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Analytical Approximations for a System of Fractional Partial Differential Equations

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Abstract: In this work, we evaluate a system of partial differential equations utilizing the Caputo fractional operator by employing the fractional natural decomposition analysis (FNDM). The approximate analytical solutions are derived by utilizing the FNDM technique, which is a form of the fractional Adomian decomposition with the natural transform. This novel algorithm's high accuracy and rapid convergence are demonstrated with illustrative cases. The obtained findings demonstrate that the proposed method is a viable tool for solving systems of nonlinear fractional differential equations. Additionally, we demonstrate that FNDM can handle a broad class of nonlinear systems using the Caputo fractional operator more efficiently, clearly, and correctly, making it extensively useful in physics and engineering.

Keywords: Fractional differential equations, Adomian decomposition method, natural transform, Caputo fractional operators.

1 Introduction

Fractional partial fractional equations (FPDEs) naturally occur in a variety of fields, including physics, biology, economics, and engineering applications such as electrostatics, fluid mechanics, astronomy, and relaxation processes. Many professionals use such models extensively to simply explain its intricate structures, simplify the regulating design without sacrificing hereditary behaviors, and produce natural concerns that are closely understood for these occurrences[1,2,3,4,5].

The most approximate and observational methods and techniques, such as fractional variational iteration technique, methodology of fractional differential transformation, technique for expanding fractional series, Iteration technique based on fractional Sumudu variation, Laplace transform with a fraction, The fractional homotopy perturbation approach, Decomposition technique using fractions of Sumudu, technique utilizing fractional Fourier series, In the meaning of Caputo, ordinary and partial differential equations, the fractional reduced differential transform procedure, fractional Adomian decomposition method, and other methods [6,7,8,9,10,11,12]have successfully been applied.

The FNDM, a coupling strategy of the FADM and NT, is what we're trying to introduce, and to use it to resolve nonlinear fractional PDEs. The following sections make up the remaining portion of this work: Some definitions for fractional calculus are provided in Section 2. Section 3 discusses some fundamental definitions and features of natural transforms. The FNDM with CFO analysis is carried out in section 4. Section 5 demonstrates FNDM applications. The study's conclusion can be found in Section 6.

2 Preliminaries

This part covers certain fractional calculus concepts and symbols that will be useful in this investigation [1,2,3,4,5].

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Definition 1. *Suppose $\psi(\tau), \psi(\tau) \in R,\ \tau > 0$, which is in the space $C_m, m \in R$ if there exists*

$$\{\rho, (\rho > m), s.t. \psi(\tau) = \tau^{\rho}\psi_1(\tau),\ where\ \psi_1(\tau) \in C[0,\infty)\}$$

and $\psi(\tau)$ is known as in the space C_m^n when $\psi^{(n)} \in C_m, n \in N$.

Definition 2. *The fractional integral operator of order $\alpha \geq 0$ for Riemann Liouville of $\psi(\tau) \in C_m, m \geq -1$ is given by the form*

$$I^{\alpha}\psi(\tau) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^{\tau} (\tau-\xi)^{\alpha-1}\psi(\xi)d\xi, & \alpha > 0, \tau > 0 \\ I^0\psi(\tau) = \psi(\tau), & \alpha = 0 \end{cases} \quad (1)$$

where $\Gamma(\cdot)$ is the well-known Gamma function.

The following are the characteristics of the operator I^{α}: For $\psi \in C_m, m \geq -1, \alpha, \beta \geq 0$, then

1. $I^{\alpha}I^{\beta}\psi(\tau) = I^{\alpha+\beta}\psi(\tau)$
2. $I^{\alpha}I^{\beta}\psi(\tau) = I^{\beta}I^{\alpha}\psi(\tau)$
3. $I^{\alpha}\tau^n = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}\tau^{\alpha+n}$

Definition 3. *In the understanding of Caputo, $\psi(\tau)$'s fractional derivative is as follows:*

$$D^{\alpha}\psi(\tau) = I^{n-\alpha}D^n\psi(\tau) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\tau}(\tau-\xi)^{n-\alpha-1}\psi^{(n)}(\xi)d\xi, \quad (2)$$

for $n-1 < \alpha \leq n, n \in N, \tau > 0$ and $\psi \in C_{-1}^n$.

