Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/160306

Convergence and Stability Results for New Random Algorithms in Separable Banach Spaces

Dhekra M. Albaqeri[1] *and Hasanen A. Hammad* [2,*]

[1]Department of Mathematics, Faculty of Education, Sanaa University, 71516, Yemen
[2]Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt

Received: 2 Jan. 2022, Revised: 12 Mar. 2022, Accepted: 1 Apr. 2022
Published online: 1 May 2022

Abstract: In this paper, we introduce new iterative schemes namely,Jungck-DI-CR random iterative scheme and Jungck-DI-Karahan-Ozdemir random iterative scheme. Also, interesting results for convergence and stability are obtained under new generalized ϕ- weakly contraction mappings. Finally, the conditions of countability finite family of the control sequences and injectivity of the operators are omitted.

Keywords: (S,T)- stable, Jungck-DI-CR random iterative scheme, Jungck-DI-Karahan-Ozdemir random iterative scheme, Bochner integrable.

1 Introduction

Over the past years, many different iterative methods have been studied to approximate fixed points with suitable contractive conditions via various spaces, see [1,2,3]

In 1976, Jungck [4] presented the following iterative process: Consider X is a Banach space, Y is arbitrary set and $S,T:Y\to X$ are given mappings so that $T(Y)\subseteq S(Y)$. For $x_0\in Y$,

$$Sx_{n+1}=Tx_n,\ n\geq 0$$

He used this method to approximate the common FPs of S and T fulfilling the Jungck contraction. Obviously, if $S=I$ (where I is the identity mapping) and $Y=X$, then it reduces to the Picard iteration. Algorithms were developed and accelerated according to the following approach:

Jungck-Mann iterative process has been shown by Singh *et al.* [5] as follows:

$$Sx_{n+1}=(1-\alpha_n)Sx_n+\alpha_nTx_n,$$

for $\alpha_n,\beta_n,\gamma_n\in[0,1]$.

Olatinwo [6] defined the Jungck-Ishikawa and Jungck-Noor iterative processes as follows:

$$\begin{aligned}Sx_{n+1}&=(1-\alpha_n)Sx_n+\alpha_nTy_n,\\ Sy_n&=(1-\beta_n)Sx_n+\beta_nTx_n,\end{aligned}$$

and

$$\begin{aligned}Sx_{n+1}&=(1-\alpha_n)Sx_n+\alpha_nTy_n,\\ Sy_n&=(1-\beta_n)Sx_n+\beta_nTz_n,\\ Sz_n&=(1-\gamma_n)Sx_n+\gamma_nTx_n,\end{aligned}$$

respectively.

In 2013, Hussain *et al.* [7] describe the Jungck-CR iterative scheme as:

$$\begin{aligned}Sx_{n+1}&=(1-\alpha_n)Sy_n+\alpha_nTy_n,\\ Sy_n&=(1-\beta_n)Tx_n+\beta_nTz_n,\\ Sz_n&=(1-\gamma_n)Sx_n+\gamma_nTx_n,\end{aligned}\tag{1}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are sequences in $[0,1]$. Olatinwo [8], introduced the following two schemes:

a)Kirk-Mann iterative scheme:

$$x_{n+1}=\sum_{i=1}^{k}\alpha_{n,i}T^ix_n,\ \sum_{i=1}^{k}\alpha_{n,i}=0,1,2,3,...,$$

where $\alpha_{n,i}\geq 0,\alpha_{n,0}\neq 0,\alpha_{n,i}\in[0,1]$ and k is a fixed integer.

b)Kirk-Ishikawa iterative scheme:

$$x_{n+1}=\alpha_{n,0}x_n+\sum_{i=1}^{k}\alpha_{n,i}T^iy_n,\ \sum_{i=1}^{k}\alpha_{n,i}=1,$$

$$y_n=\sum_{j=1}^{s}\beta_{n,j}T^jx_n,\ \sum_{j=1}^{s}\beta_{n,j}=1,\ n\geq 0,$$

* Corresponding author e-mail: hassanein_hamad@science.sohag.edu.eg

where $k \geq s$, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,j} \geq 0$, $\beta_{n,0} \neq 0$, $\alpha_{n,i}, \beta_{n,j} \in [0,1]$ and k,s are fixed integers.

After that, Chugh and Kumar [9] generalized Kirk-Ishikawa algorithm to Kirk-Noor procedure as follows:

$$x_{n+1} = \alpha_{n,0}x_n + \sum_{i=1}^{k} \alpha_{n,i}T^i y_n, \ \sum_{i=1}^{k} \alpha_{n,i} = 1,$$
$$y_n = \beta_{n,0}x_n + \sum_{j=1}^{s} \beta_{n,j}T^j z_n, \ \sum_{j=1}^{s} \beta_{n,j} = 1,$$
$$z_n = \sum_{l=1}^{t} \gamma_{n,l}T^l x_n, \ \sum_{l=1}^{t} \gamma_{n,l} = 1, \ n = 0,1,2,..., \quad (2)$$

where $k \geq s \geq t, \alpha_{n,i} \geq 0, \alpha_{n,0} \neq 0, \beta_{n,j} \geq 0, \beta_{n,0} \neq 0, \alpha_{n,i}, \beta_{n,j}, \gamma_{n,l} \in [0,1]$ and k,s,t are fixed integers.

Karahan-Ozdemir [10], introduced the following method:

$$x_{n+1} = (1-\alpha_n)Ty_n + \alpha_n Ty_n,$$
$$y_n = (1-\beta_n)Tx_n + \beta_{n,j}Tz_n,$$
$$z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \quad (3)$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences of positive numbers in $[0,1]$.

On the other hand, the concept of stable fixed point iterative scheme was introduced and studied by Harder [11], Harder and Hicks [12,13]. Many other stability results for several fixed point iterative schemes and various classes of nonlinear mappings were obtained. (see e.g. [14,15,16,17,18,19]).

