\Big||x|-|y|\Big| \\
&\leq \frac{1}{19+e^{2t}}\Phi(t)|x-y|.
\end{aligned}$$

So (H_2) is satisfied with

$$\|p\|_\infty = \frac{1}{20}, N=\frac{1}{10\sqrt{\pi}},\; \eta=\frac{2}{5\sqrt{\pi}}.$$

Concerning the hypothesis (H_3), we have

$$\begin{aligned}
|\sigma(t,x)-\sigma(t,y)| &= \frac{1}{5}\big|sin(t+x)-sin(t+y)\big| \\
&\leq \frac{1}{5}|x-y|.
\end{aligned}$$

Then (H_3) is satisfied with

$$\zeta=\frac{1}{5},\; \sigma_0 = 0 \; and \; \delta=\frac{1}{5}.$$

Furthermore, we can get

$$|g(x)-g(y)| \leq \frac{1}{4}|x-y|,$$

it is obvious that

$$G=\frac{1}{4}.$$

Also, we have $\alpha = \frac{1}{2}$, $B(\alpha) = \frac{1}{2}(1+\frac{1}{\sqrt{\pi}})$, $\|y_0\| = 0,01$, $\|g(0)\| = 0$, $\varepsilon = \frac{\pi}{2(\sqrt{\pi}+1)}$, and $\|\sigma_0\| = 0$.
Then, the inequation (ii) of Lemma 5 becames

$$\begin{aligned}
&\big(-4\varepsilon M(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta+\varepsilon\eta)+\varepsilon(N+\gamma)^2\big)\|\Phi\|_\infty^2+\\
&\big(-4M\frac{\alpha}{B(\alpha)}\beta(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta+\varepsilon\eta)+\\
&2\frac{\alpha}{B(\alpha)}\beta\varepsilon(N+\gamma)^2+2(N+\gamma)(G+\zeta-1)\varepsilon\big)\|\Phi\|_\infty+\\
&(N+\gamma)^2(\frac{\alpha}{B(\alpha)}\beta)^2+2(N+\gamma)\frac{\alpha}{B(\alpha)}\beta(G+\zeta-1)+\\
&(G+\zeta-1)^2-4\frac{\alpha}{B(\alpha)}\beta\eta\\
&=-0.14766811<0.
\end{aligned}$$

Therefore, in this case, the problem does not admit a solution.

Example 6. Consider the following $\Phi-Fractional$ functional differential equation.

$$\begin{cases} {}^{ABC}D^{\frac{1}{2}}_{0^+}(y(t)+\frac{|y(t)|e^{-t}}{2(|y(t)|+5)}) = \frac{e^{-t}ln(6+|y(t)|)}{5}\Phi(t)^2y(t)+\frac{sin(t)\Phi(t)}{5(|y(t)|^2+1)}, & t>0, \\ y(0)+\frac{1}{8}cos(y(1)) = 0,001. \end{cases} \tag{18}$$

Let

$$\begin{aligned}
\Phi(t) &= \frac{e^{-t}}{1+t}, \ \ \|\Phi\|_\infty = 1\\
A_\Phi(t,y) &= \frac{e^{-3t}ln(6+|y(t)|)}{5(1+t)^2}, \ \ t\geq 0, \ y\in\mathbb{R},\\
\sigma(t,y) &= \frac{|y|e^{-t}}{2(|y|+5)},\\
f_\Phi(t,y) &= \frac{sin(t)e^-t}{5(|y(t)|^2+1)(1+t)},\\
g(y) &= \frac{1}{8}cos(y(1)).
\end{aligned}$$

For all $x,y\in\mathbb{R}$, we have

$$\begin{aligned}
|A_\Phi(t,x)-A_\Phi(t,y)| &\leq \frac{e^{-3t}}{5(1+t)^2}|ln(6+x)-ln(6+y)|\\
&\leq \frac{e^{-3t}}{30(1+t)^2}|x-y|.
\end{aligned}$$

Thus the operator $A(t,y)$ satisfies the hypothesis (H_1), with

$$\|L\|_\infty = \frac{1}{30},\ \lambda = \frac{ln(6)}{5},\ M = \frac{1}{15\sqrt{\pi}},\ \gamma = \frac{2ln(6)}{5\sqrt{\pi}},\ \|\Phi\|_\infty = 1,\ \beta = \frac{1}{2}^{\frac{1}{2}}exp(\frac{-1}{2}).$$

Moreover, we can get

$$\begin{aligned}
|f_\Phi(t,x)-f_\Phi(t,y)| &\leq \frac{|sin(t)|e^{-t}}{5(1+t)}\left|\frac{1}{(x^2+1)}-\frac{1}{(y^2+1)}\right|\\
&\leq \frac{|sin(t)|e^{-t}}{10(1+t)}|x-y|.
\end{aligned}$$

So (H_2) is satisfied, with

$$\|p\|_\infty = \frac{1}{10}, N = \frac{1}{5\sqrt{\pi}},\ \eta = \frac{2}{5\sqrt{\pi}}.$$