The following are the essential features of the operator D^{α}:

1. $D^{\alpha}I^{\alpha}\psi(\tau) = \psi(\tau)$
2. $D^{\alpha}I^{\alpha}\psi(\tau) = \psi(\tau) - \sum_{k=0}^{n-1}\psi^{(k)}(0)\frac{\tau^k}{k!}$

Definition 4. *The following formula gives the Mittag-Leffler function E_{α} if it satisfies the following: For each $\alpha > 0$, then:*

$$E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{z^{\alpha}}{\Gamma(n\alpha+1)} \quad (3)$$

3 Natural Transform Definitions and Properties

We present some context for the natural transform approach [9] in this section.

Definition 5. *The function $\psi(\tau)$ for $\tau \in R$ has a natural transform defined by*

$$\mathbb{N}[\psi(\tau)] = R(\omega,\mu) = \int_{-\infty}^{\infty} e^{-\omega\tau}\psi(\mu\tau)d\tau, \quad \omega,\mu \in (-\infty,\infty) \quad (4)$$

We denote that the Natural transform of the time function $\psi(\tau)$ is $\mathbb{N}[\psi(\tau)]$, and the variables ω and μ are the Natural transform elements. Furthermore, define $\psi(\tau)\mathscr{H}(\tau)$ as on the axis of positive real, if $\mathscr{H}(\tau)$ is Heaviside function, and $\tau \in (0,\infty)$. Consider

$$\mathscr{A} = \{\psi(\tau) : \exists M, t_1, t_2 > 0, with\ |\psi(\tau)| \leq Me^{\frac{|\tau|}{t_j}},\ for\ \tau \in (-1)^j \times [0,\infty), j \in Z^+\}$$

The natural transform, often known as the $\mathbb{NT}$, is defined as follows:

$$\mathbb{N}[\psi(\tau)\mathscr{H}(\tau)] = \mathbb{N}^+[\psi(\tau)] = R^+(\omega,\mu) = \int_0^{\infty} e^{-\omega\tau}\psi(\mu\tau)d\tau, \quad \omega,\mu \in (-\infty,\infty) \quad (5)$$

The natural transform has the following essential characteristics:

1. $\mathbb{N}^+[1] = \frac{1}{\omega}$
2. $\mathbb{N}^+[\tau^{\alpha}] = \frac{\Gamma(\alpha+1)\mu^{\alpha}}{\omega^{\alpha+1}}$, *such that $\alpha \geq -1$*

4 Fractional Natural Adomian Decomposition Method (FNADM) Analysis

Suppose that the general fractional nonlinear PDEs with Caputo fractional operator for the system

$$D_\tau^\alpha \psi_i(\xi,\tau)+\mathscr{R}\psi_i(\xi,\tau)+\mathscr{N}\psi_i(\xi,\tau)=\Im_i(\xi,\tau), \quad 0<\alpha\leq 1 \tag{6}$$

with the initial condition

$$\psi_i(\xi,0)=\Im_i(\xi), \tag{7}$$

where $D_\tau^{(\alpha)}\psi_i(\xi,\tau)$ which is the CF derivative of $\psi_i(\xi,\tau)$, $i=1,2$ and 3 defined as:

$$D_\tau^{(\alpha)}\psi_i(\xi,\tau)=\frac{d^\alpha\psi_i(\xi,\tau)}{d\tau^\alpha}\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\int_0^\tau(\tau-\omega)^{n-\alpha-1}\frac{d^n\psi_i(\xi,\tau)}{d\tau^n}d\omega, & n-1<\alpha<n\\ \frac{d^n\psi_i(\xi,\tau)}{d\tau^n}, & \alpha=n\in N\end{cases} \tag{8}$$

$\mathscr{R}$ denotes the operator for linear differentials, $\mathscr{N}$ indicates generic *NDO*, and $\Im_i(\xi,\tau)$ is the source term.

