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Dynamics and Ulam stability for Ambartsumian equation with *k*-Generalized Ξ-Hilfer fractional derivative

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Abstract: This manuscript studies the Ambartsumian equation with k-generalized Ξ- Hilfer fractional derivative. The main aim of this research work is to establish the existence and uniqueness results as well as to analyse the Ulam-Hyers-Rassias stability using standard fixed point technique.

Keywords: Ambartsumian Equation, *k*-generalized Ξ - Hilfer fractional derivative, Existence and Uniqueness, Ulam-Hyers-Rassias Stable

1 Introduction

Fractional order calculus has not been explored in engineering and other sciences, due to its inherent complexity, as well as the fact that it lacks a completely valid geometric or physical interpretation. However, some natural behaviors related to different fields of engineering are more accurately represented by using it; Now, it is a promising tool used in physics, fluid mechanics, signal processing, thermal diffusion phenomenon, botanical electrical impedances, robotics, etc. Various analytical and numerical methods have been developed for fractional order differential equations. For more details refer to [1,7,8].

Together with Riemann Liouville and Caputo fractional derivatives, Hilfer created a completely new kind of derivative which generalized Riemann Liouville fractional derivative, for short, Hilfer fractional derivative, see [6,14,16,17]. Motivated by the definition of Hilfer fractional derivative that contains, as particular cases, the classical Riemannâ€“Liouville and Caputo fractional derivative, our main objective is propose a fractional differential operator the so-called ψ-Hilfer fractional derivative, i.e., a fractional derivative of a function with respect to another ψ function. With this fractional derivative, we recover a wide class of fractional derivatives and integrals. The advantage of the fractional operator ψ-Hilfer proposed here is the freedom of choice of the classical differentiation operator and the choice of the function ψ, i.e., from the choice of the function ψ, the operator of classical differentiation, can act on the fractional integration operator or else the fractional integration operator can act on the classical differentiation operator. In [11,12,19], the authors discussed about Hilfer type fractional derivatives of some order with various situation.

The original integer order Ambartsumian equation was introduced in the theory of surface brightness in the Milky Way. The authors in [2,3,10,15] studied the Ambartsumian equation.

The stability problem named after Ulam is currently a research trend in many applications. Mathematicians have proposed and proved many other theorems in the field of stability by changing the type of functional equation, control function, and space in the above theorem. The study of Ulam stability was initiated due to an interesting problem posed in the year 1940, by Ulam, regarding the stability for the equation of group homomorphisms. An answer was given by Hyers , in 1941, in the framework of Banach spaces, for the additive Cauchy equation. In the following years, many mathematicians were concerned with this problem, also for the case of differential

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equations, integral equations, and partial differential equations.The results of Rassias had a great impact on the issue of the stability of functional equations. Today this type of stability is called the Ulamâ€"Hyers-Rassias (U-H-R) stability. We refer readers to [9,18], references for consideration of the stability of various functional equations in different spaces.

In this work, we the generalize the Ξ-Hilfer fractional derivative by using the functions k-Gamma, k-Beta and k-Mittag-Leffler [4,5,13] then set some properties for the defined operator. Let us consider the following initial value problem with nonlinear implicit k-generalized Ξ-Hilfer type fractional differential equation for each $t \in (a,b]$:

$$ {}^{H}_{k}\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\mathscr{A}(t) = f\left(t, \mathscr{A}(t), {}^{H}_{k}\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\mathscr{A}(t)\right), \quad (1) $$

with the condition

$$ \left(\mathscr{I}^{k(1-\zeta),k;\Xi}_{a^+}\mathscr{A}\right)(a^+) = \mu, \quad (2) $$

where, ${}^{H}_{k}\mathscr{D}^{\vartheta,r;\Xi}_{a^+}$ is the k-generalized Ξ-Hilfer fractional derivative, $0 < \vartheta < 1, 0 \le r \le 1$, $\mathscr{I}^{k(1-\zeta),k;\Xi}_{a^+}\mathscr{A}$ is the k-generalized Ξ-Hilfer fractional integral $\zeta = \frac{r(k-\vartheta)+\vartheta}{k}, \mu \in E, k > 0$, $f : [a,b] \times E \times E \to E$ and $\eta > 1$ satisfies

$$ f\left(t, \mathscr{A}(t), {}^{H}_{k}\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\mathscr{A}(t)\right) = \frac{1}{\eta}\mathscr{A}\left(\frac{t}{\eta}\right) - \mathscr{A}(t) = w(t). $$

2 Axillary results

Let $0 < a < b < \infty, J = [a,b], \vartheta \in (0,1), r \in [0,1], k > 0$ with $\vartheta < k$ and $\zeta = \frac{1}{k}(r(k-\vartheta)+\vartheta)$. Let C(J,E) be the Banach space of all continuous functions from J into E with the norm

$$ \|\mathscr{A}\|_{\infty} = \sup\{\|\mathscr{A}(t)\| : t \in J\}. $$

Consider the weighted Banach space

$$ C_{\zeta,k;\Xi}(J) = \{\mathscr{A} : (a,b] \to E : t \to (\Xi(t) - \Xi(a))^{1-\zeta}\mathscr{A}(t) \in C(J,E)\Big\}, $$

with the norm

$$ \|\mathscr{A}\|_{C_{\zeta,k;\Xi}} = \sup_{t\in J}\left\|(\Xi(t) - \Xi(a))^{1-\zeta}\mathscr{A}(t)\right\|, $$

and

$$ C^{n}_{\zeta,k;\Xi}(J) = \left\{\mathscr{A} \in C^{n-1}(J) : \mathscr{A}^{n} \in C_{\zeta,k;\Xi}(J)\right\}, n \in \mathbb{N}, $$
$$ C^{0}_{\zeta,k;\Xi}(J) = C_{\zeta,k;\Xi}(J), $$

with the norm

$$ \|\mathscr{A}\|_{C^{n}_{\zeta,k;\Xi}} = \sum_{i=0}^{n-1}\left\|\mathscr{A}^{i}\right\|_{\infty} + \|\mathscr{A}^{n}\|_{C_{\zeta,k;\Xi}}. $$