Definition 1.[13] *Let (X,d) be a metric space, $T: X \to X$ be a self-mapping and $x_0 \in X$. Assume that the iterative scheme*

$$x_{n+1} = f(T,x_n), n \geq 0,$$

converges to a fixed point p of T. Let z_n be an arbitrary sequence in X and define

$$\varepsilon_n = d(z_{n+1}, f(T,z_n)), n \geq 0.$$

The iterative scheme defined by (1) is said to be T-stable or stable with respect to T if and only if

$$\lim_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim_{n\to\infty} z_n = p.$$

The definition of (S,T)-stability can be found in Singh et al. [5].

Definition 2.[5] *Let $S,T: Y \to X$ be non-self operators for an arbitrary set Y such that $T(Y) \subseteq S(Y)$ and p a point of coincidence of S and T. Let $\{Sx_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iterative procedure*

$$Sx_{n+1} = f(T,x_n), n = 0,1,2,..., \quad (4)$$

where $x_0 \in X$ is the initial approximation and f is some functions. Suppose that $\{Sx_n\}_{n=0}^{\infty}$ converges to p. Let $\{Sy_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set

$$\varepsilon_n = d(Sy_n, f(T,y_n)), n = 0,1,2,... .$$

Then, the iterative procedure (4) is said to be (S,T)-stable if and only if $\lim_{n\to\infty}\varepsilon_n = 0$ implies $\lim_{n\infty} Sy_n = p$.

2 Preliminaries

Let (Ω,Σ) be a measurable space, C be nonempty subset of a separable Banach space X. A mapping $\xi: \Omega \to C$ is called measurable if $\xi^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of X. A mapping $T: \Omega \times C \to C$ is said to be random mapping if for each fixed $x \in C$, the mapping $T(.,x): \Omega \to C$ is measurable. A measurable mapping $\xi^*: \Omega \to C$ is called a random fixed point of the random mapping $T: \Omega \times C \to C$ if $T(\omega,\xi^*(\omega)) = \xi^*(\omega)$ for each $\omega \in \Omega$. Let $S,T: \Omega \times C \to C$ be two random self-maps. A measurable map ξ^* is called a common random fixed point of the pair (S,T) if $\xi^*(\omega) = S(\omega,\xi^*(\omega)) = T(\omega,\xi^*(\omega))$, for each $\omega \in \Omega$ and some $\xi^*(\omega) \in C$.

Several authors have provided random version of many known iterative algorithms (see e.g. [20,21,22] and references therein).

Agwu et al. [20] introduced a new random scheme called Jungck-DI-SP random iterative scheme as follows:

Definition 3.[20] *Let $\Gamma,S: \Omega \times C \leftrightarrow H$ be two random mappings defined on a nonempty closed convex subset of a separable Hilbert space H. Let $x_0(\xi): \Omega \leftrightarrow C$ be arbitrary measurable mapping for $\xi \in \Omega, n = 1,2,....$ with $\Gamma(\xi,C) \subseteq S(\xi,C)$. The Jungck-DI-CR random iterative scheme is a sequence $\{S(\xi,x_n(\omega))\}_{n=0}^{\infty}$ defined by*

$$S(\xi,x_{n+1}(\xi)) = \alpha_{n,1}S(\xi,x_n(\xi)) + \sum_{i=2}^{l_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) + \prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,y_n(\xi)),$$
$$S(\xi,y_n(\xi)) = \gamma_{n,1}S(\xi,x_n(\xi)) + \sum_{t=2}^{l_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_n(\xi)) + \prod_{b=1}^{l_2}(1-\gamma_{n,b})\Gamma^{l_2}(\xi,z_n(\xi)),$$
$$S(\xi,z_n(\xi)) = \delta_{n,1}S(\xi,x_n(\xi)) + \sum_{s=2}^{l_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,c})\Gamma^{s-1}(\xi,x_n(\xi)) + \prod_{c=1}^{l_3}(1-\delta_{n,c})\Gamma^{l_3}(\xi,x_n(\xi)), \quad (5)$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ are countable finite of measurable real sequences in $[0,1]$, and $l_1,l_2,l_3 \in \mathbb{N}$.

On the other hand, different contraction mappings in multiple research were studied, for example, Albaqeri and Rashwan [23], introduced the following generalized ϕ-weakly contractive condition:

Definition 4.[23] *Let $S,T : \Omega \times C \leftrightarrow C$ be two random mappings defined on a nonempty closed convex subset C of a separable Banach space X such that $T(\xi,X) \subseteq S(\xi,X)$. Then the random operators S,T are satisfying the following generalized ϕ- weakly contractive-type if there exist $L(\xi) \geq 0$ and a continuous and non- decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t) > 0$ for each $t \in (0,\infty)$ and $\phi(0) = 0$ such that for each $x,y \in C, \xi \in \Omega$,*

$$\begin{aligned}&\|T(\xi,x) - T(\xi,y)\| \\ \leq\ & e^{L(\xi)\|S(\xi,x)-T(\xi,x)\|} \\ &\times \|S(\xi,x) - S(\xi,y)\| - \phi(\|T(\xi,x) - T(\xi,y)\|) \qquad (6)\end{aligned}$$

Recently, Okeke et al. [24] introduced the following generalized $\phi-$weakly contraction of the rational type:

Definition 5.[24] *A random operator $T : \Omega \times C \leftrightarrow C$ is a generalized $\phi-$weakly contraction of the rational type, if there exist $L(\omega), M(\omega) \geq 0$ and a continuous and non-decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t) > 0$ for each $t \in (0,\infty)$ and $\phi(0) = 0$ such that for each $x,y \in C, \xi \in \Omega$, we have*

$$\begin{aligned}&\int \|T(\xi,x) - T(\xi,y)\| d\mu(\xi) \\ \leq\ & e^{L(\xi)\|x-T(\xi,x)\|}\left(\int \frac{\|x-y\|}{1+M(\xi)\|x-T(\xi,x)\|}\right. \\ &\left. -\phi\left(\int \frac{\|x-y\|}{1+M(\xi)\|x-T(\xi,x)\|}\right)\right) \qquad (7)\end{aligned}$$