When we use $\mathbb{NT}$ on both sides of (6), we obtain

$$\mathbb{N}[D_\tau^\alpha\psi_i(\xi,\tau)]+\mathbb{N}[\mathscr{R}\psi_i(\xi,\tau)]+\mathbb{N}[\mathscr{N}\psi_i(\xi,\tau)]=\mathbb{N}[\Im_i(\xi,\tau)] \tag{9}$$

$$\mathbb{W}_i(\tau,\omega,\mu)=\frac{\mu^\alpha}{\omega^\alpha}\sum\frac{\omega^{\alpha-(k+1)}}{\mu^{\alpha-k}}[D^k\psi_i(\xi,\tau)]_{\tau=0}+\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\psi_i(\xi,\tau)]-\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\mathscr{R}\psi_i(\xi,\tau)+\mathscr{N}\psi_i(\xi,\tau)] \tag{10}$$

By taking the inverse $\mathbb{NT}$ to (10), we obtain

$$\psi_i(\xi,\tau)=\phi_i(\xi,\tau)-\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\mathscr{R}\psi_i(\xi,\tau)+\mathscr{N}\psi_i(\xi,\tau)]\right] \tag{11}$$

From the nonhomogeneous term to the essential initial condition, $\phi_i(\xi,\tau)$ is an increasing function. Now, if the unknown function $\psi_i(\xi,\tau)$ has an infinite series solution, of the kind

$$\psi_i(\xi,\tau)=\sum_{n=0}^{\infty}(\psi_i)_n(\xi,\tau) \tag{12}$$

and

$$\mathscr{N}\psi_i(\xi,\tau)=\sum_{n=0}^{\infty}(A_i)_n(\xi,\tau) \tag{13}$$

The, by using (13), we may formulate (11) as follows:

$$\sum_{n=0}^{\infty}(\psi_i)_n(\xi,\tau)=\phi_i(\xi,\tau)-\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}\left[\mathscr{R}\sum_{n=0}^{\infty}(\psi_i)_n(\xi,\tau)\right]+\sum_{n=0}^{\infty}(A_i)_n\right] \tag{14}$$

such that $(A_i)_n$ is an $\mathbb{AD}$ polynomial that is representing the nonlinear value and is defined as follows:

$$(A_i)_n=\frac{1}{n!}\frac{d^n}{d\eta^n}\mathbb{N}\left[\sum_{i=0}^{\infty}\eta^i\psi_i\right]_{\eta=0} \tag{15}$$

We can infer by comparing the two sides of (14)

$$\psi_{i0}(\xi,\tau)=\phi_i(\xi,\tau)$$

$$\psi_{i1}(\xi,\tau)=-\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\mathscr{R}\psi_{i0}(\xi,\tau)+A_{i0}]\right]$$

$$\psi_{i2}(\xi,\tau)=-\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\mathscr{R}\psi_{i1}(\xi,\tau)+A_{i1}]\right]$$

$$\vdots$$

We continue in this direction until we reach the wide form offered by

$$(\psi_i)_{n+1}(\xi,\tau) = -\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\mathscr{R}(\psi_i)_n(\xi,\tau)+(A_i)_n]\right],\ \ n\geq 1$$

Finally, we have an approximate solution

$$\psi(\xi,\tau) = \sum_{n=0}^{\infty}(\psi_i)_n(\xi,\tau)$$

5 Applications

The proposed technique (FNDM) in order to solve fractional system of PDEs will be applied in this section.