Definition 1.*[17] Let $n-1 < \vartheta \le n$ with $n \in \mathbb{N}$, $J = [a,b]$ an interval such that $-\infty \le a < b \le \infty$, Ξ is an increasing function and $\Xi'(t) \neq 0$, $\forall t \in J$.The k-generalized Ξ-Hilfer left sided fractional derivative ${}^{H}_{k}\mathscr{D}^{\vartheta,r;\Xi}_{a^+}(.)$ and k-generalized Ξ- Hilfer right sided fractional derivative ${}^{H}_{k}\mathscr{D}^{\vartheta,r;\Xi}_{b^-}(.)$ of a function $\mathscr{A}$ of order ϑ and type $0 \le r \le 1$ with $k > 0$ are defined by*

$$ {}^{H}_{k}\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\mathscr{A}(t) = $$
$$ \left(\mathscr{I}^{r(kn-\vartheta),k;\Xi}_{a^+}\left(\frac{1}{\Xi'(t)}\frac{d}{dt}\right)^{n}\left((k)^{n}\mathscr{I}^{(1-r)(kn-\vartheta),k;\Xi}_{a^+}\mathscr{A}\right)\right)(t) $$
$$ = \left(\mathscr{I}^{r(kn-\vartheta),k;\Xi}_{a^+}\delta^{n}_{\Xi}\left((k)^{n}\mathscr{I}^{(1-r)(kn-\vartheta),k;\Xi}_{a^+}\mathscr{A}\right)\right)(t), $$

and

$$ {}^{H}_{k}\mathscr{D}^{\vartheta,r;\Xi}_{b^-}\mathscr{A}(t) = $$
$$ \left(\mathscr{I}^{r(kn-\vartheta),k;\Xi}_{b^-}\left(-\frac{1}{\Xi'(t)}\frac{d}{dt}\right)^{n}\right. $$
$$ \left.\left((k)^{n}\mathscr{I}^{(1-r)(kn-\vartheta),k;\Xi}_{b^-}\mathscr{A}\right)\right)(t) $$
$$ = \left(\mathscr{I}^{r(kn-\vartheta),k;\Xi}_{b^-}(-1)^{n}\delta^{n}_{\Xi}\left((k)^{n}\mathscr{I}^{(1-r)(kn-\vartheta),k;\Xi}_{b^-}\mathscr{A}\right)\right)(t), $$

where $\delta^{n}_{\Xi} = \left(\frac{1}{\Xi'(t)}\frac{d}{dt}\right)^{n}$.

Definition 2.*[13] Let [a,b] be a finite or infinite interval on the real axis $\mathbb{R} = (-\infty,\infty)$, $\Xi(t) > 0$ be an increasing function on $(a,b]$ and $\Xi'(t) > 0$ be continuous on (a,b) and $\vartheta > 0$. The generalized left-sided k- fractional integral operator $\mathscr{I}^{\vartheta,k;\Xi}_{a^+}(.)$ and the generalized right-sided k-fractional integral operator $\mathscr{I}^{\vartheta,k;\Xi}_{b^-}(.)$ of a function $\mathscr{A}$ of order ϑ are defined by*

$$ \mathscr{I}^{\vartheta,k;\Xi}_{a^+}\mathscr{A}(t) = \frac{1}{k\Gamma_k(\vartheta)}\int_a^t \frac{\Xi'(s)\mathscr{A}(s)ds}{(\Xi(t)-\Xi(s))^{1-\frac{\vartheta}{k}}}, $$
$$ \mathscr{I}^{\vartheta,k;\Xi}_{b^-}\mathscr{A}(t) = \frac{1}{k\Gamma_k(\vartheta)}\int_t^b \frac{\Xi'(s)\mathscr{A}(s)ds}{(\Xi(s)-\Xi(t))^{1-\frac{\vartheta}{k}}}, $$

generalized fractional integral operators are defined by

$$ \mathscr{I}^{\vartheta,k;\Xi}_{G,a^+}\mathscr{A}(t) = \frac{1}{k\Gamma_k(\vartheta)}\int_a^t \frac{\Xi'(s)\mathscr{A}(s)ds}{G\left(\Xi(t)-\Xi(s),\frac{\vartheta}{k}\right)}, $$
$$ \mathscr{I}^{\vartheta,k;\Xi}_{G,b^-}\mathscr{A}(t) = \frac{1}{k\Gamma_k(\vartheta)}\int_t^b \frac{\Xi'(s)\mathscr{A}(s)ds}{G\left(\Xi(s)-\Xi(t),\frac{\vartheta}{k}\right)}. $$

Theorem 1.*[13] Let $\mathscr{A} : [a,b] \to E$ be an integrable function, and take $\vartheta > 0$ and $k > 0$. Then $\mathscr{I}^{\vartheta,k;\Xi}_{G,a^+}\mathscr{A}$ exists for all $t \in [a,b]$.*

Theorem 2.*[13] Let $\mathscr{A} \in L^1[a,b]$ and take $\vartheta > 0$ and $k > 0$. Then $\mathscr{I}^{\vartheta,k;\Xi}_{G,a^+}\mathscr{A} \in C([a,b],E)$.*

