Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/160517

Generalized Z-Entropy (Gze) and Fractal Dimensions

Ismail A. Mageed[1] *and Ashiq Hussain Bhat*[2,*]

[1]Department of Computer Science, Faculty of Engineering and Informatics, University of Bradford, United Kingdom, BD7 1DP, UK
[2]Population Research Centre (PRC), University of Kashmir, Srinagar-190006, India

Received: 2 Jul. 2022, Revised: 2 Aug. 2022, Accepted: 13 Aug. 2022
Published online: 1 Sep. 2022

Abstract: In this paper, we reveal the relation between the most general entropy, namely Generalized Z-Entropy (Gze) and the fractal dimension, a statistical index which is measuring the complexity of a given pattern, embedded in given spatial dimensions. Numerical experiments are undertaken to interpret the behaviour of the derived (Gze) fractal dimension corresponding to its parameters. This work reports the ultimate generalization in the literature. Moreover, this paper unifies information theory with fractal geometry.

Keywords: Fractal dimension, Fractal geometry, Generalized Z-Entropy, Scaling factor

1 Introduction

In 1948 [1], Claude Shannon defined the entropy $H(X)$ for a discrete random variable X, as given by

$$H(X) = \sum_i p(x_i) I(x_i) = \sum_i p(x_i) \ln(p(x_i)) \quad (1)$$

In this expression, the probability of i-event is $p(x_i)$. In information theory, this entropy defines the measure of information. Several other entropic formalisms are available for having different approaches to the measure of information, which is present in each distribution. Here, we will discuss in a simple approach, the link between entropy and the fractal dimension. The fractal dimension is a statistical index, measuring the complexity of a given pattern, which is embedded in given spatial dimensions. It has also been characterized as a measure of the space-filling capacity of a pattern that tells how a fractal scales differently from the space it is embedded in [2,3,4]. The idea of a fractional approach to calculus has a long history in mathematics (c.f., [5]), but the term became popular with the works of Benoit Mandelbrot, in particular from his 1967 paper where he discussed the fractional dimensions [6]. In [6], Mandelbrot cited a previous work by Lewis Fry Richardson, who was discussing how a coastline's measured length can change with the length of the rigid stick used for measurements. In this manner, the fractal dimension of a coastline is provided by the number of rigid sticks, required to measure the coastline, and by the scale of the used stick. [7] Several formal mathematical definitions of fractal dimension exist in this framework, following formulas are given, where N stands for the number of sticks used to cover the coastline, ε is the scaling factor, and D the fractal dimension:

$$N \propto \varepsilon^{-D} \quad (2)$$

$$\ln N = -D = \frac{\ln N}{\ln \varepsilon} \quad (3)$$

Let us see this example. We use Google Earth satellite images and GIMP (the GNU Image Manipulation Program) to have a map and rigid sticks to repeat what Richardson considered. Here, in the Figure 1, it is shown the same approach for a part of Grand Canyon. The ruler tool of Google Earth is used to establish the reference length. In the left-upper panel, we have the rulers for 6 km, 3 km, and 1 km. To determine the fractal dimension, we choose as reference length that of 6 km. In the left-lower panel, we can see that we need about 13 rigid sticks, one-half the reference length long, to follow the rim of this part of the canyon. In the case that we used a stick, which is 1/6 long, we need 44 sticks. We can go on reducing the length of sticks.

The illustrated portraits in Fig.1, are based on Google Earth satellite images of a part of Grand Canyon, Arizona. Rigid sticks are created by GIMP. The ruler tool of Google Earth is used to establish the reference length. For evaluating the fractal dimension of the rim of the canyon, we choose as reference length that of 6 km. In the