5.1 Example

$$D_\tau^\alpha\psi(\xi,\tau)-\psi_{\xi\xi}(\xi,\tau)-2\psi\psi_\xi+(\psi\varphi)_\xi = 0,\ \ 0<\alpha\leq 1$$

$$D_\tau^\beta\varphi(\xi,\tau)-\varphi_{\xi\xi}(\xi,\tau)-2\varphi\varphi_\xi+(\psi\varphi)_\xi = 0,\ \ 0<\alpha\leq 1 \tag{16}$$

subject to initial conditions

$$\psi(\xi,0) = e^{-\xi}$$

$$\varphi(\xi,0) = e^{-\xi} \tag{17}$$

Applying $\mathbb{NT}$ to each side of (16), and we have achieved this by utilizing the differential characteristic of FNDM

$$\mathbb{N}[\psi(\xi,\tau)] = \frac{1}{\omega}e^{-\xi}+\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\psi_{\xi\xi}(\xi,\tau)+2\psi\psi_\xi-(\psi\varphi)_\xi]$$

$$\mathbb{N}[\varphi(\xi,\tau)] = \frac{1}{\omega}e^{-\xi}+\frac{\mu^\beta}{\omega^\beta}\mathbb{N}[\varphi_{\xi\xi}(\xi,\tau)+2\varphi\varphi_\xi-(\psi\varphi)_\xi] \tag{18}$$

Taking the inverse Natural transform to (18), then

$$\psi(\xi,\tau) = e^{-\xi}+\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\psi_{\xi\xi}(\xi,\tau)+2\psi\psi_\xi-(\psi\varphi)_\xi]\right]$$

$$\varphi(\xi,\tau) = e^{-\xi}+\mathbb{N}^{-1}\left[\frac{\mu^\beta}{\omega^\beta}\mathbb{N}[\varphi_{\xi\xi}(\xi,\tau)+2\varphi\varphi_\xi-(\psi\varphi)_\xi]\right] \tag{19}$$

Assume the following infinite series solutions for the unknown functions $\psi(\xi,\tau)$ and $\varphi(\xi,\tau)$:

$$\psi(\xi,\tau) = \sum_{n=0}^{\infty}\psi_n(\xi,\tau),\quad \varphi(\xi,\tau) = \sum_{n=0}^{\infty}\varphi_n(\xi,\tau) \tag{20}$$

In addition, $\psi\psi_\xi = \sum_{n=0}^{\infty}A_n,\ \varphi\varphi_\xi = \sum_{n=0}^{\infty}C_n$ and $(\psi\varphi)_\xi = \sum_{n=0}^{\infty}B_n$ are nonlinear terms which are shown by the Adomian polynomials. Then, construct equation (19) as follows using equation (20)

$$\sum_{n=0}^{\infty}\psi_n(\xi,\tau) = e^{-\xi}+\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[\sum_{n=0}^{\infty}(\psi_n)_{\xi\xi}+2\sum_{n=0}^{\infty}A_n-\sum_{n=0}^{\infty}B_n]\right]$$

$$\sum_{n=0}^{\infty}\varphi_n(\xi,\tau) = e^{-\xi}+\mathbb{N}^{-1}\left[\frac{\mu^\beta}{\omega^\beta}\mathbb{N}[\sum_{n=0}^{\infty}(\varphi_n)_{\xi\xi}+2\sum_{n=0}^{\infty}C_n-\sum_{n=0}^{\infty}B_n]\right] \tag{21}$$

Where

$$A_0 = \psi_0(\psi_0)_\xi$$
$$A_1 = \psi_0(\psi_1)_\xi+\psi_1(\psi_0)_\xi$$

$$\vdots$$

$$B_0 = (\psi_0\varphi_0)_\xi$$
$$B_1 = (\psi_1\varphi_0 + \psi_0\varphi_1)_\xi$$
$$\vdots$$

$$C_0 = \varphi_0(\varphi_0)_\xi$$
$$C_1 = \varphi_0(\varphi_1)_\xi + \varphi_1(\varphi_0)_\xi$$
$$\vdots$$

