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Progressive Type-II Censoring Ailamujia distribution Under Binomial Removals with Data Analysis

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Abstract: Ailamujia distribution is a useful lifetime model for several engineering applications. The Ailamujia distribution is a versatile distribution to model the repair time and guarantee the distribution delay time. This study considers the estimation problem for Ailamujia distribution based on progressive Type-II censoring with Binomial removals. The maximum likelihood estimators (MLE's) for the model parameters were derived along with the asymptotic confidence intervals. A simulation study was performed using different values of sample sizes, parameters, and different number of removed observations to investigate the behavior of the estimators. A real life time data set was analyzed using progressive Type-II Ailamujia distribution and showed appropriate results.

Keywords: Ailamujia Distribution, Maximum Likelihood Estimation, Progressive Type-II Censoring, Survival Analysis

1 Introduction

In reliability theory and survival analysis, it is difficult to collect lifetime data for all subjects under consideration because of time and cost constraints. Various types of censoring schemes are used by the practitioners based on the model and available information using both parametric and nonparametric methods. Recently, progressive censoring is of special importance in reliability and survival analysis. Progressive censoring was first introduced by Cohen [1]. An extensive studies are available in literature related to progressive censoring; among these studies of Mann et al. [2], Balakrisnan and Aggarwala [3], and Laweless [4]. Different failure time models in literature have been used on progressive censoring under Binomial removals; including exponential distribution [5], Type-II generalized logistic distribution [6], generalized exponential distribution [7], Pareto distribution [8,9,10], exponentiated gamma [11]. Rayleigh distribution [12]. Burr Type−XII Distribution [13] and the Gompertz Distribution [14].

Type-II Progressively censored life test is conducted as follows. Consider n identical units in a test, at the time of the first failure, R_1 units from the remaining $n-1$ survival items are removed. At the time of the second failure, R_2 units from the remaining $n-R_1-1$ items are removed, and so forth. Finally, at the time of m-th failure, the reaming survival units, say R_M are removed. Note that censoring takes place here progressively in m stages. Clearly, this scheme includes, as special cases, the complete sample situation (when $m=n$ and $R_1=\cdots=R_m=0$) and the conventional Type-II right censoring situation (when $R_1=\cdots=R_{m-1}=0$ and $R_m=n-m$). The corresponding scheme $(r_1,r_2,\ldots,r_m)$ is known as progressive Type-II right censoring scheme.

Different lifetime data can be represented by several well-known continuous probability distributions as well as their generalizations. Ailamujia distribution is a useful lifetime model that has many engineering applications [15]. In some practical applications such as the repair time, guarantee the distribution delay time, it is found that the Ailamujia model is a convenient one compared to other models. Lv et al. [16] studied the different properties including mean, variance, and median and maximum likelihood estimators. This distribution has also been investigated for the interval estimation and the hypothesis testing [17]. The minimax estimation of the Ailamujia model parameter has been discussed under a non-informative prior using the three loss functions [18].

This paper considers progressive Type-II censoring for Ailamujia distribution with Binomial removals. The maximum likelihood estimators (MLE's) of the model parameters along with the asymptotic confidence intervals

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are obtained. A simulation study was performed using different combinations of sample sizes, parameters and different number of removed observations to observe the behavior of the MLE's via bias and root mean square error (RMSE). Analysis of real life time data is presented as an illustration of the developed progressive censoring scheme in this work.

2 Estimation of the Parameters

If X is the lifetime of a product and follows Ailamujia distribution, its probability density function is given by:

$$f(x,\theta) = 4\theta^2 x e^{-2\theta x}; x \geq 0, \theta > 0 \tag{1}$$

while the corresponding cumulative distribution function is given as:

$$F(x,\theta) = 1-(1+2\theta x)e^{-2\theta x}; x \geq 0, \theta > 0 \tag{2}$$

where, θ is the unknown parameter. It can be easily concluded that

$$E(X) = \frac{1}{\theta} \text{ and } \sigma^2 = \frac{1}{2\theta}$$

The maximum likelihood estimator for θ is given by

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} x_i}$$

The survival function and the hazard function are, respectively, given by:

$$r(x) = (1+2\theta x)e^{-2\theta x}$$

$$h(x) = \frac{4\theta^2 x}{1+2\theta x}$$

Let (X_1,R_1), (X_2,R_3),$\dots$,(X_m,R_m), be progressively censored sample, where $X_1 < X_2 < \cdots < X_m$. With predetermined number of removals, such as $R_1 = r_1, R_2 = r_2, \dots, R_m = r_m$, the conditional likelihood function can be written as (Cohen [1]):

$$L(\theta;x|R=r) = A\prod_{i=1}^{m} f(x_i)\left(1-F(x_i)\right)^{r_i} \tag{3}$$

where

$$A = n(n-r-1)\dots\left(n-\sum_{i=1}^{m-1} r_i+1\right)$$

Substituting (1) and (2) into equation (3), the likelihood function becomes

$$L(\theta;x|R=r) = A\prod_{i=1}^{m} 4\theta^2 x_i e^{-2\theta x_i}\left((1+2\theta x_i)e^{-2\theta x_i}\right)^{r_i} \tag{4}$$