* Corresponding author e-mail: ashiqhb14@gmail.com

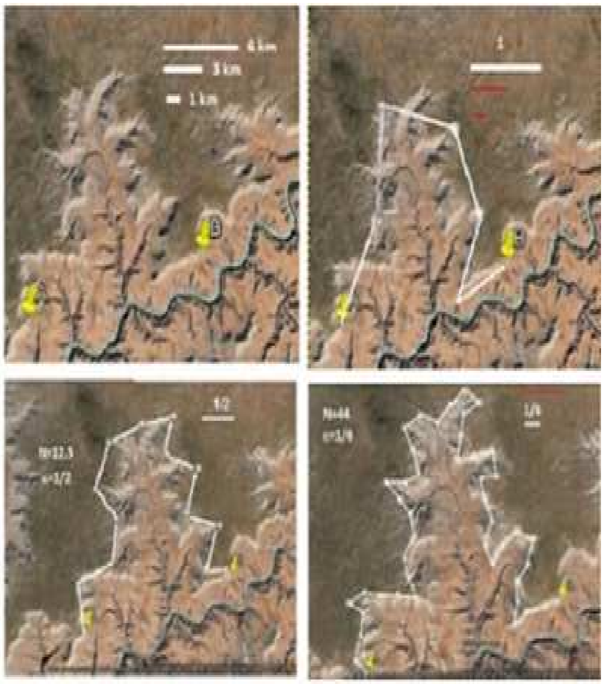


Fig. 1: Google Earth Satellite images and GNU Image Manipulation Program of a part of Grand Canyon, Arizona

left-lower panel, we can see that we need about 13 rigid sticks, one-half the reference length long, to follow its rim. With a stick 1/6 long, we need 44 sticks.

Table 1

N	ε	D
13	$1/2$	3.70
44	$1/6$	2.11
119	$1/12$	1.92
405	$1/30$	1.72
871	$1/60$	1.65

In the Table 1, considering the case of the Fig. 1, we give the fractal dimension of the rim. Of course, when the scaled sticks are smaller, we need more images, here not shown. The process should be further iterated, to reach the limit of smaller scales. Therefore, the fractal dimension of the rim of the canyon, defined as the boundary between flat soil and steep terrain, is a number between 1 and 2. The proposed approach illustrates an example to show the method to evaluate experimentally a fractal dimension. In the given framework, let us consider the role of probability. Each rigid stick has the same probability to be used and then sticks have a uniform distribution. In probability, the discrete uniform distribution is a probability distribution of a finite number N of values, which are equally likely to be observed; every one of N values have then the equal probability 1/N. An example of discrete uniform distribution is that we obtain by throwing a die. If the die has 6 faces, the possible values are $1,2,3,4,5,6$ each time the die is thrown the probability of a given score is 1/6. In Fig. 2, the significant impact of N (the number of sticks used to cover the coastline) on ε (the scaling factor) is illustrated. More interestingly, Fig.3, provides strong supporting evidence of the impact of FD (fractal dimension) on ε (the scaling factor). Clearly, by looking at Fig. 2, we can see that the scaling factor decreases with a very heavy tailed trend by the increase of N, whereas, in Fig. 3, the portrayed data shows that both ε(the scaling factor) and FD(fractal dimension) are decreasing at the same time.

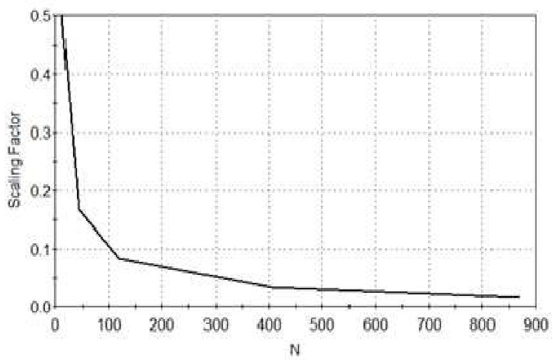


Fig. 2: An illustrative data portrait of how N impacts ε (Scaling Factor)

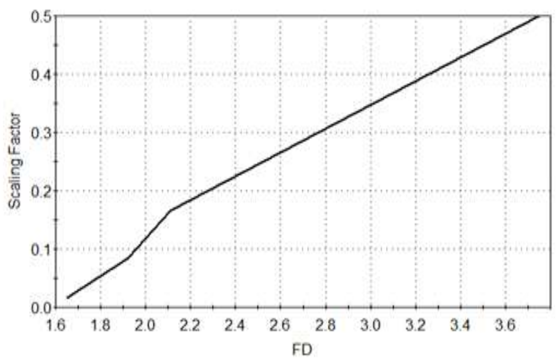


Fig. 3: An illustrative data portrait of how FD impacts ε (Scaling Factor)