Now, comparing both sides of (21), we get

$$\psi_0(\xi,\tau) = e^{-\xi}, \quad \text{and} \quad \varphi_0(\xi,\tau) = e^{-\xi}$$
$$\psi_1(\xi,\tau) = \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[(\psi_0)_{\xi\xi} + 2A_0 - B_0]\right]$$
$$= e^{-\xi}\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^{\alpha+1}}\right]$$
$$= e^{-\xi}\frac{\tau^\alpha}{\Gamma(\alpha+1)}$$

$$\varphi_1(\xi,\tau) = \mathbb{N}^{-1}\left[\frac{\mu^\beta}{\omega^\beta}\mathbb{N}[(\varphi_0)_{\xi\xi} + 2C_0 - B_0]\right]$$
$$= e^{-\xi}\mathbb{N}^{-1}\left[\frac{\mu^\beta}{\omega^{\beta+1}}\right]$$
$$= e^{-\xi}\frac{\tau^\beta}{\Gamma(\beta+1)}$$

$$\psi_2(\xi,\tau) = \mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[(\psi_1)_{\xi\xi} + 2A_1 - B_1]\right]$$
$$= \mathbb{N}^{-1}\left[e^{-\xi}\frac{\mu^{2\alpha}}{\omega^{2\alpha+2}} - 2e^{-2\xi}\frac{\mu^{2\alpha}}{\omega^{2\alpha+2}} + 2e^{-2\xi}\frac{\mu^{\alpha+\beta}}{\omega^{\alpha+\beta+2}}\right]$$
$$= e^{-\xi}\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} - 2e^{-2\xi}\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} + 2e^{-2\xi}\frac{\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$

$$\varphi_2(\xi,\tau) = \mathbb{N}^{-1}\left[\frac{\mu^\beta}{\omega^\beta}\mathbb{N}[(\varphi_1)_{\xi\xi} + 2C_1 - B_1]\right]$$
$$= \mathbb{N}^{-1}\left[e^{-\xi}\frac{\mu^{2\beta}}{\omega^{2\beta+2}} - 2e^{-2\xi}\frac{\mu^{2\beta}}{\omega^{2\beta+2}} + 2e^{-2\xi}\frac{\mu^{\alpha+\beta}}{\omega^{\alpha+\beta+2}}\right]$$
$$= e^{-\xi}\frac{\tau^{2\beta}}{\Gamma(2\beta+1)} - 2e^{-2\xi}\frac{\tau^{2\beta}}{\Gamma(2\beta+1)} + 2e^{-2\xi}\frac{\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$

$$\vdots$$

We continue to get the following approximate solutions:

$$\begin{aligned}\psi(\xi,\tau) &= \psi_0(\xi,\tau)+\psi_1(\xi,\tau)+\psi_2(\xi,\tau)+.....\\ &= e^{-\xi}+e^{-\xi}\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}+e^{-\xi}\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}-2e^{-2\xi}\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}+2e^{-2\xi}\frac{\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+...\\ \varphi(\xi,\tau) &= \varphi_0(\xi,\tau)+\varphi_1(\xi,\tau)+\varphi_2(\xi,\tau)+.....\\ &= e^{-\xi}+e^{-\xi}\frac{\tau^{\beta}}{\Gamma(\beta+1)}+e^{-\xi}\frac{\tau^{2\beta}}{\Gamma(2\beta+1)}-2e^{-2\xi}\frac{\tau^{2\beta}}{\Gamma(2\beta+1)}+2e^{-2\xi}\frac{\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+...\end{aligned} \tag{22}$$