Suppose that an individual unit being removed from the test at the i^{th} failure, $i = 1,2,\dots,m-1$, is independent of the others but with the same probability p. Therefore, $R_i, i = 1,2,\dots,m-1$, follows a binomial distribution with parameters

$$n-m-\sum_{k=1}^{i-1} r_k \text{ and } P$$

Thus,

$$P(R_1 = r_1) = \binom{n-m}{r_1} p^{r_1}(1-p)^{n-m-r_1}$$
$$P(R_i = r_i|R_{i-1} = r_{i-1},\dots,R_1 = r_1)$$
$$= \binom{n-m-\sum_{k=1}^{i-1} r_k}{r_i} p^{r_i}(1-p)^{n-m-\sum_{k=1}^{i-1} r_k},$$
$$\forall i = 1,2,\dots,m-1$$

where

$$0 \leq r_i \leq n-m-\sum_{j=1}^{i-1} r_j (i = 1,\dots,m-1)$$

The full likelihood function takes the following form

$$L(\theta,p;x,r) = L(\theta;x|R=r)P(R=r) \tag{5}$$

where $P(\underline{R}=\underline{r})$ is the joint distribution and is given by:

$$\begin{aligned} P(\underline{R}=\underline{r}) = \;& P(R_1 = r_1)P(R_2 = r_2|R_1 = r_1)\dots \\ & P(R_{m-1} = r_{m-1}|R_{m-2} = r_{m-2},\dots,R_1 = r_1) \end{aligned}$$

and

$$P(\underline{R}=\underline{r}) = \frac{(n-m)!\,p^{\sum_{i=1}^{m-1} r_i}(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}}{(n-m-\sum_{i=1}^{m-1} r_i)!\prod_{i=1}^{m-1} r_i!} \tag{6}$$

Using equations (4), (5) and (6), it is possible to write the full likelihood function as shown in the following form

$$L(\theta,p;x,r) = AL_1(\theta)L_2(p)$$

where

$$A = \frac{c(n-m)!}{(n-m-\sum_{i=1}^{m-1} r_i)!\prod_{i=1}^{m-1} r_i!}$$

$$L_1(\theta) = \prod_{i=1}^{m} 4\theta^2 x_i e^{-2\theta x_i}\left((1+2\theta x_i)e^{-2\theta x_i}\right)^{r_i}$$

$$L_2(p) = p^{\sum_{i=1}^{m-1} r_i}(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}$$

The MLE of θ can be obtained by maximizing

$$L_1(\theta) = \prod_{i=1}^{m} 4\theta^2 x_i e^{-2\theta x_i}\left((1+2\theta x_i)e^{-2\theta x_i}\right)^{r_i}$$

$$L_1(\theta) = 4^m\theta^{2m}\left(\prod_{i=1}^{m} x_i\right) e^{-2\theta\sum_{i=1}^{m} x_i}$$
$$\times\left(\prod_{i=1}^{m}(1+2\theta x_i)^{r_i}\right) e^{-2\theta\sum_{i=1}^{m} r_i x_i}$$

$$L_1(\theta) = 4^m\theta^{2m}\left(\prod_{i=1}^{m} x_i\right)\left(\prod_{i=1}^{m}(1+2\theta x_i)^{r_i}\right)$$
$$\times e^{-2\theta\left(\sum_{i=1}^{m} x_i+\sum_{i=1}^{m} r_i x_i\right)}$$

The ln likelihood function

$$\ln L_1(\theta) = m\ln 4 + 2m\ln\theta + \sum_{i=1}^{m}\ln x_i + \sum_{i=1}^{m} r_i \ln(1+2\theta x_i)$$
$$-2\theta\left(\sum_{i=1}^{m} x_i + \sum_{i=1}^{m} r_i x_i\right)$$

The MLE of θ can be obtained by solving the following equation:

$$\frac{nL_1(\theta)}{d\theta} = g(\theta) = 0$$

$$\frac{m}{\theta} + \theta\sum_{i=1}^{m}\frac{r_i}{1+2\theta x_i} - \left(\sum_{i=1}^{m} x_i + \sum_{i=1}^{m} r_i x_i\right) = 0$$

Therefore, the MLE of θ, can be obtained by solving $g(\theta) = 0$ using fixed point iterative numerical method.

