Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/100103

Construction by the Tikhonov Method of a Nonzero Solution of the Homogeneous Cauchy Problem for one Equation with a Fractional Derivative

Bakhrom Irgashev[1,2]

[1] Namangan Engineering Construction Institute, Namangan, 160103 Uzbekistan
[2] Romanovskiy Institute of Mathematics, Academy of Sciences of Uzbekistan, Tashkent, 100174 Uzbekistan

Received: 2 Sep. 2023, Revised: 18 Oct. 2023, Accepted: 20 Nov. 2023
Published online: 1 Jan. 2024

Abstract: In this article, for a homogeneous equation of high even order with a fractional derivative in the sense of Caputo, a non-trivial solution of the homogeneous Cauchy problem in the upper half-plane is constructed by the method of A.N.Tikhonov. The idea of the method is that the solution is constructed as a series of infinitely differentiable functions with certain estimates. The values of the functions themselves and derivatives of any order at the initial point are equal to 0. The existence of such functions follows from the works of Carleman on quasi-analytic functions.

Keywords: Equation, even order, fractional derivative, initial value problem, non-uniqueness, series.

1 Introduction

Consider in the domain $\Omega = \{(x,y) : -\infty < x < +\infty, 0 < y < T\}$ the following initial value problem of problem type Cauchy

$$\begin{cases} {}_C D_{0y}^{\frac{1}{p}} u(x,y) = \frac{\partial^{2n} u(x,y)}{\partial x^{2n}}, \\ u(x,0) = 0, \end{cases} \tag{1}$$

where

$$1 \neq p \in N, n = 2m-1, m \in N,$$

$${}_C D_{0y}^{\frac{1}{p}} u(x,y) = \frac{1}{\Gamma\left(1-\frac{1}{p}\right)} \int_0^y \frac{u_t(x,t)\,dt}{(y-t)^{\frac{1}{p}}}$$

is a fractional derivative in Caputo's sense.

In works [1,2] an equation of type (1) for $n = 1$ was studied and it was shown that it describes the process of diffusion in a medium with fractal geometry. The article [3] shows the uniqueness of the solution to the Cauchy problem for the diffusion equation of fractal order, in the class of the following functions:

$$|u(x,y)| \leqslant M_1 \exp\left(M_2 |x|^{\frac{2}{2-\alpha}}\right), M_1, M_2 > 0. \tag{2}$$

Also in the work [3] a nontrivial solution to the homogeneous Cauchy problem was constructed when condition (2) is violated. Note also that problems with initial conditions for equations of even order, involving fractional derivatives, were considered in the works [4,5,6], where a uniqueness class similar to the Tikhonov classes [7] was obtained.

In this article, using methods from works [7,8], a non-zero solution to the Cauchy problem is constructed.

* Corresponding author e-mail: bahromirgasev@gmail.com

2 Main Results

The following lemma is true.

Lemma. Let the $F(y)$ satisfy the following conditions:
1. $F(y)$ has a derivative of any integer order;
2. $\left.\frac{d^kF(y)}{dy^k}\right|_{y=0}=0, \forall k\in N\cup\{0\}$;
3. $\left|\frac{d^kF(y)}{dy^k}\right|<[\alpha k]!, \forall k\in N\cup\{0\}, 1<\alpha$;

then
1. ${}_CD_{0y}^{\frac{1}{p}+k}F(y)=\frac{1}{\Gamma\left(1-\frac{1}{p}\right)}\frac{d^{k+1}}{dy^{k+1}}\int\limits_0^y\frac{F(t)dt}{(y-t)^{\frac{1}{p}}}=D_{0y}^{\frac{1}{p}+k}F(y)$;
2. ${}_CD_{0y}^{\frac{1}{p}}\left({}_CD_{0y}^{k+\frac{m}{p}}F(y)\right)=\begin{cases}\frac{d^{k+1}F(y)}{dy^{k+1}}, m=p-1,\\ D_{0y}^{k+\frac{m+1}{p}}(y), 0\le m\le p-1;\end{cases}$
3. $\lim\limits_{y\to+0}{}_CD_{0y}^{k+\frac{m}{p}}F(y)=0$;
4. $\left|{}_CD_{0y}^{k+\frac{m}{p}}F(y)\right|\leqslant M[\alpha(k+1)]!$,

here $[a]$ is the integer part of the number a; $m,k\in N\cup\{0\}, 0\leqslant m\leqslant p-1, 0<M-const$.

Proof of the lemma.