Keeping in mind the generalized ϕ-weakly contractive conditions (6) and (7), we introduce the following generalized ϕ-weakly contractive condition:

Definition 6.*Let $S,T : \Omega \times C \leftrightarrow C$ be two random mappings defined on a nonempty closed convex subset C of a separable Banach space X such that $T(\xi,X) \subseteq S(\xi,X)$. Then the random operators S and T are satisfying the following generalized ϕ- weakly contractive-type if there exist $L(\xi), \eta(\xi) \geq 0, v^i \in [0,1), \forall i \in \mathbb{N}$ and a continuous and non- decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t) > 0$ for each $t \in (0,\infty)$ and $\phi(0) = 0$ such that for each $x,y \in C, \xi \in \Omega$,*

$$\begin{aligned}&\|T^i(\xi,x) - T^i(\xi,y)\| \\ \leq\ & e^{\sum_{j=1}^{i}\binom{i}{j}v^{i-1}L^j(\xi)\|S^j(\xi,x)-T^j(\xi,x)\|} \\ &\times\left(\frac{v^i\|S(\xi,x)-S(\xi,y)\|}{1+\eta^i\|S(\xi,x)-T(\xi,x)\|}\right. \\ &\left. -\phi^i\left(\frac{\|S(\xi,x)-S(\xi,y)\|}{1+\eta^i\|S(\xi,x)-T(\xi,x)\|}\right)\right) \qquad (8)\end{aligned}$$

Proposition 1.([2]) *Let $\{\alpha_n\}_{i=1}^N \subseteq \mathbb{N}$ be a countable subset of the set of real numbers $\mathbb{R}$, where k is a fixed nonnegative integer and N is any integer with $k+1 \leq N$. Then the following holds:*

$$\alpha_k + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j) + \prod_{j=k}^{N}(1-\alpha_j) = 1.$$

Proposition 2.([2])*Let u,v be arbitrary elements of the real Hilbert space H. Let k be a fixed nonnegative integer and $N \in \mathbb{N}$ such that $k+1 \leq N$. Let $\{v_i\}_{i=1}^{N-1} \subseteq H$, and $\{\alpha_n\}_{i=1}^N \subseteq [0,1]$ be a countable finite subset of H and $\mathbb{R}$, respectively. Define*

$$\begin{aligned}y = {}& \alpha_k t + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)v_{i-1} \\ &+ \prod_{j=k}^{N}(1-\alpha_j)v.\end{aligned}$$

Then,

$$\begin{aligned}\|y-u\|^2 = {}& \alpha_k\|t-u\|^2 \\ &+ \sum_{i=k=1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-u\|^2 \\ &+ \prod_{j=k}^{i-1}(1-\alpha_j)\|v-u\|^2, \\ &-\alpha_k\left[\sum_{i=k=1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)\|t-v_{i-1}\|^2\right. \\ &\left.+ \prod_{j=k}^{i-1}(1-\alpha_j)\|t-v\|^2\right], \\ &-(1-\alpha_k)\left[\sum_{i=k=1}^{N} \alpha_i \times\right. \\ &\prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-(\alpha_{i+1}+w_{i+1})\|^2 \\ &\left.+\alpha_N \prod_{j=k}^{i-1}(1-\alpha_j)\|v-v_{N-1}\|^2\right],\end{aligned}$$

where $w_k = \sum_{i=k=1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)v_{i-1} + \prod_{j=k}^{i-1}(1-\alpha_j)v$, $k = 1,2,...N$ and $w_n = (1-c_n)v$.

The following lemma is useful for proving our results.

Lemma 1.[25] *If $\{\lambda_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty}\lambda_n = 0$, and $0 \leq \delta < 1$, then for any sequence of positive numbers $\{x_n\}$ satisfying $x_{n+1} \leq \delta x_n + \lambda_n$, $n = 0,1,2,....$ Then, $\lim_{n\to\infty} x_n = 0$.*

3 Convergence results

The following section contains some convergence results for the new random iterative schemes under the new generalized ϕ-weakly contraction defined in (8). First of all, motivated by iterative schemes (2), (3) and (5), we will define new random iterative schemes as follows:

Definition 7.*let $\Gamma, S : \Omega \times C \leftrightarrow C$ be two random mappings defined on a nonempty closed convex subset of a separable Banach space X. Let $x_0(\xi) : \Omega \leftrightarrow C$ be* *arbitrary measurable mapping for $\xi \in \Omega, n = 1,2,..$ with $\Gamma(\xi,X) \subseteq S(\xi,X)$. The Jungck-DI-CR random iterative scheme is a sequence $\{S(\xi,x_n(\omega))\}_{n=0}^{\infty}$ defined by*

$$\begin{aligned} S(\xi,x_{n+1}(\xi)) &= \alpha_{n,1}S(\xi,y_n(\xi)) \\ &\quad + \sum_{i=2}^{l_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) \\ &\quad + \prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,y_n(\xi)), \\ S(\xi,y_n(\xi)) &= \gamma_{n,1}S(\xi,x_n(\xi)) \\ &\quad + \sum_{t=2}^{l_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_n(\xi)) \\ &\quad + \prod_{b=1}^{l_2}(1-\gamma_{n,b})\Gamma^{l_2}(\xi,z_n(\xi)), \\ S(\xi,z_n(\xi)) &= \delta_{n,1}S(\xi,x_n(\xi)) \\ &\quad + \sum_{s=2}^{l_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,c})\Gamma^{s-1}(\xi,x_n(\xi)) \\ &\quad + \prod_{c=1}^{l_3}(1-\delta_{n,c})\Gamma^{l_3}(\xi,x_n(\xi)), \end{aligned} \tag{9}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ are countable finite of measurable real sequences in $[0,1]$, and $l_1,l_2,l_3 \in \mathbb{N}$.