Maximizing $L_2(p)$, the MLE of p can be obtained as:

$$\hat{p} = \frac{\sum_{i=1}^{m-1} r_i}{\sum_{i=1}^{m-1} r_i + (m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i}$$

The elements of the Fisher information matrix can be obtained using the following likelihood function

$$L(\theta,p) \propto \prod_{i=1}^{m}\left[4\theta^2 x_i e^{-2\theta x_i}\left((1+2\theta x_i)e^{-2\theta x_i}\right)^{r_i}\right]$$
$$\left[p^{\sum_{i=1}^{m-1} r_i}(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}\right]$$

$$\log L(\theta,p) \propto 2m\ln\theta + \sum_{i=1}^{m}\ln x_i + \sum_{i=1}^{m} r_i\ln(1+2\theta x_i)$$
$$-2\theta\left(\sum_{i=1}^{m} x_i + \sum_{i=1}^{m} r_i x_i\right) + \sum_{i=1}^{m-1} r_i \log p$$
$$+\left((m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i\right)\log(1-p)$$

Thus, the variance covariance matrix is now approximated as

$$\begin{bmatrix} -\frac{\partial^2 \log L(\theta,p)}{\partial\theta^2} & 0 \\ 0 & -\frac{\partial^2 \log L(\theta,p)}{\partial p^2}\end{bmatrix}^{-1}$$

where

$$\frac{\partial^2 \log L(\theta,p)}{\partial\theta^2} = \sum_{i=1}^{m}\frac{r_i}{1+2\theta x_i} - 2\theta\sum_{i-1}^{m}\frac{r_i x_i}{(1+2\theta x_i)^2} - \frac{m}{\theta^2}$$

$$\frac{\partial^2 \log L(\theta,p)}{\partial p^2} = \frac{\sum_{i=1}^{m-1} r_i}{p^2} + \frac{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i}{(1-p)^2}$$

The asymptotic distribution of the maximum likelihood estimators is a biverate normal (BVN) given as

$$\begin{pmatrix}\hat{\theta}\\ \hat{p}\end{pmatrix} \approx \text{BVN}\left(\begin{pmatrix}\theta\\ p\end{pmatrix}, V\right)$$

An estimate of V is obtained by using the observed Fisher information matrix where the parameters θ and p are replaced by the corresponding maximum likelihood estimates.

The $100(1-\alpha)\%$ asymptotic confidence intervals for θ and p are

$$\hat{\theta} \pm z_{\frac{\alpha}{2}}\sqrt{var(\hat{\theta})},\;\; \hat{p} \pm z_{\frac{\alpha}{2}}\sqrt{var(\hat{p})}$$

3 Simulation Results

A simulation study is performed to assess the final sample behavior of the maximum likelihood estimator. Different sample sizes; namely $n = 25, 50$, and 100 are used. Different combinations of the parameter values of θ were considered. The values of the parameter p used in the simulation study are 0.25 and 0.5. The simulation results are based on 1000 replicates. The means and root mean square errors (RMSE) of the maximum likelihood estimators for the two parameters p and θ are shown in Table 1.

The following remarks can be drawn based on the results shown in Table 1:

1. For fixed m as n increase, the bias and RMSE always show a decreasing trend.
2. For fixed n as m increase, the bias and RMSE decrease.
3. As the scale parameter θ increase, the bias and RMSE increase.
4. As the value of the parameter p increase, the bias and RMSE increase.

Table 1: Mean and RMSE of the MLEs for p and θ for different values of n, m, θ and p