Note that for the existence of F(y), it is sufficient to satisfy the following condition (see [7,9]):

$$\sum_{k=1}^{\infty}\frac{1}{\sqrt[k]{[\alpha k]!}}<\infty.$$

Let's check this condition, we have

$$n!=\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\exp\frac{1}{12n+\theta_n}, 0<\theta_n<1,$$

from here we get

$$\sum_{k=1}^{\infty}\frac{1}{\sqrt[k]{[\alpha k]!}}=\sum_{k=1}^{\infty}\frac{1}{\left(\sqrt{2\pi[\alpha k]}\left(\frac{[\alpha k]}{e}\right)^{[\alpha k]}\exp\left(\frac{1}{12[\alpha k]+\theta_n}\right)\right)^{\frac{1}{k}}}\sim$$

$$\sim\sum_{k=1}^{\infty}\frac{1}{\left(\left(\frac{[\alpha k]}{e}\right)^{\alpha k-\{\alpha k\}}\right)^{\frac{1}{k}}}=\sum_{k=1}^{\infty}\frac{\left(\frac{[\alpha k]}{e}\right)^{\frac{\{\alpha k\}}{k}}}{\left(\left(\frac{[\alpha k]}{e}\right)^{\alpha}\right)}\sim\sum_{k=1}^{\infty}\frac{1}{\left(\left(\frac{[\alpha k]}{e}\right)^{\alpha}\right)}<\infty.$$

Now let's prove the lemma.

1. Known formula [10]

$${}_CD_{0y}^{\frac{1}{p}+k}F(y)=D_{0y}^{\frac{1}{p}+k}F(y)-\sum_{i=0}^{k}\frac{F^{(i)}(0)}{\Gamma\left(i+1-\frac{1}{p}-k\right)}y^{i-\frac{1}{p}-k},$$

from here

$${}_CD_{0y}^{\frac{1}{p}+k}F(y)=D_{0y}^{\frac{1}{p}+k}F(y), k\in N\cup\{0\}.$$

2.

$${}_CD_{0y}^{\frac{1}{p}}\left({}_CD_{0y}^{\frac{m}{p}+k}F(y)\right)={}_CD_{0y}^{\frac{1}{p}}\left(\frac{1}{\Gamma\left(1-\frac{m}{p}\right)}\int\limits_0^y\frac{F^{(k+1)}(t)\,dt}{(y-t)^{\frac{m}{p}}}\right)=$$

$$=\frac{1}{\Gamma\left(1-\frac{1}{p}\right)\Gamma\left(1-\frac{m}{p}\right)}\int\limits_0^y\frac{\frac{d}{dz}\int\limits_0^z\frac{F^{(k+1)}(t)dt}{(z-t)^{\frac{m}{p}}}}{(y-z)^{\frac{1}{p}}}dz=$$

$$=\frac{1}{\Gamma\left(1-\frac{1}{p}\right)\Gamma\left(1-\frac{m}{p}\right)}\int\limits_0^y F^{(k+2)}(t)\,dt\int\limits_t^y \frac{dz}{(y-z)^{\frac{1}{p}}(z-t)^{\frac{m}{p}}}=$$

$$=\frac{1}{\Gamma\left(2-\frac{1+m}{p}\right)}\int\limits_0^y (y-t)^{1-\frac{m+1}{p}}F^{(k+2)}(t)\,dt=\begin{cases}F^{(k+1)}(y)\,,m=p-1,\\ {}_CD_{0y}^{k+\frac{m+1}{p}}F(y)\,,0\leqslant m\leqslant p-2.\end{cases}$$

3. Let us prove the third property

$$\left|{}_CD_{0y}^{k+\frac{m}{p}}F(y)\right|<\frac{[\alpha(k+1)]!}{\Gamma\left(1-\frac{m}{p}\right)}\int\limits_0^y (y-t)^{-\frac{m}{p}}dt=$$

$$=\frac{[\alpha(k+1)]!}{\Gamma\left(1-\frac{m}{p}\right)}\frac{y^{1-\frac{m}{p}}}{1-\frac{m}{p}},$$

from here

$$\lim_{y\to+0}{}_CD_{0y}^{k+\frac{m}{p}}F(y)=0.$$

4. From property 3 we have

$$\left|{}_CD_{0y}^{k+\frac{m}{p}}F(y)\right|\leqslant M[\alpha(k+1)]!,$$

where

$$M=\frac{T^{1-\frac{m}{p}}}{\Gamma\left(2-\frac{m}{p}\right)}.$$

The lemma is proved.