Also, the Jungck-DI-Karahan-Ozdemir random iterative scheme is a sequence $\{S(\xi,x_n(\omega))\}_{n=0}^{\infty}$ as follows:

Definition 8. *let $\Gamma, S : \Omega \times C \leftrightarrow C$ be two random mappings defined on a nonempty closed convex subset of a separable Banach space X. Let $x_0(\xi) : \Omega \leftrightarrow C$ be arbitrary measurable mapping for $\xi \in \Omega, n = 1,2,..$ with $\Gamma(\xi,X) \subseteq S(\xi,X)$. The Jungck-DI-Karahan-Ozdemir random iterative scheme is a sequence $\{S(\xi,x_n(\omega))\}_{n=0}^{\infty}$ defined by*

$$\begin{aligned} S(\xi,x_{n+1}(\xi)) &= \alpha_{n,1}\Gamma^{i}(\xi,x_n(\xi)) \\ &\quad + \sum_{i=2}^{l_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) \\ &\quad + \prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,y_n(\xi)), \end{aligned}$$

$$\begin{aligned} S(\xi,y_n(\xi)) &= \gamma_{n,1}\Gamma^{t}(\xi,x_n(\xi)) \\ &\quad + \sum_{t=2}^{l_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_n(\xi)) \\ &\quad + \prod_{b=1}^{l_2}(1-\gamma_{n,b})\Gamma^{l_2}(\xi,z_n(\xi)), \end{aligned}$$

$$\begin{aligned} S(\xi,z_n(\xi)) &= \delta_{n,1}S(\xi,x_n(\xi)) \\ &\quad + \sum_{s=2}^{l_3} \delta_{n,s} \prod_{c=1}^{s-1}(1-\delta_{n,c})\Gamma^{s-1}(\xi,x_n(\xi)) \\ &\quad + \prod_{c=1}^{l_3}(1-\delta_{n,c})\Gamma^{l_3}(\xi,x_n(\xi)), \end{aligned} \tag{10}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ are countable finite of measurable real sequences in $[0,1]$ and $l_1,l_2,l_3 \in \mathbb{N}$.

Remark. 1.If Ω is a singleton in (9) and (10), we get the nonrandom version of (9) and (10), respectively.

2.(a) If $l_3 = 0$ in (9), we get the following iterative scheme:

$$\begin{aligned} S(\xi,x_{n+1}(\xi)) &= \alpha_{n,1}S(\xi,y_n(\xi)) \\ &\quad + \sum_{i=2}^{l_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) \\ &\quad + \prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,y_n(\xi)), \end{aligned}$$

$$\begin{aligned} S(\xi,y_n(\xi)) &= \gamma_{n,1}S(\xi,x_n(\xi)) \\ &\quad + \sum_{t=2}^{l_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_n(\xi)) \\ &\quad + \prod_{b=1}^{l_2}(1-\gamma_{n,b})\Gamma^{l_2}(\xi,z_n(\xi)). \end{aligned} \tag{11}$$

(b) If $l_2 = l_3 = 0$ in (9), we get the following iterative scheme:

$$\begin{aligned} S(\xi,x_{n+1}(\xi)) &= \alpha_{n,1}S(\xi,y_n(\xi)) \\ &\quad + \sum_{i=2}^{l_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) \\ &\quad + \prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,y_n(\xi)), \end{aligned} \tag{12}$$

3.If S is an identity mapping in (9) and (10), we obtain the following iterative schemes:

$$\begin{aligned} x_{n+1}(\xi) &= \alpha_{n,1}y_n(\xi) \\ &\quad + \sum_{i=2}^{l_1} \alpha_{n,i} \prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) \\ &\quad + \prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,y_n(\xi)), \end{aligned}$$

$$\begin{aligned} y_n(\xi) &= \gamma_{n,1}x_n(\xi) \\ &\quad + \sum_{t=2}^{l_2} \gamma_{n,t} \prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_n(\xi)) \\ &\quad + \prod_{b=1}^{l_2}(1-\gamma_{n,b})\Gamma^{l_2}(\xi,z_n(\xi)), \end{aligned}$$

$$z_n(\xi) = \delta_{n,1}x_n(\xi) + \sum_{s=2}^{l_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})\Gamma^{s-1}(\xi,x_n(\xi)) + \prod_{c=1}^{l_3}(1-\delta_{n,c})\Gamma^{l_3}(\xi,x_n(\xi)), \tag{13}$$

$$\begin{aligned} x_{n+1}(\xi) &= \alpha_{n,1}\Gamma^i(\xi,x_n(\xi)) + \sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) + \prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,y_n(\xi)),\\ y_n(\xi) &= \gamma_{n,1}\Gamma^t(\xi,x_n(\xi)) + \sum_{t=2}^{l_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_n(\xi)) + \prod_{b=1}^{l_2}(1-\gamma_{n,b})\Gamma^{l_2}(\xi,z_n(\xi)),\\ z_n(\xi) &= \delta_{n,1}x_n(\xi) + \sum_{s=2}^{l_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})\Gamma^{s-1}(\xi,x_n(\xi)) + \prod_{c=1}^{l_3}(1-\delta_{n,c})\Gamma^{l_3}(\xi,x_n(\xi)), \end{aligned} \tag{14}$$

where $\{\alpha_n\}_{n=0}^{\infty},\{\gamma_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ are countable finite of measurable real sequences in $[0,1]$ and $l_1,l_2,l_3\in\mathbb{N}$.