			$p = 0.25$				$p = 0.5$			
n	m	θ	$\hat{P}$		$\hat{\theta}$		$\hat{P}$		$\hat{\theta}$	
			Mean	RMSE	mean	RMSE	Mean	RMSE	mean	RMSE
25	15	0.1	0.26404	0.07685	0.07809	0.0155	0.52325	0.11605	0.0872	0.01709
		0.3	0.26404	0.07685	0.23905	0.04899	0.52325	0.11605	0.27329	0.05645
		0.5	0.26404	0.07685	0.42122	0.10494	0.52325	0.11605	0.49265	0.13228
	20	0.1	0.28675	0.1217	0.09289	0.01549	0.5407	0.16602	0.09711	0.01592
		0.3	0.28675	0.1217	0.28249	0.04875	0.5407	0.16602	0.29681	0.05025
		0.5	0.28675	0.1217	0.48277	0.08689	0.5407	0.16602	0.50868	0.09652
50	30	0.1	0.25695	0.04949	0.08634	0.01185	0.50982	0.08153	0.09099	0.01277
		0.3	0.25695	0.04949	0.26526	0.0386	0.50982	0.08153	0.28311	0.04111
		0.5	0.25695	0.04949	0.47325	0.08167	0.50982	0.08153	0.51738	0.09547
	40	0.1	0.26506	0.07859	0.09563	0.01092	0.52325	0.11605	0.09711	0.01059
		0.3	0.26506	0.07859	0.29089	0.03464	0.52325	0.11605	0.29791	0.0344
		0.5	0.26506	0.07859	0.49787	0.06312	0.52325	0.11605	0.51391	0.06324
100	80	0.1	0.25693	0.0494	0.09679	0.0081	0.50982	0.08153	0.09787	0.00758
		0.3	0.25693	0.0494	0.29368	0.02346	0.50982	0.08153	0.29992	0.02491
		0.5	0.25693	0.0494	0.50834	0.04481	0.50982	0.08153	0.51799	0.0444
	90	0.1	0.26506	0.07859	0.09922	0.00755	0.52325	0.11605	0.09883	0.00731
		0.3	0.26506	0.07859	0.30063	0.02371	0.52325	0.11605	0.30065	0.02359
		0.5	0.26506	0.07859	0.50687	0.03909	0.52325	0.11605	0.50833	0.03765

4 Data Analysis:

A real life data set is considered which represents the number of million revolutions before failure for each of the 23 ball bearing in life tests, the data arose in tests on endurance of deep groove ball bearings (Lawless [[4]]). The observations are shown as follows:
17.88, 28.92, 33, 41.52, 42.12, 45.6, 48.8, 51.84, 51.96, 54.12, 55.56, 67.8, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4

The Kolmogorov-Smirnov (K-S) test was used to show that the Ailamujia distribution is appropriate for analyzing this data. The P-value= 0.34 which is statistically significant and suggests that the Ailamujia distribution is appropriate for analyzing this data. The maximum likelihood estimate of θ using complete data with $n = 23$ is 0.01384.

Three progressively censored samples were generated from the above data with $m = 20, 17$ and 14.

Progressive censoring with $m = 20$:

(0, 17.88)	(0, 28.92)	(2, 33)	(1, 41.52)
(0, 42.12)	(0, 45.6)	(0, 48.8)	(0, 51.84)
(0, 51.96)	(0, 54.12)	(0, 55.56)	(0, 68.64)
(0, 68.88)	(0, 93.12)	(0, 98.64)	(0, 105.12)
(0, 105.84)	(0, 127.92)	(0, 128.04)	(0, 173.4)

Progressive censoring with $m = 17$:

(1, 17.88)	(1, 28.92)	(0, 33)	(0, 41.52)
(0, 42.12)	(1, 45.6)	(0, 48.8)	(0, 51.96)
(0, 54.12)	(0, 55.56)	(0, 67.8)	(0, 68.44)
(1, 68.64)	(1, 68.88)	(0, 93.12)	(0, 98.64)
(1, 105.84)			

Progressive censoring with $m = 14$:

(2, 17.88)	(0, 28.92)	(0, 41.52)	(1, 42.12)
(0, 45.6)	(0, 48.8)	(1, 51.84)	(1, 51.96)
(2, 54.12)	(0, 55.56)	(1, 68.64)	(1, 68.88)
(0, 93.12)	(0, 173.4)		

The maximum likelihood estimates for the model parameters for the three generated progressive censoring schemes for different values of m are as follows:

Censoring scheme 1: ($m = 20$, the number of removed observations is $n - m = 3$): The maximum likelihood estimate for θ is 0.013491.

Censoring scheme 2: ($m = 17$, the number of removed observations is $n - m = 6$): The maximum likelihood estimate for θ is 0.012815.

Censoring scheme 3: ($m = 14$, the number of removed observations is $n - m = 9$): The maximum likelihood estimate for θ is 0.011026.

5 Conclusion

We develop some results on Ailamujia distribution when progressive type II censoring with binomial removals is performed. The maximum likelihood estimators for the model parameters were derived along with asymptotic confidence intervals. The simulation results showed that as the sample size increases the performance of the estimators improves in term of bias and RMSE. The biases and RMSEs decrease as m increases. As the scale parameter θ increases, the bias and RMSE increase. Also, as the value of the parameter p increase, the bias and RMSE increase. A real lifetime data set was analyzed using progressive Type-II censoring of Ailamujia distribution and showed appropriate results.

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