The solution to the Cauchy problem (1) will be constructed in the form of the following series:

$$u(x,y)=V_0(x,y)+V_1(x,y)+\ldots+V_{p-1}(x,y)\,, \tag{3}$$

where

$$V_0(x,y)=\sum_{j=0}^{\infty}\frac{x^{2npj}}{(2npj)!}F^{(j)}(y)\,, \tag{4}$$

$$V_1(x,y)=\sum_{j=0}^{\infty}\frac{x^{2n(jp+1)}}{(2n(jp+1))!}{}_CD_{0y}^{j+\frac{1}{p}}F(y)\,, \tag{5}$$

$$\ldots$$

$$V_{p-2}(x,y)=\sum_{j=0}^{\infty}\frac{x^{2n(jp+p-2)}}{(2n(jp+p-2))!}{}_CD_{0y}^{j+\frac{p-2}{p}}F(y)\,, \tag{6}$$

$$V_{p-1}=\sum_{j=0}^{\infty}\frac{\left({}_CD_{0y}^{j+\frac{p-1}{p}}F(y)\right)x^{2n(jp+p-1)}}{(2n(jp+p-1))!}. \tag{7}$$

Using the lemma, we formally have

$$\frac{\partial^{2n}V_0(x,y)}{\partial x^{2n}}=\sum_{j=1}^{\infty}\frac{x^{2n(pj-1)}}{(2n(pj-1))!}F^{(j)}(y)\,, \tag{8}$$

$$\frac{\partial^{2n}V_1(x,y)}{\partial x^{2n}}=\sum_{j=0}^{\infty}\frac{x^{2njp}}{(2njp)!}{}_CD_{0y}^{j+\frac{1}{p}}F(y)\,, \tag{9}$$

$$\ldots$$

$$\frac{\partial^{2n}V_{p-2}(x,y)}{\partial x^{2n}}=\sum_{j=0}^{\infty}\frac{x^{2n(jp+p-3)}}{(2n(jp+p-3))!}{}_CD_{0y}^{j+\frac{p-2}{p}}F(y),\tag{10}$$

$$\frac{\partial^{2n}V_{p-1}(x,y)}{\partial x^{2n}}=\sum_{j=0}^{\infty}\frac{x^{2n(jp+p-2)}}{(2n(jp+p-2))!}{}_CD_{0y}^{j+\frac{p-1}{p}}F(y),\tag{11}$$

$${}_CD_{0y}^{\frac{1}{p}}V_0(x)=\sum_{j=0}^{\infty}\frac{x^{2npj}}{(2npj)!}D_{0y}^{\frac{1}{p}+j}F(y),\tag{12}$$

$${}_CD_{0y}^{\frac{1}{p}}V_1(x,y)=\sum_{j=0}^{\infty}\frac{x^{2n(jp+1)}}{(2n(jp+1))!}{}_CD_{0y}^{\frac{2}{p}+j}F(y),\tag{13}$$

$$\ldots$$

$${}_CD_{0y}^{\frac{1}{p}}V_{p-2}(x,y)=\sum_{j=0}^{\infty}\frac{x^{2n(jp+p-2)}}{(2n(jp+p-2))!}{}_CD_{0y}^{\frac{p-1}{p}+j}F(y),\tag{14}$$

$${}_CD_{0y}^{\frac{1}{p}}V_{p-1}(x,y)=\sum_{j=1}^{\infty}\frac{x^{2n(jp-1)}}{(2n(jp-1))!}F^{(j)}(y),\tag{15}$$

Next we will show that the series (4) -(15) will converge uniformly, then we get that (3) is a non-zero solution to the problem (1). In what follows we will assume that $|x|\geq L>1$, where L is some fixed number and for any fixed value $\varepsilon>0$ the relation

$$\frac{2n}{2n-\frac{\alpha}{p}}\leqslant\frac{2n}{2n-\frac{1}{p}}+\varepsilon,\ 1<\alpha<2n.\tag{16}$$

It is not difficult to check the correctness of the relation that we will use further

$$\frac{k!}{n!}\leqslant\frac{1}{(n-k)!},k,n\in N,k\leqslant n.\tag{17}$$

Using relations (16)-(17), we begin to study the convergence of series. Let us show the uniform convergence of series (4) and (7), the convergence of others is shown in a similar way. We have

$$|V_0(x)|<\sum_{j=0}^{\infty}\frac{x^{2npj}[\alpha j]!}{(2npj)!}\leqslant\sum_{j=0}^{\infty}\frac{x^{2npj}}{(2npj-[\alpha j])!},$$

let be

$$m(j)=2npj-[\alpha j],$$

then

$$|V_0(x)|<\sum_{j=0}^{\infty}\frac{\left(|x|^{\frac{2npj}{2npj-[\alpha j]}}\right)^m}{m(j)!}\leqslant\sum_{j=0}^{\infty}\frac{\left(|x|^{\frac{2npj}{2npj-\alpha j}}\right)^m}{m(j)!}\leqslant$$

$$\leqslant\sum_{m=0}^{\infty}\frac{\left(|x|^{\frac{2np}{2np-\alpha}}\right)^m}{m!}=\exp\left(|x|^{\frac{2n}{2n-\frac{\alpha}{p}}}\right)\leqslant\exp\left(|x|^{\frac{2n}{2n-\frac{1}{p}}+\varepsilon}\right).$$