Theorem 1.*Let C be a nonempty closed and convex subset of separable Banach space X, and let $\Gamma,S:\Omega\times C\leftrightarrow C$ be two random operators satisfying the generalized ϕ- weakly contraction defined in (8)with $\Gamma(\xi,X)\subseteq S(\xi,X)$. Let $q(\xi)$ be a common random fixed point of (S,Γ,S^i,Γ^i)(i.e.,$S(\xi,q(\xi))=\Gamma(\xi,q(\xi))=S^i(\xi,q(\xi))=\Gamma^i(\xi,q(\xi))=q(\xi)$, and for $x_0\in C$, the sequence $\{S(\xi,x_n(\xi))\}_{n=0}^{\infty}$ is the random Jungck-DI-CR iterative scheme defined by (9). Then the random common fixed point $q(\xi)$ is Bochner integrable.*

*Proof.*To prove that $q(\xi)$ is Bochner integrable, it suffices to prove that

$$\lim_{n\to\infty}\|S(\xi,x_n(\xi))-q(\xi)\|=0.$$

Using the Jungck-DI-CR random iterative scheme (9). Applying contractive condition (8) and using Proposition 2, we get

$$\begin{aligned} &\|S(\xi,x_{n+1}(\xi))-q(\xi)\|^2\\ &= \|\alpha_{n,1}S(\xi,y_n(\xi)) + \sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi)) + \prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,y_n(\xi))-q(\xi)\|^2\\ &\le \alpha_{n,1}\|S(\xi,y_n(\xi))-q(\xi)\|^2 + \sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\|\Gamma^{i-1}(\xi,y_n(\xi))-q(\xi)\|^2 + \prod_{a=1}^{l_1}(1-\alpha_{n,a})\|\Gamma^{l_1}(\xi,y_n(\xi))-q(\xi)\|^2, \end{aligned}$$

it follows that

$$\begin{aligned} &\le \alpha_{n,1}\|S(\xi,y_n(\xi))-q(\xi)\|^2 + \sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\\ &\times\left(e^{\sum_{j=1}^{i-1}\binom{i-1}{j}\nu^{i-2}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)^2\\ &\times\left(\frac{\nu^{i-1}\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|}{1+\eta^{i-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|} - \phi^{i-1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|}{1+\eta^{i-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\right)^2\\ &+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\left(e^{\sum_{j=1}^{l_1}\binom{l_1}{j}\nu^{l_1-1}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)^2\\ &\times\left(\frac{\nu^{l_1}\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|}{1+\eta^{l_1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|} - \phi^{l_1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|}{1+\eta^{l_1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\right)^2, \end{aligned}$$

it follows that

$$\begin{aligned} &\|S(\xi,x_{n+1}(\xi))-q(\xi)\|^2\\ &\le \alpha_{n,1}\|S(\xi,y_n(\xi))-q(\xi)\|^2\\ &+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\left(e^{\sum_{j=1}^{i-1}\binom{i-1}{j}\nu^{i-2}L^j(\xi)\|0\|}\right)^2\\ &\times\left(\frac{\nu^{i-1}\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|}{1+\eta^{i-1}\|0\|} - \phi^{i-1}(\frac{\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|}{1+\eta^{i-1}\|0\|}\right)^2\\ &+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\left(e^{\sum_{j=1}^{l_1}\binom{l_1}{j}\nu^{l_1-1}L^j(\xi)\|0\|}\right)^2\\ &\times\left(\frac{\nu^{l_1}\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|}{1+\eta^{l_1}\|0\|} - \phi^{l_1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|}{1+\eta^{l_1}\|0\|}\right)\right)^2, \end{aligned}$$

it follows that

$$\begin{aligned}&\alpha_{n,1}\|S(\xi,y_n(\xi))-q(\xi)\|^2\\ \leq\ &\alpha_{n,1}\|S(\xi,y_n(\xi))-q(\xi)\|^2+\sum_{i=2}^{l_1}\alpha_{n,i}\times\\ &\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^{i-1})^2\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|^2\\ &+\prod_{a=1}^{l_1}(1-\alpha_{n,a})(\nu^{l_1})^2\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|^2\\ =\ &\Bigg(\alpha_{n,1}+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^{i-1})^2\\ &+\prod_{a=1}^{l_1}(1-\alpha_{n,a})(\nu^{l_1})^2\Bigg)\\ &\times\|S(\xi,y_n(\xi))-q(\xi)\|^2.\end{aligned}$$

Since $\nu^{i-1},\nu^{l_1}\in[0,1)$, we have by Proposition 1,

$$\begin{aligned}&\alpha_{n,1}+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^{i-1})^2\\ &+\prod_{a=1}^{l_1}(1-\alpha_{n,a})(\nu^{l_1})^2\\ <\ &\alpha_{n,1}+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\\ =\ &1,\end{aligned}$$

then we can apply this fact above to get the following:

$$\|S(\xi,x_{n+1}(\xi))-q(\xi)\|^2<\|S(\xi,y_n(\xi))-q(\xi)\|^2.$$

By using (8) and (9), we have:

$$\begin{aligned}&\|S(\xi,y_n(\xi))-q(\xi)\|^2=\\ &\|\gamma_{n,1}S(\xi,x_n(\xi))+\sum_{t=2}^{l_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\times\\ &\Gamma^{t-1}(\xi,z_n(\xi)+\prod_{b=1}^{l_2}(1-\gamma_{n,b})\Gamma^{l_2}(\xi,z_n(\xi))-q(\xi)\|^2\\ \leq\ &\gamma_{n,1}\|S(\xi,x_n(\xi))-q(\xi)\|^2+\sum_{t=2}^{l_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\times\\ &\|\Gamma^{t-1}(\xi,z_n(\xi)-q(\xi)\|^2+\prod_{b=1}^{l_2}(1-\gamma_{n,b})\times\\ &\|\Gamma^{l_2}(\xi,z_n(\xi))-q(\xi)\|^2,\end{aligned}$$