Lemma 1.[13] *Let $\vartheta > 0, r > 0$ and $k > 0$, then we have the following semigroup property given by*

$$\begin{aligned}\mathscr{I}_{a^+}^{\vartheta,k;\Xi}\mathscr{I}_{a^+}^{r,k;\Xi}\mathscr{A}(t) &= \mathscr{I}_{a^+}^{\vartheta+r,k;\Xi}\mathscr{A}(t)\\ &= \mathscr{I}_{a^+}^{r,k;\Xi}\mathscr{I}_{a^+}^{\vartheta,k;\Xi}\mathscr{A}(t),\end{aligned}$$

and

$$\begin{aligned}\mathscr{I}_{b^-}^{\vartheta,k;\Xi}\mathscr{I}_{b^-}^{r,k;\Xi}\mathscr{A}(t) &= \mathscr{I}_{b^-}^{\vartheta+r,k;\Xi}\mathscr{A}(t)\\ &= \mathscr{I}_{b^-}^{r,k;\Xi}\mathscr{I}_{b^-}^{\vartheta,k;\Xi}\mathscr{A}(t).\end{aligned}$$

Lemma 2.[13] *Let $\vartheta > 0, r > 0$ and $k > 0$, then we have*

$$\mathscr{I}_{a^+}^{\vartheta,k;\Xi}\left[\Xi(t)-\Xi(a)\right]^{\frac{r}{k}-1} = \frac{\Gamma_k(r)}{\Gamma_k(\vartheta+r)}\left[\Xi(t)-\Xi(a)\right]^{\frac{\vartheta+r}{k}-1},$$

and

$$\mathscr{I}_{b^-}^{\vartheta,k;\Xi}\left[\Xi(b)-\Xi(t)\right]^{\frac{r}{k}-1} = \frac{\Gamma_k(r)}{\Gamma_k(\vartheta+r)}\left[\Xi(b)-\Xi(t)\right]^{\frac{\vartheta+r}{k}-1}.$$

Theorem 3.[13,17] *Let $0 < a < b < \infty$, $\vartheta > 0, 0 \le \zeta < 1, k > 0$ and $\mathscr{A} \in C_{(\zeta,k,\Xi)}(J)$. If $\frac{\vartheta}{k} > 1-\zeta$, then*

$$\left(\mathscr{I}_{a^+}^{\vartheta,k;\Xi}\mathscr{A}\right)(a) = \lim_{t\to a^+}\left(\mathscr{I}_{a^+}^{\vartheta,k;\Xi}\mathscr{A}\right)(t) = 0.$$

Lemma 3.[13,17] *Let $t > a, \vartheta > 0, 0 \le r \le 1, k > 0$. Then for $0 < \zeta < 1; \zeta = \frac{1}{k}(r(k-\vartheta)+\vartheta)$, then we have*

$$\left[{}_k^H\mathscr{D}_{a^+}^{\vartheta,r;\Xi}(\Xi(s)-\Xi(a))^{\zeta-1}\right](t) = 0.$$

Theorem 4.[13,17] *If $\mathscr{A} \in C^n_{\zeta,k,\Xi}[a,b]$, $n-1 < \vartheta < n, 0 \le r \le 1$, where $n \in \mathbb{N}$ and $k > 0$, then* $\left(\mathscr{I}_{a^+}^{\vartheta,k;\Xi}\ {}_k^H\mathscr{D}_{a^+}^{\vartheta,r;\Xi}\mathscr{A}\right)(t) =$

$$\mathscr{A}(t) - \sum_{i=1}^{n}\frac{(\Xi(t)-\Xi(a))^{\zeta-i}}{k^{i-n}\Gamma_k(k(\zeta-i+1))}\left[\delta_\Xi^{n-i}\left(\mathscr{I}_{a^+}^{k(n-\zeta),k;\Xi}\mathscr{A}(a)\right)\right],$$

where

$$\zeta = \frac{1}{k}(r(kn-\vartheta)+\vartheta).$$

Lemma 4.[13,17] *Let $\vartheta > 0, 0 \le r \le 1$ and $x \in C^1_{\zeta,k;\Xi}(J)$, where $k > 0$, then for $t \in [a,b)$, we have*

$$\left(\mathscr{I}_{a^+}^{\vartheta,k;\Xi}\,{}_k^H\mathscr{D}_{a^+}^{\vartheta,r;\Xi}\mathscr{A}(t)\right) = \mathscr{A}(t).$$

Definition 3.[4] *Let X be a Banach space and let Ω_X be the family of bounded subsets of X. The Kuratowski measure of non compactness is the map $\lambda : \Omega_X \to [0,\infty)$ defined by*

$$\lambda(M) = \inf\left\{\varepsilon > 0 : M \subset \bigcup_{j=1}^{m} M_j, diam(M_j) \le \varepsilon\right\},$$

where $M \in \Omega_X$.
The map λ satisfies the following properties:

(i)$\lambda(M) = 0 = \overline{M}$ is compact.
(ii)$\lambda(M) = \lambda(\overline{M})$.
(iii)$M_1 \subset M_2 \implies \lambda(M_1) \le \lambda(M_2)$.
(iv)$\lambda(M_1+M_2) \le \lambda(B_1)+\lambda(B_2)$.
(v)$\lambda(cM) = |c|\lambda(M), c \in \mathbb{R}$.
(vi)$\lambda(convM) = \lambda(M)$.

Lemma 5.[4] *Let $D \subset C_{\zeta,k,\Xi}(J)$ be a bounded and equicontinuous set, then*

(i)The function $t \to \lambda(D(t))$ is continuous on (a,b] and

$$\lambda_{C_{\zeta,k,\Xi}}(D) = \sup_{t\in J}\lambda\left((\Xi(t)-\Xi(a))^{1-\zeta}D(t)\right).$$

(ii)$\lambda\left[\int_a^b u(s)ds : u \in D\right] \le \int_a^b \lambda(D(s))ds$, where

$$D(t) = \{u(t) : t \in D\}, t \in (a,b].$$

Theorem 5.[4][*Mönch's fixed point Theorem*] *Let D be a closed , bounded and convex subset of a Banach space X such that 0 $\in$ D, and let T be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}T(V) \quad or \quad V = T(V)\cup\{0\} \implies \lambda(V) = 0, \quad (3)$$

holds for every subset V of D. Then T has a fixed point.