Using $\alpha=1,\beta=1$, and a Taylor series expansion, the above approximation yields

$$\psi(\xi,\tau)=e^{-\xi}+e^{-\xi}\tau+e^{-\xi}\frac{\tau^2}{2!}-2e^{-2\xi}\frac{\tau^2}{2!}+2e^{-2\xi}\frac{\tau^2}{2!}+\cdots$$

$$\varphi(\xi,\tau)=e^{-\xi}+e^{-\xi}\tau+e^{-\xi}\frac{\tau^2}{2!}-2e^{-2\xi}\frac{\tau^2}{2!}+2e^{-2\xi}\frac{\tau^2}{2!}+\cdots$$

Therefore,

$$\psi(\xi,\tau)=e^{-\xi+\tau}$$

$$\psi(\xi,\tau)=e^{-\xi+\tau}$$

These are the precise solutions to equation (16) for $\alpha=1,\beta=1$. As a result, the estimate rapidly approaches the solution.

5.2 Example

$$\begin{aligned}D_\tau^{\alpha}\zeta(\xi_1,\xi_2,\tau)-\varphi_{\xi_1}\psi_{\xi_2}(\xi_1,\xi_2,\tau)&=1\\ D_\tau^{\beta}\varphi(\xi_1,\xi_2,\tau)-\psi_{\xi_1}\zeta_{\xi_2}(\xi_1,\xi_2,\tau)&=5\\ D_\tau^{\gamma}\psi(\xi_1,\xi_2,\tau)-\zeta_{\xi_1}\varphi_{\xi_2}(\xi_1,\xi_2,\tau)&=5\end{aligned} \tag{23}$$

Subject to the initial conditions

$$\begin{aligned}\zeta(\xi_1,\xi_2,0)&=\xi_1+2\xi_2,\\ \varphi(\xi_1,\xi_2,0)&=\xi_1-2\xi_2,\\ \psi(\xi_1,\xi_2,0)&=-\xi_1+2\xi_2\end{aligned} \tag{24}$$

Applying $\mathbb{NT}$ to each side of (23), and by using the differential property of FNDM, we have

$$\begin{aligned}\mathbb{N}[\zeta(\xi_1,\xi_2,\tau)]&=\frac{1}{\omega}(\xi_1+2\xi_2)+\frac{\mu^{\alpha}}{\omega^{\alpha+1}}+\frac{\mu^{\alpha}}{\omega^{\alpha}}\mathbb{N}[\varphi_{\xi_1}\psi_{\xi_2}(\xi_1,\xi_2,\tau)]\\ \mathbb{N}[\zeta(\xi_1,\xi_2,\tau)]&=\frac{1}{\omega}(\xi_1-2\xi_2)+5\frac{\mu^{\beta}}{\omega^{\beta+1}}+\frac{\mu^{\beta}}{\omega^{\beta}}\mathbb{N}[\psi_{\xi_1}\zeta_{\xi_2}(\xi_1,\xi_2,\tau)]\\ \mathbb{N}[\psi(\xi_1,\xi_2,\tau)]&=\frac{1}{\omega}(-\xi_1+2\xi_2)+5\frac{\mu^{\gamma}}{\omega^{\gamma+1}}+\frac{\mu^{\gamma}}{\omega^{\gamma}}\mathbb{N}[\zeta_{\xi_1}\varphi_{\xi_2}(\xi_1,\xi_2,\tau)]\end{aligned} \tag{25}$$

Taking the inverse Natural transform to (18), then

$$\zeta(\xi_1,\xi_2,\tau)=\xi_1+2\xi_2+\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}+\mathbb{N}^{-1}\Big[\frac{\mu^{\alpha}}{\omega^{\alpha}}\mathbb{N}[\varphi_{\xi_1}\psi_{\xi_2}(\xi_1,\xi_2,\tau)]\Big]$$

$$\varphi(\xi_1,\xi_2,\tau)=\xi_1-2\xi_2+5\frac{\tau^{\beta}}{\Gamma(\beta+1)}+\mathbb{N}^{-1}\Big[\frac{\mu^{\beta}}{\omega^{\beta}}\mathbb{N}[\psi_{\xi_1}\zeta_{\xi_2}(\xi_1,\xi_2,\tau)]\Big]$$

$$\psi(\xi_1,\xi_2,\tau)=-\xi_1+2\xi_2+5\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}+\mathbb{N}^{-1}\Big[\frac{\mu^{\gamma}}{\omega^{\gamma}}\mathbb{N}[\zeta_{\xi_1}\varphi_{\xi_2}(\xi_1,\xi_2,\tau)]\Big] \tag{26}$$