it follows that

$$\begin{aligned}&\|S(\xi,y_n(\xi))-q(\xi)\|^2\\ \leq\ &\gamma_{n,1}\|S(\xi,x_n(\xi))-q(\xi)\|^2+\sum_{t=2}^{l_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\\ &\times\left(e^{\sum_{j=1}^{t-1}\binom{t-1}{j}\nu^{t-2}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)^2\\ &\times\Bigg(\frac{\nu^{t-1}\|S(\xi,q(\xi))-S(\xi,z_n(\xi))\|}{1+\eta^{t-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\\ &-\phi^{t-1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,z_n(\xi))\|}{1+\eta^{t-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\Bigg)^2\\ &+\prod_{b=1}^{l_2}(1-\gamma_{n,b})\\ &\times\left(e^{\sum_{j=1}^{l_2}\binom{l_2}{j}\nu^{l_2-1}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)^2\\ &\times\Bigg(\frac{\nu^{l_2}\|S(\xi,q(\xi))-S(\xi,z_n(\xi))\|}{1+\eta^{l_2}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\\ &\phi^{l_2}\left(\frac{\|S(\xi,q(\xi))-S(\xi,z_n(\xi))\|}{1+\eta^{l_2}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\Bigg)^2,\end{aligned}$$

it follows that

$$\begin{aligned}&\|S(\xi,y_n(\xi))-q(\xi)\|^2\\ \leq\ &\gamma_{n,1}\|S(\xi,x_n(\xi))-q(\xi)\|^2\\ &+\sum_{t=2}^{l_2}\gamma_{n,b}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\left(e^{\sum_{j=1}^{t-1}\binom{t-1}{j}\nu^{t-2}L^j(\xi)\|0\|}\right)^2\\ &\times\Bigg(\frac{\nu^{t-1}\|S(\xi,q(\xi))-S(\xi,z_n(\xi))\|}{1+\eta^{t-1}\|0\|}\\ &-\phi^{t-1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,z_n(\xi))\|}{1+\eta^{t-1}\|0\|}\right)\Bigg)^2\\ &+\prod_{b=1}^{l_2}(1-\gamma_{n,b})\left(e^{\sum_{j=1}^{l_2}\binom{l_2}{j}\nu^{l_2-1}L^j(\xi)\|0\|}\right)^2\\ &\times\Bigg(\frac{\nu^{l_2}\|S(\xi,q(\xi))-S(\xi,z_n(\xi))\|}{1+\eta^{l_2}\|0\|}\\ &-\phi^{l_2}\left(\frac{\|S(\xi,q(\xi))-S(\xi,z_n(\xi))\|}{1+\eta^{l_2}\|0\|}\right)\Bigg)^2,\end{aligned}$$

it follows that

$$\begin{aligned}&\|S(\xi,y_n(\xi))-q(\xi)\|^2\\ \leq\ &\gamma_{n,1}\|S(\xi,x_n(\xi))-q(\xi)\|^2+\sum_{t=2}^{l_2}\gamma_{n,t}\times\\ &\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^{t-1})^2\|S(\xi,z_n(\xi))-q(\xi)\|^2\\ &+\prod_{b=1}^{l_2}(1-\gamma_{n,b})(\nu^{l_2})^2\|S(\xi,z_n(\xi))-q(\xi)\|^2.\end{aligned}\tag{15}$$

Again, we compute the last estimate of (15) by using (8) and (9) with Proposition 2 as follows:

$$
\begin{aligned}
&\|(\xi, z_n(\xi) - q(\xi)\|^2 = \\
&\|\delta_{n,1} S(\xi, x_n(\xi)) + \sum_{s=2}^{l_3} \delta_{n,s} \times \\
&\prod_{c=1}^{s-1} (1-\delta_{n,s}) \Gamma^{s-1}(\xi, x_n(\xi)) \\
&\quad + \prod_{c=1}^{l_3} (1-\delta_{n,c}) \Gamma^{l_3}(\xi, x_n(\xi)) - q(\xi)\|^2 \\
&\le \delta_{n,1} \|S(\xi, x_n(\xi)) - q(\xi)\|^2 \\
&\quad + \sum_{s=2}^{l_3} \delta_{n,s} \prod_{c=1}^{s-1} (1-\delta_{n,s}) \|\Gamma^{s-1}(\xi, x_n(\xi)) - q(\xi)\|^2 \\
&\quad + \prod_{c=1}^{l_3} (1-\delta_{n,c}) \|\Gamma^{l_3}(\xi, x_n(\xi)) - q(\xi)\|^2,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\|(\xi, z_n(\xi) - q(\xi)\|^2 \\
&\le \delta_{n,1} \|S(\xi, x_n(\xi)) - q(\xi)\|^2 + \sum_{s=2}^{l_3} \delta_{n,c} \prod_{c=1}^{s-1} (1-\delta_{n,c}) \\
&\quad \times \left(e^{\sum_{j=1}^{s-1} \binom{s-1}{j} \nu^{s-2} L^j(\xi) \|S^j(\xi, q(\xi)) - \Gamma^j(\xi, q(\xi))\|} \right)^2 \\
&\quad \times \left(\frac{\nu^{s-1} \|S(\xi, q(\xi)) - S(\xi, x_n(\xi))\|}{1 + \eta^{s-1} \|S(\xi, q(\xi)) - \Gamma(\xi, q(\xi))\|} \right)^2 \\
&-\phi^{s-1} \left(\frac{\|S(\xi, q(\xi)) - S(\xi, x_n(\xi))\|}{1 + \eta^{s-1} \|S(\xi, q(\xi)) - \Gamma(\xi, q(\xi))\|} \right) \Bigg)^2 \\
&+ \prod_{c=1}^{l_3} (1-\delta_{n,c}) \times \\
&\left(e^{\sum_{j=1}^{l_3} \binom{l_3}{j} \nu^{l_3-1} L^j(\xi) \|S^j(\xi, q(\xi)) - \Gamma^j(\xi, q(\xi))\|} \right)^2 \\
&\times \left(\frac{\nu^{l_3} \|S(\xi, q(\xi)) - S(\xi, x_n(\xi))\|}{1 + \eta^{l_3} \|S(\xi, q(\xi)) - \Gamma(\xi, q(\xi))\|} \right. \\
&-\phi^{l_3} \left(\frac{\|S(\xi, q(\xi)) - S(\xi, x_n(\xi))\|}{1 + \eta^{l_3} \|S(\xi, q(\xi)) - \Gamma(\xi, q(\xi))\|} \right) \Bigg)^2,
\end{aligned}
$$