Further

$$\left|{}_CD_{0y}^{\frac{1}{p}}V_{p-2}(x,y)\right|<M\sum_{j=0}^{\infty}\frac{x^{2n(jp+p-2)}[\alpha(j+1)]!}{(2n(jp+p-2))!}\leqslant$$

$$\leqslant Mx^{2n(p-2)}\sum_{j=0}^{\infty}\frac{x^{2njp}}{(2n(jp+p-2)-[\alpha(j+1)])!},$$

we introduce the notation

$$m(j)=2n(jp+p-2)-[\alpha(j+1)],$$

then

$$\left| {}_C D_{0y}^{\frac{1}{p}} V_{p-2}(x,y) \right| < Mx^{2n(p-2)} \sum_{j=0}^{\infty} \frac{\left(|x|^{\frac{2njp}{2n(jp+p-2)-[\alpha(j+1)]}} \right)^m}{m(j)!} <$$

$$< Mx^{2n(p-2)} \sum_{j=0}^{\infty} \frac{\left(|x|^{\frac{2njp}{2njp-\alpha j}} \right)^m}{m(j)!} < Mx^{2n(p-2)} \exp\left(|x|^{\frac{2n}{2n-\frac{\alpha}{p}}} \right).$$

Now let $0 < \delta < \frac{2n}{2n-\frac{\alpha}{p}}$ an arbitrary fixed number, then the number L is chosen so that for all $|x| > L,$ the following inequalities were simultaneously satisfied:

$$\frac{|x|^{\delta}}{\ln|x|} \geqslant 2n(p-2), |x| > \left(\frac{2n(p-2)}{\delta} \right)^{\frac{1}{\delta}},$$

then

$$\left| {}_C D_{0y}^{\frac{1}{p}} V_{p-2}(x,y) \right| < M \exp\left(|x|^{\frac{2n}{2n-\frac{\alpha}{p}}} + |x|^{\delta} \right) <$$

$$< M \exp\left(2|x|^{\frac{2n}{2n-\frac{\alpha}{p}}} \right) < M \exp\left(2|x|^{\frac{2n}{2n-\frac{1}{p}}+\varepsilon} \right).$$

Theorem. The Cauchy problem (1) has a non-zero solution from the class

$$|u(x,y)| \leqslant M_1 \exp\left(M_2 |x|^{\frac{2n}{2n-\frac{1}{p}}+\varepsilon} \right),$$

where

$$0 < M_1, M_2, \varepsilon - const.$$

References

[1] R. R. Niqmatullin, Theoretical Explanation of the Universal Response, *Phys. Stat. Sol. (b)* **123** (2), 739–745 (1984).
[2] R. R. Niqmatullin, The realization of the generalized transfer equation in a medium with fractal geometry, *Phys. Stat. Sol. (b)* **133** (1), 425–430 (1986).
[3] A. N. Kochubei, Diffusion of fractional order,*Differ. Equ.* **26** (4), 485–492 (1990).
[4] A. V. Pskhu, The fundamental solution of a diffusion-wave equation of fractional order,*Izv. Math.* **73** (2), 351–392 (2009).
[5] A. V. Pskhu, Fundamental solutions and cauchy problems for an odd-order partial differential equation with fractional derivative, *Elect. J. Differ. Equ.* **2019**, 1–13 (2019).
[6] L. L. Karasheva, Cauchy problem for high even order parabolic equation with time fractional derivative, *Siberian Electr. Math. Rep.* **15**, 696–706 (2018).[In Russian].
[7] A. N. Tychonoff , Theoremes dunicite pour lequation de la chaleur, *Mat. Sb.* **42**(2), 199–216 (1935).
[8] O. A. Ladyzenskaya, On the uniqueness of the solution of Cauchy's problem for a linear parabolic equation, *Mat. Sbornik* **69**, 175–184 (1950).[Russian].
[9] T. Carleman, *Les fonctions quasi analytiques*, Gauthier-Villars, North-Holland Mathematical Studies, 1926.
[10] Y. Luchko and R. Gorenflo, An operational method for solving fractional differential equations with the Caputo derivatives, *Acta Math. Vietnamica* **24** (2), 207–233 (1999).