The current paper is organized as follows. Section 2 overviews the available work done in the entropic derivation of fractal dimension. Section 3 mainly deals with the derivation of the new resultsand provides numerical portrait which clearly supports the strong evidence of the significant impact of the (Gze)?s parameters on the behaviour of the (Gze) factional dimension. Section 4 is devoted to conclusion and future work.

2 Materials and Methods

In [8], the author has derived the corresponding fractal dimension to Shannon entropy [1], Renyi entropy [9,10] (with q entropic index), Tsallis entropy [11] (with q entropic index) and Kaniadakis entropy [12] (with entropic index k). All these derivations were undertaken in the case of equiprobable distribution.
The Shannonian entropic [8] definition of the fractal dimension is given by:

$$D_s = \lim_{\varepsilon \to 0} \frac{\ln N}{\frac{1}{\varepsilon}} \tag{4}$$

The "generalized dimension" or the "Renyi dimension" of order $q \in (0.5, 1)$, is defined in the following manner [10]

$$D_R = \lim_{\varepsilon \to 0} \frac{\ln N}{\frac{1}{\varepsilon}} \tag{5}$$

The "generalized dimension" or the "Tsallisian dimension" of order $q \in (0.5, 1)$, is defined in the following manner [8]

$$D_T = \lim_{\varepsilon \to 0} \frac{\frac{1}{1-q}\left(N^{1-q} - 1\right)}{\frac{1}{\varepsilon}}, \quad D_T > 0 \tag{6}$$

Now, we can define the Kaniadakis generalized dimension [8] for Kaniadakis entropy K (k is the entropic index):

$$D_K = \lim_{\varepsilon \to 0} \frac{\frac{1}{2k}\left(N^k - N^{-k}\right)}{\frac{1}{\varepsilon}} \tag{7}$$

The behaviour of these generalized dimension when their indices are varied can be seen. The case of the Koch snowflake (N = 4 and $\varepsilon = 1/3$) was proposed in [8] to determine the corresponding fractal dimension to each entropy. Notably, it was conjectured in [8], that Tsallis generalized statistics seem to be the natural frame for studying fractal systems.

3 Results and Discussion

The proposed Generalized Z-Entropy (GZE) [13] is a non-extensive entropy functional defined by:

$$\mathrm{H}_{\mathrm{q,a,b,Z}}(\mathrm{p}) = Z_{a,b} = \frac{\mathrm{c}}{(1-\mathrm{q})(\mathrm{a}-\mathrm{b})}\left[\left(\sum_n p_{q,\mathrm{Z}}(n)^q\right)^a - \left(\sum_n p_{q,\mathrm{Z}}(n)^q\right)^b\right] \tag{8}$$

Where, c is a positive constant, $1 > q > 0.5, a > 0, b \in \mathbb{R}$ or $b > 0, a \in \mathbb{R}$ with $a \neq b$.

The following proposition is of great importance as it solidifies our choice of using the proposed GZE in our study.

Proposition 1.*(c.f., P. Tempesta, [13])*

i. The $Z_{a,b}$-entropy reduces to the Renyi entropy in the double limit, $a \to 0, b \to 0$.
ii. The $Z_{a,b}$-entropy reduces to the Tsallis entropy in the double limit, $a \to 1, b \to 0$.
iii. The $Z_{a,b}$-entropy reduces to the $Z_{k,q}$= k-entropy (Kaniadakisian entropy functional) in the limit $a = -b = k$.
iv. The $Z_{a,b}$-entropy reduces to the Sharma-Mittal entropy [14] in the limit $b \to 0$.
v. The $Z_{a,b}$-entropy reduces to the Shannonian entropy functional in the triple limit $a \to 0, b \to 0, q \to 1$.