Theorem 6.[4][*Darbo's fixed point theorem*] *Let D be a non empty closed , bounded and convex subset of a Banach space X, and let T be a continuous mapping of D into itself such that for any non empty subset C of D,*

$$\lambda(T(C)) \le l\lambda(C), \quad (4)$$

where $0 \le l < 1$, and λ is the Kuratowski measure of non compactness. Then T has a fixed point in D.

3 Existence Theory

Theorem 7.*If $w(.) \in C^1_{\zeta,k,\Xi}(J)$,then $\mathscr{A}$ satisfies (3)-(4) if and only if it satisfies*

$$\begin{aligned}\mathscr{A}(t) &= \frac{(\Xi(t)-\Xi(a))^{\zeta-1}}{\Gamma_k(k\zeta)}\mu\\ &+\left(\mathscr{I}_{a^+}^{\vartheta,r;\Xi}\left[\frac{1}{\eta}\mathscr{A}\left(\frac{t}{\eta}\right)-\mathscr{A}(t)\right]\right). \quad (5)\end{aligned}$$

Proof. Assume that $\mathscr{A} \in C^1_{\zeta,k,\Xi}(J)$ satisfies the equation (1) and (2) , and applying $\left(\mathscr{I}^{\vartheta,r;\Xi}_{a^+}(.)\right)$ on both sides of equation (1),hence we get

$$\left(\mathscr{I}^{\vartheta,r;\Xi}_{a^+}\ {}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\mathscr{A}\right)(t) = \mathscr{I}^{\vartheta,r;\Xi}_{a^+}\left[\frac{1}{\eta}\mathscr{A}\left(\frac{t}{\eta} - \mathscr{A}(t)\right]\right),$$

and by using Theorem (4) and equation (2), we get

$$\begin{aligned}\mathscr{A}(t) &= \frac{(\Xi(t)-\Xi(a))^{\zeta-1}}{\Gamma_k(k\zeta)}\mathscr{I}^{k(1-\zeta),k;\Xi}_{a^+}\mathscr{A}(a^+) \\ &+ \left(\mathscr{I}^{\vartheta,r;\Xi}_{a^+}\left[\frac{1}{\eta}\mathscr{A}\left(\frac{t}{\eta}\right) - \mathscr{A}(t)\right]\right) \\ &= \frac{(\Xi(t)-\Xi(a))^{\zeta-1}}{\Gamma_k(k\zeta)}\mu \\ &+ \left(\mathscr{I}^{\vartheta,r;\Xi}_{a^+}\left[\frac{1}{\eta}\mathscr{A}\left(\frac{t}{\eta}\right) - \mathscr{A}(t)\right]\right).\end{aligned}$$

Let us now prove that if $\mathscr{A}$ satisfies equation (5),then it satisfies (1) and (2). Now we have to apply ${}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}$ on both sides of equation (5), then we get

$$\begin{aligned}&{}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\mathscr{A}(t) \\ &= {}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\left(\frac{(\Xi(t)-\Xi(a))^{\zeta-1}}{\Gamma_k(k\zeta)}\mu\right) \\ &+ {}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\left(\mathscr{I}^{\vartheta,r;\Xi}_{a^+}\left[\frac{1}{\eta}\mathscr{A}\left(\frac{t}{\eta}\right) - \mathscr{A}(t)\right]\right).\end{aligned}$$

Now using the Lemma 3 and the Lemma 4 , we get

$${}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\mathscr{A}(t) = \frac{1}{\eta}\mathscr{A}\left(\frac{t}{\eta}\right) - \mathscr{A}(t).$$

Here, we obtained the equation (1).
Now we apply the operator $\mathscr{I}^{k(1-\zeta),k;\Xi}_{a^+}(.)$ on equation (5), then we get

$$\begin{aligned}&\left(\mathscr{I}^{k(1-\zeta),k;\Xi}_{a^+}\mathscr{A}\right)(t) = \\ &\frac{\mu}{\Gamma_k(k\zeta)}\mathscr{I}^{k(1-\zeta),k;\Xi}_{a^+}(\Xi(t)-\Xi(a))^{\zeta-1} \\ &+ \left(\mathscr{I}^{k(1-\zeta),k;\Xi}_{a^+}\mathscr{I}^{\vartheta,r;\Xi}_{a^+}\left[\frac{1}{\eta}\mathscr{A}\left(\frac{t}{\eta}\right) - \mathscr{A}(t)\right]\right).\end{aligned}$$

By the Lemma 2,

$$\begin{aligned}&\left(\mathscr{I}^{k(1-\zeta),k;\Xi}_{a^+}\mathscr{A}\right)(t) = \\ &\frac{\mu}{\Gamma_k(1)}(\Xi(t)-\Xi(a))^0 \\ &+ \left(\mathscr{I}^{k(1-\zeta),k;\Xi}_{a^+}\mathscr{I}^{\vartheta,r;\Xi}_{a^+}\left[\frac{1}{\eta}\mathscr{A}\left(\frac{t}{\eta}\right) - \mathscr{A}(t)\right]\right).\end{aligned}$$

By the Lemma 1,

$$\begin{aligned}&\left(\mathscr{I}^{k(1-\zeta),k;\Xi}_{a^+}\mathscr{A}\right)(t) = \\ &\mu + \left(\mathscr{I}^{k(1-\zeta)+\vartheta,k;\Xi}_{a^+}\left[\frac{1}{\eta}\mathscr{A}\left(\frac{t}{\eta}\right) - \mathscr{A}(t)\right]\right).\end{aligned}$$

By the Theorem (3), with $t \to a^+$

$$\left(\mathscr{I}^{k(1-\zeta),k;\Xi}_{a^+}\mathscr{A}\right)(a^+) = \mu.$$

Hence we have the equation (2).