Consider $\varphi_{\xi_1}\psi_{\xi_2}=\sum_{n=0}^{\infty}A_n$, $\psi_{\xi_1}\zeta_{\xi_2}=\sum_{n=0}^{\infty}B_n$ and $\zeta_{\xi_1}\varphi_{\xi_2}=\sum_{n=0}^{\infty}C_n$ are the Adomian polynomials indicate the nonlinear terms. Now, equation (26) may be rewritten as follows:

$$\zeta(\xi_1,\xi_2,\tau)=\xi_1+2\xi_2+\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}+\mathbb{N}^{-1}\Big[\frac{\mu^{\alpha}}{\omega^{\alpha}}\mathbb{N}[\sum_{n=0}^{\infty}A_n]\Big]$$

$$\varphi(\xi_1,\xi_2,\tau)=\xi_1-2\xi_2+5\frac{\tau^{\beta}}{\Gamma(\beta+1)}+\mathbb{N}^{-1}\Big[\frac{\mu^{\beta}}{\omega^{\beta}}\mathbb{N}[\sum_{n=0}^{\infty}B_n]\Big]$$

$$\psi(\xi_1,\xi_2,\tau)=-\xi_1+2\xi_2+5\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}+\mathbb{N}^{-1}\Big[\frac{\mu^{\gamma}}{\omega^{\gamma}}\mathbb{N}[\sum_{n=0}^{\infty}C_n]\Big] \tag{27}$$

Where

$$A_0=(\varphi_0)_{\xi_1}(\psi_0)_{\xi_2}$$
$$A_1=(\varphi_1)_{\xi_1}(\psi_0)_{\xi_2}+(\varphi_0)_{\xi_1}(\psi_1)_{\xi_2}$$
$$\vdots$$

$$B_0=(\psi_0)_{\xi_1}(\zeta_0)_{\xi_2}$$
$$B_1=(\psi_1)_{\xi_1}(\zeta_0)_{\xi_2}+(\psi_0)_{\xi_1}(\zeta_1)_{\xi_2}$$
$$\vdots$$

$$C_0=(\zeta_0)_{\xi_1}(\varphi_0)_{\xi_2}$$
$$C_1=(\zeta_1)_{\xi_1}(\varphi_0)_{\xi_2}+(\zeta_0)_{\xi_1}(\varphi_1)_{\xi_2}$$
$$\vdots$$

Now, comparing both sides of (27), we get

$$\zeta_0(\xi_1,\xi_2,\tau)=\xi_1+2\xi_2+\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}$$

$$\varphi_0(\xi_1,\xi_2,\tau)=\xi_1-2\xi_2+5\frac{\tau^{\beta}}{\Gamma(\beta+1)}$$

$$\psi_0(\xi_1,\xi_2,\tau)=-\xi_1+2\xi_2+5\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}$$

$$\zeta_1(\xi_1,\xi_2,\tau)=\mathbb{N}^{-1}\Big[\frac{\mu^{\alpha}}{\omega^{\alpha}}\mathbb{N}[A_0]\Big]=2\mathbb{N}^{-1}\Big[\frac{\mu^{\alpha}}{\omega^{\alpha+1}}\Big]=2\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}$$