Theorem 1.*In the case of equi-probable distribution, the GZE fractal dimension, $D_{a,b}$ is devised by*

$$D_{Z_{a,b}} = \lim_{\varepsilon \to 0} \frac{\frac{1}{(1-q)(a-b)}\left(N^{(1-q)a} - N^{(1-q)b}\right)}{\frac{1}{\varepsilon}} \tag{9}$$

Provided that, $1 > q > 0.5, a > 0, b \in R$ or $b > 0, a \in R$ with $a \neq b$.

*Proof.*By the definition, we have

$$\begin{aligned}
D_{Z_{a,b}} &= \frac{1}{(1-q)(a-b)} \lim_{\varepsilon \to 0} \frac{\left(\sum_n p_{q,Z}(n)^q\right)^a - \left(\sum_n p_{q,Z}(n)^q\right)^b}{\frac{1}{\varepsilon}} \\
&= \frac{1}{(1-q)(a-b)} \lim_{\varepsilon \to 0} \frac{\left(\sum_{i=1}^N \left(\frac{1}{N}\right)^q\right)^a - \left(\sum_{i=1}^N \left(\frac{1}{N}\right)^q\right)^b}{\frac{1}{\varepsilon}} \\
&= \frac{1}{(1-q)(a-b)} \lim_{\varepsilon \to 0} \frac{\left(\frac{1}{N}\right)^{aq}\left(\sum_{i=1}^N 1\right)^a - \left(\frac{1}{N}\right)^{bq}\left(\sum_{i=1}^N 1\right)^b}{\frac{1}{\varepsilon}} \\
&= \frac{1}{(1-q)(a-b)} \lim_{\varepsilon \to 0} \frac{\left(\frac{1}{N}\right)^{aq}(N)^a - \left(\frac{1}{N}\right)^{bq}(N)^b}{\frac{1}{\varepsilon}} \\
&= \frac{1}{(1-q)(a-b)} \lim_{\varepsilon \to 0} \frac{\left(N^{(1-q)a} - N^{(1-q)b}\right)}{\frac{1}{\varepsilon}}, \text{ as claimed, } \left(\text{c.f., } (9)\right)
\end{aligned}$$

This completes the proof.

Corollary 1.*The GZE fractal dimension, $D_{Z_{a,b}}$ satisfies the following:*

i. $\lim_{a \to 0, b \to 0} D_{Z_{a,b}} = D_R$
ii. $\lim_{q \to 1}\left(\lim_{a \to 0, b \to 0} D_{Z_{a,b}}\right) = D_s$
iii. $\lim_{a \to 1, b \to 0} D_{Z_{a,b}} = D_T$
iv. $\lim_{a \to k, b \to -k} D_{Z_{a,b}} = D_K$
v. $\lim_{b \to 0} D_{Z_{a,b}} = D_{\text{Sharma-Mittal}}$

*Proof.*It could be verified that $D_{Z_{a,b}}$(c.f., (9)) satisfies the following:

$$\text{i. } \lim_{a\to 0,b\to 0} D_{Z_{a,b}} = \lim_{\varepsilon\to 0} \frac{\lim_{a\to 0,b\to 0} \frac{1}{(1-q)(a-b)} \left(N^{(1-q)a} - N^{(1-q)b}\right)}{\frac{1}{\varepsilon}}$$

$$= \lim_{\varepsilon\to 0} \frac{\lim_{a\to 0} \frac{1}{(1-q)(1)} \left((1-q)N^{(1-q)a} \ln N\right)}{\frac{1}{\varepsilon}}$$

$$= \lim_{\varepsilon\to 0} \frac{\ln N}{\frac{1}{\varepsilon}} = D_R \text{ (c.f .,(5))} \quad (3.1)$$

ii.Taking the limit both sides of (3.1) as $q \to 1$, we have

$$\lim_{q\to 1}\left(\lim_{a\to 0,b\to 0} D_{Z_{a,b}}\right) = \lim_{q\to 1}\left(\lim_{\varepsilon\to 0} \frac{\ln N}{\frac{1}{\varepsilon}}\right)$$