yields

$$
\begin{aligned}
&\|(\xi, z_n(\xi) - q(\xi)\|^2 \\
&\le \delta_{n,1} \|S(\xi, x_n(\xi)) - q(\xi)\|^2 \\
&\quad + \sum_{s=2}^{l_3} \delta_{n,s} \prod_{c=1}^{s-1} (1-\delta_{n,c}) \left(e^{\sum_{j=1}^{s-1} \binom{s-1}{j} \nu^{s-2} L^j(\xi) \|0\|} \right)^2 \\
&\quad \times \left(\frac{\nu^{s-1} \|S(\xi, q(\xi)) - S(\xi, x_n(\xi))\|}{1 + \eta^{s-1} \|0\|} \right. \\
&\quad -\phi \left({}^{s-1} \frac{\|S(\xi, q(\xi)) - S(\xi, x_n(\xi))\|}{1 + \eta^{s-1} \|0\|} \right) \Bigg)^2 \\
&\quad + \prod_{c=1}^{l_3} (1-\delta_{n,c}) \left(e^{\sum_{j=1}^{l_3} \binom{l_3}{j} \nu^{l_3} L^j(\xi) \|0\|} \right)^2 \\
&\quad \times \left(\frac{\nu^{l_3} \|S(\xi, q(\xi)) - S(\xi, x_n(\xi))\|}{1 + \eta^{l_3} \|0\|} \right. \\
&\quad -\phi^{l_3} \left(\frac{\|S(\xi, q(\xi)) - S(\xi, x_n(\xi))\|}{1 + \eta^{l_3} \|0\|} \right) \Bigg)^2,
\end{aligned}
$$

this implies that

$$
\begin{aligned}
&\|(\xi, z_n(\xi) - q(\xi)\|^2 \\
&\le \delta_{n,1} \|S(\xi, x_n(\xi)) - q(\xi)\|^2 + \sum_{s=2}^{l_3} \delta_{n,s} \times \\
&\prod_{c=1}^{s-1} (1-\delta_{n,c}) (\nu^{s-1})^2 \|S(\xi, q(\xi)) - S(\xi, x_n(\xi))\|^2 \\
&+ \prod_{c=1}^{l_3} (1-\delta_{n,c}) (\nu^{l_3})^2 \|S(\xi, q(\xi)) - S(\xi, x_n(\xi))\|^2 \\
&= \left(\delta_{n,1} + \sum_{s=2}^{l_3} \delta_{n,c} \prod_{c=1}^{s-1} (1-\delta_{n,c}) (\nu^{s-1})^2 \right. \\
&\quad \left. + \prod_{c=1}^{l_3} (1-\delta_{n,c}) (\nu^{l_3})^2 \right) \\
&\quad \times \|S(\xi, q(\xi)) - S(\xi, x_n(\xi))\|^2. \qquad (16)
\end{aligned}
$$

Since $\nu^{s-1}, \nu^{l_3}, \nu^{l_3} \in [0,1)$, then by Proposition 1, we obtain

$$\|S(\xi, z_n) - q(\xi)\|^2 < \|S(\xi, x_n) - q(\xi)\|^2.$$

Applying this in (15), we have

$$
\begin{aligned}
&\|S(\xi, y_n(\xi)) - q(\xi)\|^2 \\
&\le \gamma_{n,1} \|S(\xi, x_n(\xi)) - q(\xi)\|^2 + \sum_{t=2}^{l_2} \gamma_{n,t} \times \\
&\prod_{b=1}^{t-1} (1-\gamma_{n,b}) (\nu^{t-1})^2 \|S(\xi, z_n(\xi)) - q(\xi)\|^2 \\
&+ \prod_{b=1}^{l_2} (1-\gamma_{n,b}) (\nu^{l_2})^2 \|S(\xi, q(\xi)) - S(\xi, z_n(\xi))\|^2 \\
&\le \gamma_{n,1} \|S(\xi, x_n(\xi)) - q(\xi)\|^2 + \sum_{t=2}^{l_2} \gamma_{n,t} \times \\
&\prod_{b=1}^{t-1} (1-\gamma_{n,b}) (\nu^{t-1})^2 \|S(\xi, x_n(\xi)) - q(\xi)\|^2 \\
&+ \prod_{b=1}^{l_2} (1-\gamma_{n,b}) (\nu^{l_2})^2 \|S(\xi, x_n(\xi)) - q(\xi)\|^2,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\|S(\xi,y_n(\xi))-q(\xi)\|^2\\
\le &\left(\gamma_{n,1}+\sum_{t=2}^{l_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^{t-1})^2\right.\\
&\left.+\prod_{b=1}^{l_2}(1-\gamma_{n,b})(\nu^{l_2-1})^2\right)\\
&\times\|S(\xi,x_n(\xi))-q(\xi)\|^2.\\
< &\left(\gamma_{n,1}+\sum_{t=2}^{l_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,t})+\prod_{b=1}^{l_2}(1-\gamma_{n,b})\right)\\
&\times\|S(\xi,x_n(\xi))-q(\xi)\|^2\\
= &\|S(\xi,x_n(\xi))-q(\xi)\|^2. \qquad (17)
\end{aligned}
$$

Applying (17) in (16), we get

$$
\begin{aligned}
\|S(\xi,x_{n+1}(\xi))-q(\xi)\| &< \|S(\xi,y_n(\xi))-q(\xi)\|\\
&< \|S(\xi,x_n(\xi))-q(\xi)\|.
\end{aligned}
$$

Using Lemma 1, we obtain that $\lim_{n\to\infty}\|S(\xi,x_n(\xi))-q(\xi)\|=0$. The proof is completed.