As a consequence of Theorem 7, we have the following result.

Lemma 6. *Let $\zeta = \frac{r(k-\vartheta)+\vartheta}{k}$ where $0 < \vartheta < 1, 0 \le r \le 1, k > 0$. Let $f : J \times E \times E \to E$ be a continuous function such that $f(.,\mathscr{A}(.),\overline{\mathscr{A}}(.)) \in C^1_{\zeta,k;\Xi}(J)$, for any $\mathscr{A},\overline{\mathscr{A}} \in C_{\zeta,k;\Xi}(J)$. Then $\mathscr{A}$ satisfies the problem (1)-(2) if and only if $\mathscr{A}$ is the fixed point of the operator $\mathscr{T} : C_{\zeta,k;\Xi}(J) \to C_{\zeta,k;\Xi}(J)$ defined by*

$$\begin{aligned}\mathscr{T}\mathscr{A}(t) =& \frac{(\Xi(t)-\Xi(a))^{\zeta-1}}{\Gamma_k(k\zeta)}\mu \\ &+ \frac{1}{k\Gamma_k(\vartheta)}\int_a^t \frac{\Xi'(s)\varphi(s)ds}{(\Xi(t)-\Xi(s))^{1-\frac{\vartheta}{k}}},\end{aligned} \qquad (6)$$

where, φ be a function satisfying the functional equation

$$\varphi(t) = f\left(t, \mathscr{A}(t), \left[\frac{1}{\eta}\mathscr{A}\left(\frac{t}{\eta}\right) - \mathscr{A}(t)\right]\right).$$

In the sequel, the following hypotheses are included :
(H_1) : The function $t \to f(t,\mathscr{A},\overline{\mathscr{A}})$ is measurable on $(a,b]$ for each $\mathscr{A},\overline{\mathscr{A}} \in E$, the functions $\mathscr{A} \to f(t,\mathscr{A},\overline{\mathscr{A}})$ and $\overline{\mathscr{A}} \to f(t,\mathscr{A},\overline{\mathscr{A}})$ are continuous on E for $t \in (a,b]$ and

$$f(.,\mathscr{A}(.),\overline{\mathscr{A}}(.)) = C^1_{\vartheta,k;\Xi}(J), \quad \text{for any} \in C_{\vartheta,k;\Xi}(J).$$

(H_2) : There exists a continuous function $p : J \to [0,\infty)$ such that
$\left\|f(t,\mathscr{A},\overline{\mathscr{A}})\right\| \le p(t)$, for $t \in (a,b]$ and for each $\mathscr{A},\overline{\mathscr{A}} \in E$.
(H_3) : For each bounded set $B \subset E$ and for each $t \in (a,b]$, we have

$$\lambda\left(f\left(t,B,\left({}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}B\right)\right)\right) \le (\Xi(t)-\Xi(a))^{1-\zeta}p(t)\lambda(B),$$

where,

$${}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}B = \left\{{}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}w : w \in B\right\},$$

$$\text{Set} \quad p^* = \sup_{t\in J} p(t).$$

Now we can derive the existence result of our proposed problem (1)-(2) by using Theorem 5.

Theorem 8. *Assume that (H_1) - (H_3) are hold. If*

$$\mathscr{L} = \frac{p^*(\Xi(b)-\Xi(a))^{1-\zeta+\frac{\vartheta}{k}}}{\Gamma_k(\vartheta+k)} < 1, \qquad (7)$$

then the problem (1)-(2) has at least one solution in $C_{\zeta,k;\Xi}(J)$.

Proof. First we have to show that the operator $\mathscr{T}$, transforms the ball $B_R = B(0,R) = \left\{w \in C_{\zeta,k;\Xi}(J) : \|w\|_{C_{\zeta,k;\Xi}(J)} \leq R\right\}$ into itself.
For any $\vartheta \in C_{\zeta,k;\Xi}(J)$, and each $t \in (a,b]$ we have

$$\left\|(\Xi(t)-\Xi(a))^{1-\zeta}(\mathscr{T}\mathscr{A})(t)\right\|$$

$$\leq \frac{\mu}{\Gamma_k(k\zeta)} + \frac{(\Xi(t)-\Xi(a))^{1-\zeta}}{k\Gamma_k(\vartheta)} \int_a^t \frac{\Xi'(s)\|\varphi(s)\|}{(\Xi(t)-\Xi(a))^{1-\frac{\vartheta}{k}}}$$
$$\leq \frac{\mu}{\Gamma_k(k\zeta)} + p^*(\Xi(t)-\Xi(a))^{1-\zeta}\left(\mathscr{I}_{a^+}^{\vartheta,k;\Xi}\varphi\right)(t).$$

By using the Lemma 2 we have

$$\left\|(\Xi(t)-\Xi(a))^{1-\zeta}(\mathscr{T}\mathscr{A})(t)\right\| \leq \frac{\mu}{\Gamma_k(k\zeta)} + \frac{p^*(\Xi(t)-\Xi(a))^{1-\zeta+\frac{\vartheta}{k}}}{\Gamma_k(\vartheta+k)}.$$

Hence for any $\mathscr{A} \in C_{\zeta,k;\Xi}(J)$and each $t \in (a,b]$, we get

$$\|\mathscr{T}\mathscr{A}\|_{C_{\zeta,k;\Xi}} \leq \frac{\mu}{\Gamma_k(k\zeta)} + \frac{p^*(\Xi(t)-\Xi(a))^{1-\zeta+\frac{\vartheta}{k}}}{\Gamma_k(\vartheta+k)} = R,$$
$$\|\mathscr{T}\mathscr{A}\|_{C_{\zeta,k;\Xi}} \leq R.$$