$$= \lim_{\varepsilon\to 0} \frac{\ln N}{\frac{1}{\varepsilon}} = D_s \text{ (c . f .,(4))} \quad (3.2)$$

$$\text{iii. } \lim_{a\to 1,b\to 0} D_{Z_{a,b}} = \lim_{\varepsilon\to 0} \frac{\lim_{a\to 1,b\to 0} \frac{1}{(1-q)(a-b)} \left(N^{(1-q)a} - N^{(1-q)b}\right)}{\frac{1}{\varepsilon}}$$

$$= \lim_{\varepsilon\to 0} \frac{\lim_{a\to 1} \frac{1}{(1-q)a} \left(N^{(1-q)a} - 1\right)}{\frac{1}{\varepsilon}}$$

$$= \lim_{\varepsilon\to 0} \frac{\frac{1}{(1-q)} \left(N^{(1-q)} - 1\right)}{\frac{1}{\varepsilon}} = D_T \text{ (c . f .,(6))} \quad (3.3)$$

$$\text{iv. } \lim_{a\to k,b\to -k} D_{Z_{a,b}} = \lim_{\varepsilon\to 0} \frac{\lim_{a\to k,b\to -k} \frac{1}{(1-q)(a-b)} \left(N^{(1-q)a} - N^{(1-q)b}\right)}{\frac{1}{\varepsilon}}$$

$$= \lim_{\varepsilon\to 0} \frac{\frac{c}{(1-q)(k+k)} \left((1-q)N^{(1-q)k} - (1-q)N^{-(1-q)k}\right)}{\frac{1}{\varepsilon}}$$

$$= \lim_{\varepsilon\to 0} \frac{\frac{c}{(2k)} \left(N^{(1-q)k} - N^{-(1-q)k}\right)}{\frac{1}{\varepsilon}}$$

Define $(1-q)k = \lambda_k, c = \frac{1}{1-q}$,$q \in (0.5, 1)$
Hence,it follows that

$$\lim_{a\to k,b\to -k} D_{\mathbb{Z}_{a,b}} = \lim_{\varepsilon\to 0} \frac{\frac{1}{(2\lambda_k)} \left(N^{\lambda_k} - N^{-\lambda_k}\right)}{\frac{1}{\varepsilon}} = D_K\text{(c.f., (7))} \quad (3.4)$$

$$\text{v. } \lim_{b\to 0} D_{\mathbb{Z}_{a,b}} = \lim_{\varepsilon\to 0} \frac{\lim_{b\to 0} \frac{1}{(1-q)(a-b)} \left(N^{(1-q)a} - N^{(1-q)b}\right)}{\frac{1}{\varepsilon}}$$

$$= \lim_{\varepsilon\to 0} \frac{\frac{1}{(1-q)} \left(N^{(1-q)a} - 1\right)}{\frac{1}{\varepsilon}} = D_{\text{Sharma-Mittal}} \quad (3.5)$$

4 Numerical experiments

We have determined that:

$$D_{Z_{a,b}} = \lim_{\varepsilon\to 0} \frac{\frac{1}{(1-q)(a-b)} \left(N^{(1-q)a} - N^{(1-q)b}\right)}{\frac{1}{\varepsilon}} \text{ (c.f., (9))}$$

Following the Koch snowflake ($N = 4$ and $\varepsilon = 1/3$), we have

$$D_{\mathbb{Z}_{a,b}} = \frac{3}{(1-q)(a-b)} \left(4^{(1-q)a} - 4^{(1-q)b}\right) \quad (10)$$

It can be seen that

$$D_{\mathbb{Z}_{2,-1}}(q) = \frac{1}{(1-q)} \left(4^{2(1-q)} - 4^{(q-1)}\right) \quad (11)$$

and

$$D_{\mathbb{Z}_{-2,1}}(q) = -\frac{1}{(1-q)} \left(4^{-2(1-q)} - 4^{(1-q)}\right) \quad (12)$$