$$\varphi_1(\xi_1,\xi_2,\tau)=\mathbb{N}^{-1}\left[\frac{\mu^\beta}{\omega^\beta}\mathbb{N}[B_0]\right]=-2\mathbb{N}^{-1}\left[\frac{\mu^\beta}{\omega^{\beta+1}}\right]=-2\frac{\tau^\beta}{\Gamma(\beta+1)}$$

$$\psi_1(\xi_1,\xi_2,\tau)=\mathbb{N}^{-1}\left[\frac{\mu^\gamma}{\omega^\gamma}\mathbb{N}[C_0]\right]=-2\mathbb{N}^{-1}\left[\frac{\mu^\gamma}{\omega^{\gamma+1}}\right]=-2\frac{\tau^\gamma}{\Gamma(\gamma+1)}$$

$$\zeta_2(\xi_1,\xi_2,\tau)=\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[A_1]\right]=\mathbb{N}^{-1}\left[\frac{\mu^\alpha}{\omega^\alpha}\mathbb{N}[0]\right]=0$$

$$\varphi_2(\xi_1,\xi_2,\tau)=\mathbb{N}^{-1}\left[\frac{\mu^\beta}{\omega^\beta}\mathbb{N}[B_1]\right]=\mathbb{N}^{-1}\left[\frac{\mu^\beta}{\omega^\beta}\mathbb{N}[0]\right]=0$$

$$\psi_2(\xi_1,\xi_2,\tau)=\mathbb{N}^{-1}\left[\frac{\mu^\gamma}{\omega^\gamma}\mathbb{N}[C_1]\right]=\mathbb{N}^{-1}\left[\frac{\mu^\gamma}{\omega^\gamma}\mathbb{N}[0]\right]=0$$

$$\vdots$$

We continue to get

$$\sum_{n=1}^{\infty}\zeta_n(\xi_1,\xi_2,\tau)=\xi_1+2\xi_2+\frac{\tau^\alpha}{\Gamma(\alpha+1)}+2\frac{\tau^\alpha}{\Gamma(\alpha+1)}+0+\ldots$$

$$\sum_{n=1}^{\infty}\varphi_n(\xi_1,\xi_2,\tau)=\xi_1-2\xi_2+5\frac{\tau^\beta}{\Gamma(\beta+1)}-2\frac{\tau^\beta}{\Gamma(\beta+1)}+0+\ldots$$

$$\sum_{n=1}^{\infty}\psi_n(\xi_1,\xi_2,\tau)=-\xi_1+2\xi_2+5\frac{\tau^\gamma}{\Gamma(\gamma+1)}-2\frac{\tau^\gamma}{\Gamma(\gamma+1)}+0+\ldots$$

Therefore, we have

$$\zeta(\xi_1,\xi_2,\tau)=\xi_1+2\xi_2+3\frac{\tau^\alpha}{\Gamma(\alpha+1)}$$

$$\varphi(\xi_1,\xi_2,\tau)=\xi_1-2\xi_2+3\frac{\tau^\beta}{\Gamma(\beta+1)}$$

$$\psi(\xi_1,\xi_2,\tau)=-\xi_1+2\xi_2+3\frac{\tau^\gamma}{\Gamma(\gamma+1)} \tag{28}$$

If $\alpha=1,\beta=1,\gamma=1$, and by applying Taylor, the approximation yields

$$\zeta(\xi_1,\xi_2,\tau)=\xi_1+2\xi_2+3\tau$$

$$\varphi(\xi_1,\xi_2,\tau)=\xi_1-2\xi_2+3\tau$$

$$\psi(\xi_1,\xi_2,\tau)=-\xi_1+2\xi_2+3\tau \tag{29}$$

These are the precise solutions to equation (23) for $\alpha=1,\beta=1,\gamma=1$. As a result, the estimate rapidly approaches the solution.

6 Conclusion

The FNDM was effectively used in this research to provide the analytical approximation solution to the nonlinear systems of FPDEs. The FNDM provides solutions in the form of convergent series with simply computed elements, as well as the elements of the precise solution.

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