Next, we have to prove that $B_R \to B_R$ is continuous.
Let $\{\mathscr{A}_n\}_n \in \mathbb{N}$ be a sequence Such that $\mathscr{A}_n \to \mathscr{A}$ in B_R.
Then for each $t \in (a,b]$, we have

$$\left\|(\Xi(t)-\Xi(a))^{1-\zeta}[(\mathscr{T}\mathscr{A}_n)(t)-(\mathscr{T}\mathscr{A})(t)]\right\|$$

$$\leq \frac{(\Xi(t)-\Xi(a))^{1-\zeta}}{k\Gamma_k(\vartheta)} \int_a^t \frac{\Xi'(s)\|\varphi_n(s)-\varphi(s)\|ds}{(\Xi(t)-\Xi(s))^{1-\frac{\vartheta}{k}}},$$

where, $\varphi_n, \varphi \in C_{\zeta,k;\Xi}(J)$ such that

$$\varphi_n(t) = f(t,\mathscr{A}_n(t),\varphi_n(t)),$$
$$\varphi(t) = f(t,\mathscr{A}(t),\varphi(t)).$$

Since $\mathscr{A}_n \to \mathscr{A}$ as $n \to \infty$, then by the Lebesque dominated convergence theorem, we have

$$\|\mathscr{T}\mathscr{A}_n - \mathscr{T}\mathscr{A}\|_{C_{\zeta,k;\Xi}} \to 0 \quad \text{as} \quad n \to \infty.$$

Next we have to prove that $\mathscr{T}(B_R)$ is bounded and equicontinuous.
Since $\mathscr{T}(B_R) \subset (B_R)$ and (B_R) is bounded, then $\mathscr{T}(B_R)$ is bounded.
Let $t_1, t_2 \in (a,b]$ such that $a < t_1 < t_2 \leq b$ and let $u \in (B_R)$. Thus, we have

$$\left\|(\Xi(t_2)-\Xi(a))^{1-\zeta}(\mathscr{T}\mathscr{A})(t_2)\right.$$
$$\left.-(\Xi(t_1)-\Xi(a))^{1-\zeta}(\mathscr{T}\mathscr{A})(t_1)\right\|$$

$$\leq \left\|\frac{(\Xi(t_2)-\Xi(a))^{1-\zeta}}{k\Gamma_k(\vartheta)} \int_a^{t_2} \frac{\Xi'(s)\varphi(s)ds}{(\Xi(t_2)-\Xi(s))^{1-\frac{\vartheta}{k}}} - \frac{(\Xi(t_1)-\Xi(a))^{1-\zeta}}{k\Gamma_k(\vartheta)} \int_a^{t_1} \frac{\Xi'(s)\varphi(s)ds}{(\Xi(t_1)-\Xi(s))^{1-\frac{\vartheta}{k}}}\right\|,$$
$$\leq \frac{(\Xi(t_2)-\Xi(a))^{1-\zeta}}{k\Gamma_k(\vartheta)} \int_{t_1}^{t_2} \frac{\Xi'(s)\varphi(s)ds}{(\Xi(t_2)-\Xi(s))^{1-\frac{\vartheta}{k}}}$$
$$+\frac{1}{k\Gamma_k(\vartheta)} \int_a^{t_1} \left|\frac{(\Xi(t_2)-\Xi(a))^{1-\zeta}}{(\Xi(t_2)-\Xi(s))^{1-\frac{\vartheta}{k}}} - \frac{(\Xi(t_1)-\Xi(a))^{1-\zeta}}{(\Xi(t_1)-\Xi(s))^{1-\frac{\vartheta}{k}}}\right| \Xi'(s)\|\varphi(s)\|ds,$$
$$\leq p^*(\Xi(b)-\Xi(a))^{1-\zeta}\left(\mathscr{I}_{t_1}^{\vartheta,k;\Xi}(1)\right)(t_2)$$
$$+\frac{p^*}{k\Gamma_k(\vartheta)} \int_a^{t_1} \left|\frac{(\Xi(t_2)-\Xi(a))^{1-\zeta}}{(\Xi(t_2)-\Xi(s))^{1-\frac{\vartheta}{k}}} - \frac{(\Xi(t_1)-\Xi(a))^{1-\zeta}}{(\Xi(t_1)-\Xi(s))^{1-\frac{\vartheta}{k}}}\right| \Xi'(s)ds.$$

By using the Lemma 2, we get

$$\left\|(\Xi(t_2)-\Xi(a))^{1-\zeta}(\mathscr{T}\mathscr{A})(t_2)\right.$$
$$\left.-(\Xi(t_1)-\Xi(a))^{1-\zeta}(\mathscr{T}\mathscr{A})(t_1)\right\|$$

$$\leq \frac{p^*(\Xi(b)-\Xi(a))^{1-\zeta}(\Xi(t_2)-\Xi(t_1))^{1-\frac{\vartheta}{k}}}{k\Gamma_k(\vartheta+k)}$$
$$+\frac{p^*}{k\Gamma_k(\vartheta)} \int_a^{t_1} \left|\frac{(\Xi(t_2)-\Xi(a))^{1-\zeta}}{(\Xi(t_2)-\Xi(s))^{1-\frac{\vartheta}{k}}} - \frac{(\Xi(t_1)-\Xi(a))^{1-\zeta}}{(\Xi(t_1)-\Xi(s))^{1-\frac{\vartheta}{k}}}\right| \Xi'(s)ds.$$

$$\left\|(\Xi(t_2)-\Xi(a))^{1-\zeta}(\mathscr{T}\mathscr{A})(t_2)\right.$$
$$\left.-(\Xi(t_1)-\Xi(a))^{1-\zeta}(\mathscr{T}\mathscr{A})(t_1)\right\| \to 0 \quad \text{as} \quad t_1 \to t_2.$$