Moreover, it holds that

$$\lim_{q\to 1} D_{\mathbb{Z}_{2,-1}}(q) = \lim_{q\to 1} \frac{1}{(1-q)} \left(4^{2(1-q)} - 4^{(q-1)}\right)$$
$$= \ln 4 \lim_{q\to 1} \left(2\left(4^{2(1-q)}\right) + 4^{(q-1)}\right)$$
$$= 3\ln 4 = 4.158883083 \quad (4.1)$$

Also, we have

$$\lim_{q\to 1} D_{\mathbb{Z}_{-2,1}}(q) = \lim_{q\to 1} -\frac{1}{(1-q)} \left(4^{-2(1-q)} - 4^{(1-q)}\right)$$
$$= \ln 4 \lim_{q\to 1} \left(2\left(4^{-2(1-q)}\right) + 4^{(1-q)}\right)$$
$$= 3\ln 4 = 4.158883083 \quad (4.2)$$

Thus, we see that

$$\lim_{q\to 1} D_{\mathbb{Z}_{2,-1}}(q) = \lim_{q\to 1} D_{\mathbb{Z}_{-2,1}}(q) = 4.15888308$$

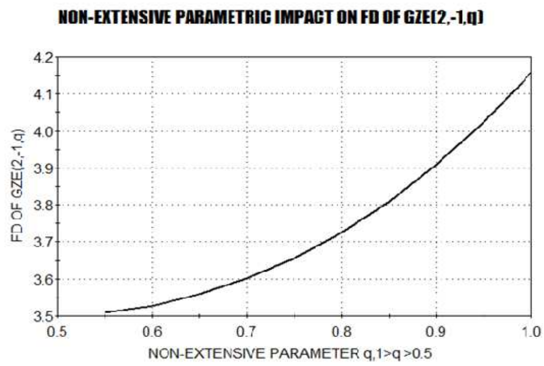


Fig. 4: IncreasabilityDecreasability of $D_{\mathbb{Z}_{2,-1}}(q)$ against against the non-extensive information theoretic parameter $q, 1 > q > 0.5$

It is observed from both the figures 4 and 5, that $D_{\mathbb{Z}_{2,-1}}(q)$ and $D_{\mathbb{Z}_{-2,1}}(q)$ are increasing in q, $q \in (0.5, 1)$ .This provides a strong evidence to support the significant impact of the non-extensive information theoretic parameter q,($1 > q > 0.55$) on the overall behaviour of both $D_{\mathbb{Z}_{2,-1}}(q)$ and $D_{\mathbb{Z}_{-2,1}}(q)$.

5 Conclusion and future work

In this paper, an exposition to reveal the relation between the most general entropy, namely Generalized Z-Entropy (GZE) and the fractal dimension, a statistical index which is measuring the complexity of a given pattern, embedded in given spatial dimensions is undertaken. Numerical experiments are to interpret the behaviour of the derived

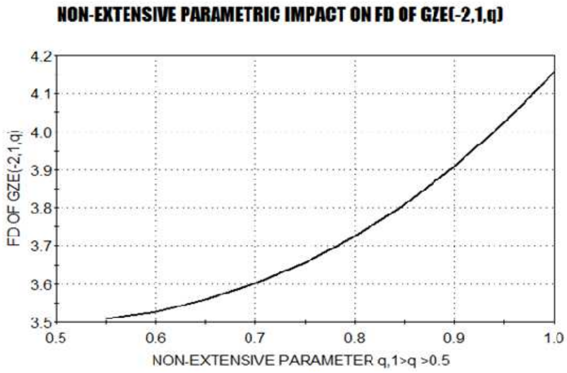


Fig. 5: Increasability of $D_{\mathbb{Z}_{-2,1}}(q)$ against against the non-extensive information theoretic parameter $q, 1 > q > 0.5$

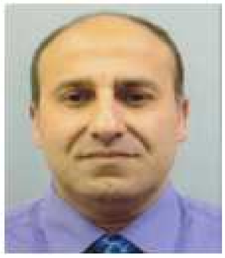
(GZE) fractal index corresponding to its parameters. This work reports the ultimate generalization in the literature. This paper is a giant step towards the unification of information theory with fractal geometry. Future work involves finding the fractal dimension of available entropies in the literature to draw a detailed comparison between these derived fractal dimensions, which will open new grounds towards Information Theoretic Fractal Geometry (ITFG).

Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] C. E. Shannon, A Mathematical Theory of Communication, *Bell System Technical Journal*, **27** , 379–423 (1948).
[2] K. Falconer, *Fractal Geometry* Wiley, New York, (2003).
[3] H. Sagan, *Space - Filling Curves* Springer-Verlag, Berlin, (1994).
[4] T. Vicsek, *Fractal growth phenomena* World Scientific, Singapore New Jersey, (1992).
[5] A. C. Sparavigna, *Fractional differentiation-based image processing*, arXiv preprint arXiv, (2009).
[6] B. B. Mandelbrot, *The fractal geometry of nature*, W.H. Freeman, New York, (1983).
[7] D. Harte , *Multifractals* , Chapman and Hall, London, (2001).
[8] A. C. Sparavigna, *Entropies and fractal dimensions*, Philica, (2016).
[9] A. Renyi, *On Measures of Information and Entropy*, Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability,**4**, 547–562 (1961).
[10] E. Ott, *Attractor dimensions*, Scholarpedia, **3**(**3**), 2110 (2008)
[11] C. Tsallis, Possible Generalization of Boltzmann-Gibbs Statistics, *Journal of Statistical Physics*, **52(1)**, 479-487 (1988).
[12] G. Kaniadakis, Statistical Mechanics in the Context of Special Relativity, *Physical Review E*, **66(5)**, 056-125 (2002).
[13] P. Tempesta, *Formal groups and Z-entropies*, Proceedings of the Royal Society A:Mathematical, Physical and Engineering Sciences, **472(2195)** (2006).
[14] B. D. Sharma, D. P. Mittal, New nonadditive measures of entropy for discrete probability distributions, *J. Math. Sci.*,**10** 28-40 (1975).



Ismail A Mageed completed his doctorate in Applied Probability at The University of Bradford, United Kingdom. His current research interests include the unification of queueing theory with information theory and information geometry. His leading research on the relativisation of queuing theory and discovering the geodesic equation of motion for transient queues was greatly received by the world research community. Mageed's research on finding the analytic solutions of the longstanding simulative approach of The Pointwise Stationary Fluid Flow Approximation theory (PSFFA) was an exceptional discovery to advance PSFFA theory. Dr Mageed has published numerous papers in many highly reputablejournals and conferences. He is also a reviewer to several prestigious journals. Mageed?s research has been internationally recognized as being revolutionary by providing several breakthroughs. He is currently an active member at the NetPen Research Group, which is the strongest research group in queueing networks in the world. Dr Mageed has published a chapter in a book of the best eight queueing theorists in the world by ISTE WILLEY. He is currently leading several research teams in UK, India, Africa and Saudi Arabia. He is also a fellow of the Royal Statistical Society (RSS), a member of INTISCC (Austria), IEANG (world council of engineering) and a life member of the Islamic Society of Statistical Sciences.

Ashiq Hussain Bhat is a researcher in the field of Statistics (Information Theory, Coding theory, Fuzzy measures) and presently working as Research Fellow at Population Research Centre (PRC) University of Kashmir, Srinagar, (J and K), India. He has completed his Ph.D from Department of Statistics, University of Kashmir, Srinagar (J and K), India on topic entitled "Generalization of Fuzzy Divergence Measures and Their Applications". He has published several research articles in reputed international journals of statistical and mathematical sciences with good indexing like Scopus, Copernicus, Zentralblatt and so on. He has attended more than five training programs and workshops regarding Statistical Software learning for biological and other applied fields of data analysis which include R Studio, SPSS, Minitab, Advanced Excel, Latex software learning for different practical fields. He has attended and presented in more than ten national and international conferences.