As for the hypothesis (H_3), we have

$$\begin{aligned}|\sigma(t,x)-\sigma(t,y)| &= \frac{e^{-t}}{2}\left|\frac{|x|}{(|x|+5)}-\frac{|y|}{(|y|+5)}\right| \\ &\leq \frac{1}{10}|x-y|.\end{aligned}$$

Then (H_3) is satisfied with

$$\zeta=\frac{1}{10},\ \sigma_0=0\ et\ \delta=0.$$

For the condition (H_4), we have

$$|g(x)-g(y)|\leq\frac{1}{8}|x-y|,$$

and therefore

$$G=\frac{1}{8}.$$

In addition, we have $\alpha=\frac{1}{2},\ \ B(\alpha)=\frac{1}{2}(1+\frac{1}{\sqrt{\pi}}),\ \ \|y_0\|=0,001,\ \ \|g(0)\|=\frac{1}{8},\ \ \varepsilon=\frac{\pi}{2(\sqrt{\pi}+1)}$, and $\|\sigma_0\|=0$. Then the inequation (ii) of Lemma 5 becames

$$\begin{aligned}&\big(-4\varepsilon M(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta+\varepsilon\eta)+\varepsilon(N+\gamma)^2\big)\|\Phi\|_\infty^2+\\ &\big(-4M\frac{\alpha}{B(\alpha)}\beta(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta+\varepsilon\eta)+\\ &2\frac{\alpha}{B(\alpha)}\beta\varepsilon(N+\gamma)^2+2(N+\gamma)(G+\zeta-1)\varepsilon\big)\|\Phi\|_\infty+\\ &(N+\gamma)^2(\frac{\alpha}{B(\alpha)}\beta)^2+2(N+\gamma)\frac{\alpha}{B(\alpha)}\beta(G+\zeta-1)+\\ &(G+\zeta-1)^2-4\frac{\alpha}{B(\alpha)}\beta\eta\\ &=-0.04339712468866<0.\end{aligned}$$

In this case, we conclude that the problem does not admit a solution. But if we change $\|\Phi\|_\infty=1$ by $\|\Phi\|_\infty=3$ (for example, we take $\Phi(t)=3\frac{e^{-t}}{1+t}$) then, we get

$$\begin{aligned}&\big(-4\varepsilon M(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta+\varepsilon\eta)+\varepsilon(N+\gamma)^2\big)\|\Phi\|_\infty^2\\ &+\big(-4M\frac{\alpha}{B(\alpha)}\beta(\|y_0\|+\|\sigma_0\|+\|g(0)\|+\delta+\varepsilon\eta)+\\ &2\frac{\alpha}{B(\alpha)}\beta\varepsilon(N+\gamma)^2+2(N+\gamma)(G+\zeta-1)\varepsilon\big)\|\Phi\|_\infty+\\ &(N+\gamma)^2(\frac{\alpha}{B(\alpha)}\beta)^2+2(N+\gamma)\frac{\alpha}{B(\alpha)}\beta(G+\zeta-1)+\\ &(G+\zeta-1)^2-4\frac{\alpha}{B(\alpha)}\beta\eta \qquad =0.17>0.\end{aligned}$$

In this case, the $\Delta=0,17>0$ of the equation $E(\rho')=0$. Hence the problem admit a solution.

Remark

From Lemmas 4 and 5, and Theorems 2 and 3, we remark that our $\Phi-$fractional differential equations problems depend only on $\|\Phi\|_\infty$, which is illustrated by our examples. This leads us to introduce for any bounded, continuous and decreasing function $\Phi:\mathbb{R}^+\to(0,a]$, where $a>0$, a set for each problems (6) and (7) as follows:

$$H_\Phi=\{\varphi:\mathbb{R}^+\to(0,a] \text{ is a bounded, continuous and decreasing function such that } \|\Phi\|_\infty=\|\varphi\|_\infty\}.$$

Then, it is evident to see that if we change Φ by any $\varphi\in H_\Phi$ which satisfies the hypotheses of Lemma 4 and Theorem 2, in our problem (6), (respectively of Lemma 5 and Theorem 3 in our problem (7)), then this last admits an unique solution in B_ρ, where $\rho\in[\rho_1,\rho_2]$ the set of solutions of $E(\rho)\leq 0$, (respectively $\rho\in[\rho_1',\rho_2']$ the set of solutions of $E(\rho')\leq 0$).

4 Conclusion

As we have seen, we are introducing a new class of fractional differential equations named by $\Phi-$fractional differential equations and we are interested in the study of the existence and uniqueness of the solution of this class for the both Caputo and Atangana–Baleanu Caputo derivative operator. Moreover, we successfully proved the global existence and uniqueness of the solution that depend on $\|\Phi\|_\infty$ see Lemma 4 and Theorem 2 for the study with the Caputo derivative operator and Lemma 5 and Theorem 3 for the study with Atangana–Baleanu Caputo derivative operator. In addition, in the above remark, we presented the set H_Φ associated to our problems. Finally, we gave some examples that confirmed the applicability of the assumptions defined in Theorem 2 and 3. Examples 4 and 5 (respectively 2 and 6) provide a comparison between the study of the same problem with the Caputo derivative operator and the Atangana–Baleanu Caputo derivative operator.

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