Hence $\mathscr{T}(B_R)$ is bounded and equicontinuous.
Let us Assume that Theorem 5 holds and let D be an equicontinuous subset of B_R such that $D \subset \overline{\mathscr{T}D} \cup \{0\}$, therefore the function $t \to d(t) = \lambda(D(t))$ is continuous on J.
By using (H_3) and the properties of measure λ for each $t \in (a,b]$, we have,

$$
\begin{aligned}
&(\Xi(t)-\Xi(a))^{1-\zeta} d(t) \\
&\quad\leq \lambda\left((\Xi(t)-\Xi(a))^{1-\zeta}(\mathscr{T}D)(t)\cup\{0\}\right), \\
&\quad\leq \lambda\left((\Xi(t)-\Xi(a))^{1-\zeta}(\mathscr{T}D)(t)\right), \\
&\quad\leq \frac{(\Xi(b)-\Xi(a))^{1-\zeta}}{k\Gamma_k(\vartheta)} \\
&\quad\int_a^t \frac{(\Xi(s)-\Xi(a))^{1-\zeta}}{(\Xi(t)-\Xi(s))^{1-\frac{\vartheta}{k}}}\Xi'(s)p(s)\lambda(D(s))ds, \\
&\quad\leq p^*(\Xi(b)-\Xi(a))^{1-\zeta}\|d\|_{C_{\zeta,k;\Xi}}\left(\mathscr{I}_{a^+}^{\vartheta,k;\Xi}(1)\right)(t), \\
&\quad\leq \frac{p^*(\Xi(b)-\Xi(a))^{1-\zeta+\frac{\vartheta}{k}}}{\Gamma_k(\vartheta+k)}\|d\|_{C_{\zeta,k;\Xi}} \\
&\|d\|_{C_{\zeta,k;\Xi}} \leq \mathscr{L}\|d\|_{C_{\zeta,k;\Xi}},
\end{aligned}
$$

From (7), we get $\|d\|_{C_{\zeta,k;\Xi}}=0$, that is $d(t)=\lambda(D(t))=0$, for each $t\in J$, and then $D(t)$ is relatively compact in B_R. Now applying Theorem 5, we conclude that $\mathscr{T}$ has a fixed point, which is a solution of the problem $(1)-(2)$.

Our next existence result for the problem $(1)-(2)$ is based on Theorem 6.

Theorem 9.*Assume that the hypotheses $(H_1)-(H_3)$ and the condition (7) hold. Then the problem $(1)-(2)$ has a solution define on $(a,b]$.*

*Proof.*Let us consider the operator $\mathscr{T}$ defined in (6).
We have to show that $\mathscr{T}$ satisfies the assumption of Darbo's fixed point theorem.
We know that $\mathscr{T}: B_R\to B_R$ is bounded and continuous and that $\mathscr{T}(B_R)$ is equicontinuous.
Now, we have to prove that the operator $\mathscr{T}$ is a $\mathscr{L}$ set contraction.
Let $D\subset B_R$ and $t\in J$.Then we have

$$
\begin{aligned}
&\lambda\left((\Xi(t)-\Xi(a))^{1-\zeta}(\mathscr{T}D)\right)(t) \\
&\quad=\lambda\left((\Xi(t)-\Xi(a))^{1-\zeta}(\mathscr{T}\mathscr{A})(t):\mathscr{A}\in D\right), \\
&\quad=(\Xi(b)-\Xi(a))^{1-\zeta}\left\{\frac{1}{k\Gamma_k(\vartheta)}\int_a^t\frac{(\Xi(s)-\Xi(a))^{1-\zeta}}{(\Xi(t)-\Xi(s))^{1-\frac{\vartheta}{k}}}\right. \\
&\quad\left.\Xi'(s)p(s)\lambda(D(s))ds\right\}, \\
&\quad\leq p^*(\Xi(b)-\Xi(a))^{1-\zeta}\lambda_{C_{\zeta,k;\Xi}}(D)\left(\mathscr{I}_{a^+}^{\vartheta,k;\Xi}(1)\right)(t), \\
&\quad\leq \frac{p^*(\Xi(b)-\Xi(a))^{1-\zeta+\frac{\vartheta}{k}}}{\Gamma_k(\vartheta+k)}\lambda_{C_{\zeta,k;\Xi}}(D),
\end{aligned}
$$

Therefore,

$$
\lambda_{C_{\zeta,k;\Xi}}(\mathscr{T}D)\leq\frac{p^*(\Xi(b)-\Xi(a))^{1-\zeta+\frac{\vartheta}{k}}}{\Gamma_k(\vartheta+k)}\lambda_{C_{\zeta,k;\Xi}}(D).
$$

So by (7) the operator $\mathscr{T}$ is a $\mathscr{L}$ set contraction.

4 Stability Theory

In this section we introduce the concept of the Ulam stability for the problem $(1)-(2)$.

Let $\mathscr{A}\in C^1_{\vartheta,k;\Xi}(J),\varepsilon>0$ and $\nu:(a,b]\to[0,\infty)$ be a continuous function, we consider the following inequality for all $t\in(a,b]$:

$$
\|({}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\mathscr{A})(t)-f(t,\mathscr{A}(t),{}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\mathscr{A}(t)\|\leq\varepsilon\nu(t). \tag{8}
$$

Definition 4.*[18] Problem (1)-(2) is Ulam-Hyers-Rassias $(U-H-R)$ stable with respect to ν if there exists a real number $\alpha_{f,\nu}>0$ such that for each $\varepsilon>0$ and for each solution $\mathscr{A}\in C^1_{\vartheta,k;\Xi}(J)$ of the inequality (8) there exists a solution $\overline{\mathscr{A}}\in C^1_{\vartheta,k;\Xi}(J)$ of (1),(2) with*

$$
\left\|\mathscr{A}(t)-\overline{\mathscr{A}}(t)\right\|\leq\varepsilon\alpha_{f,\nu}\nu(t),\quad t\in J.
$$

*Remark.*A function $\mathscr{A}\in C^1_{\vartheta,k;\Xi}(J)$ is a solution of inequality (8) if and only if there exist a $\sigma\in C_{\vartheta,k;\Xi}(J)$ such that
$\|\sigma(t)\|\leq\varepsilon\nu(t), t\in(a,b]$.
$\left({}^H_k\mathscr{D}^{\vartheta,k;\Xi}_{a^+}\mathscr{A}\right)(t)$
$=f\left(t,\mathscr{A}(t),\left({}^H_k\mathscr{D}^{\vartheta,k;\Xi}_{a^+}\mathscr{A}\right)(t)\right)+\sigma(t),\quad t\in(a,b].$

Theorem 10.*Assume that in addition to $(H_1)-(H_2)$ and (7), the following Hypotheses hold*
(H_4) :There exists a non-decreasing function $\nu\in C^1_{\vartheta,k;\Xi}(J)$and $K_\nu>0$ such that for each $t\in J$ we have

$$
(\mathscr{I}^{\vartheta,k;\Xi}_{a^+}\nu)\leq K_\nu\nu(t).
$$

(H_5): There exists a continous function $q:J\to[0,\infty)$ such that for each $t\in(a,b]$. We have

$$
p(t)\leq q(t)\nu(t).
$$

Then, our proposed problem Ulam-Hyers-stable.

$$
Set\quad q^*=\sup_{t\in J}q(t).
$$

*Proof.*Let $\mathscr{A}\in C^1_{\vartheta,k;\Xi}(J)$ be a solution of $\|({}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\mathscr{A})(t)-f(t,\mathscr{A}(t),{}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\mathscr{A}(t)\|\leq\varepsilon\nu(t)$ and let us assume that $\overline{\mathscr{A}}$ is the unique solution of the problem

$$
\begin{cases}
\left({}^H_k\mathscr{D}^{\nu,r;\Xi}_{a^+}\overline{\mathscr{A}}\right)(t)=f(t,\overline{\mathscr{A}},{}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\overline{\mathscr{A}})(t); & t\in(a,b] \\
(\mathscr{I}^{k(1-\zeta)k;\Xi}_{a^+}\overline{\mathscr{A}})(a^+)=(\mathscr{I}^{k(1-\zeta)k;\Xi}_{a^+}x)(a^+)=\mu.
\end{cases}
$$

By the Lemma 6

$$
\overline{\mathscr{A}}(t)=\frac{(\Xi(t)-\Xi(a))^{\vartheta-1}}{\Gamma_{k\vartheta}}\mathscr{I}^{k(1-\zeta),k;\Xi}_{a^+}\mu+(\mathscr{I}^{\nu,k;\Xi}_{a^+}w)(t),
$$

where, the function $w \in C^1_{\vartheta,k;\Xi}(J)$ satisfying the functional equation

$$w(t) = f(t, \mathscr{A}(t), w(t)).$$

Since, the $\mathscr{A}$ is a solution of the inequality (8) , by Remark 4 , we have

$$\left({}^H_k\mathscr{D}^{\vartheta,r;\Xi}_{a^+}\overline{\mathscr{A}}\right)(t) = f\left(t, \overline{\mathscr{A}}, \overline{\mathscr{A}}\left(\frac{t}{\eta}\right)\right)(t) + \sigma(t), \quad t \in (a,b]. \tag{9}$$

Clearly, the solution of is given by

$$\mathscr{A}(t) = \frac{(\Xi(t) - \Xi(a))^{\vartheta-1}}{\Gamma_{k\vartheta}} \mathscr{I}^{sk(1-\zeta),k;\Xi}_{a^+}\mu + (\mathscr{I}^{\vartheta,k;\Xi}_{a^+}(\overline{w} + \sigma))(t),$$

where, the function $\overline{w} \in C^1_{\vartheta,k;\Xi}(J)$ satisfying the functional equation

$$\overline{w}(t) = (t, \mathscr{A}(t), \overline{w}(t)).$$

Hence, for each $t \in (a,b]$, we have

$$\begin{aligned}
\left\|\mathscr{A}(t) - \overline{\mathscr{A}}(t)\right\| &\leq \left(\mathscr{I}^{\vartheta,k;\Xi}_{a^+}\|\overline{w}(s) - w(s)\|\right)(t) \\
&+ \left(\mathscr{I}^{\vartheta,k;\Xi}_{a^+}\|\sigma(s)\|\right)(t), \\
&\leq \varepsilon k_\nu \nu(t) \\
&+ \frac{1}{k\Gamma_k(\vartheta)}\int_a^t \frac{\Xi'(s)2q(s)\nu(s)ds}{(\Xi(t) - \Xi(s))^{1-\frac{\vartheta}{k}}}, \\
&\leq \varepsilon k_\nu \nu(t) + 2q^*\left(\mathscr{I}^{\vartheta,k;\Xi}_{a^+}\right)(t), \\
&\leq (\varepsilon + 2q^*)k_\nu \nu(t).
\end{aligned}$$

Then, for each $t \in (a,b]$, we have

$$\left\|\mathscr{A}(t) - \overline{\mathscr{A}(t)}\right\| \leq \alpha_{f,\nu}\varepsilon\nu(t),$$

where

$$\alpha_{f,\nu} = k_\nu\left(1 + \frac{2q^*}{\varepsilon}\right).$$

Hence our proposed problem (1) − (2) is U-H-R stable.

5 Conclusion

We have successfully studied fractional type Ambartsumian equations with k-generalized Hilfer fractional derivative. Also, we have provided some sufficient conditions guaranteeing the existence of solutions for a class of fractional order Ambartsumian equations. We will apply the numerical algorithms to the proposed problem in future.

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The authors declare no conflicts of interest.

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