Theorem 2. *Let C be a non-empty closed and convex subset of a separable Banach space X, and let $\Gamma,S:\Omega\times C\leftrightarrow C$ be two random operators satisfying the generalized ϕ- weakly contraction defined in (8) with $\Gamma(\xi,X)\subseteq S(\xi,X)$. Let $q(\xi)$ be a common random fixed point of (S,Γ,S^i,Γ^i) (i.e., $S(\xi,q(\xi))=\Gamma(\xi,q(\xi))=S^i(\xi,q(\xi))=\Gamma^i(\xi,q(\xi))=q(\xi)$), and for $x_0\in X$, the sequence $\{S(\xi,x_n(\xi))\}_{n=0}^{\infty}$ is the random Jungck-DI-Karahan-Ozdemir iterative scheme defined by (10). Then the random common fixed point $q(\xi)$ is Bochner integrable.*

Proof. To prove that $q(\xi)$ is Bochner integrable, it suffices to prove that

$$\lim_{n\to\infty}\|S(\xi,x_n(\xi))-q(\xi)\|=0.$$

Using the Jungck-DI-Karahan-Ozdemir random iterative scheme (10). Using contractive condition (8) and Proposition 2, we get

$$
\begin{aligned}
&\|S(\xi,x_{n+1}(\xi))-q(\xi)\|^2\\
\le &\|\alpha_{n,1}\Gamma^i(\xi,x_n(\xi))+\sum_{i=2}^{l_1}\alpha_{n,i}\times\\
&\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,y_n(\xi))\\
&+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,y_n(\xi))-q(\xi)\|^2\\
\le &\alpha_{n,1}\|\Gamma^i(\xi,x_n(\xi))-q(\xi)\|^2+\sum_{i=2}^{l_1}\alpha_{n,i}\times\\
&\prod_{a=1}^{i-1}(1-\alpha_{n,a})\|\Gamma^{i-1}(\xi,y_n(\xi))-q(\xi)\|^2\\
&+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\|\Gamma^{l_1}(\xi,y_n(\xi))-q(\xi)\|^2,
\end{aligned}
$$

this leads to

$$
\begin{aligned}
&\|S(\xi,x_{n+1}(\xi))-q(\xi)\|^2\\
\le &\alpha_{n,1}\left(e^{\sum_{j=1}^{i}\binom{i}{j}\nu^{i-1}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)^2\\
&\times\left(\frac{\nu^i\|S(\xi,q(\xi))-S(\xi,x_n(\xi))\|}{1+\eta^i\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right.\\
&\left.-\phi^i\left(\frac{\|S(\xi,q(\xi))-S(\xi,x_n(\xi))\|}{1+\eta^i\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\right)^2\\
&+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\\
&\times\left(e^{\sum_{j=1}^{i-1}\binom{i-1}{j}\nu^{i-2}L^j(\xi)\|\Gamma^j(\xi,q(\xi))-q(\xi)\|}\right)^2\\
&\times\left(\frac{\nu^{i-1}\|S(\xi,q(\xi))-S(\xi,x_n(\xi))\|}{1+\eta^{i-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right.\\
&\left.-\phi^{i-1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|}{1+\eta^{i-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\right)^2\\
&+\prod_{a=1}^{l_1}(1-\alpha_{n,a})e^{\sum_{j=1}^{l_1}\binom{l_1}{j}\nu^{l_1-1}L^j(\xi)\|\Gamma^j(\xi,q(\xi))-q(\xi)\|}\\
&\times\left(\frac{\nu^{l_1}\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|}{1+\eta^{l_1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right.\\
&\left.-\phi^{l_1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|}{1+\eta^{l_1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\right)^2,
\end{aligned}
$$

hence

$$
\begin{aligned}
&\|S(\xi,x_{n+1}(\xi))-q(\xi)\|^2\\
\le &\alpha_{n,1}\nu^i\|S(\xi,x_n(\xi))-q(\xi)\|^2\\
&+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^{i-1})^2\times\\
&\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|^2\\
&+\prod_{a=1}^{l_1}(1-\alpha_{n,a})(\nu^{l_1})^2\|S(\xi,q(\xi))-S(\xi,y_n(\xi))\|^2. \qquad (18)
\end{aligned}
$$

Now, we compute the last estimate of (18). Using (8), (10) and Proposition 2, we obtain that

$$
\begin{aligned}
&\|S(\xi,y_n(\xi))-q(\xi)\|^2\\
= &\|\gamma_{n,1}\Gamma^i(\xi,x_n(\xi))\\
&+\sum_{t=2}^{l_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,z_n(\xi)
\end{aligned}
$$

$$+\prod_{b=1}^{l_2}(1-\gamma_{n,b})\Gamma^{l_2}(\xi,z_n(\xi))-q(\xi)\|^2$$

$$\leq \gamma_{n,1}\|\Gamma^i(\xi,x_n(\xi))-q(\xi)\|^2+\sum_{t=2}^{l_2}\gamma_{n,t}\times \prod_{b=1}^{t-1}(1-\gamma_{n,b})\|\Gamma^{t-1}(\xi,z_n(\xi)-q(\xi)\|^2 +\prod_{b=1}^{l_2}(1-\gamma_{n,b})\|\Gamma^{l_2}(\xi,z_n(\xi))-q(\xi)\|^2,$$

hence

$$\|S(\xi,y_n(\xi))-q(\xi)\|^2 \leq \gamma_{n,1}\left(e^{\Sigma_{j=1}^{i}\binom{i}{j}\nu^{i-1}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)^2 \times\left(\frac{\nu^i\|S(\xi,q(\xi))-S(\xi,x_n(\xi))\|}{1+\eta^i\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}-\phi^i\left(\frac{\|S(\xi,q(\xi))-S(\xi,x_n(\xi))\|}{1+\eta^i\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\right)^2$$

$$+\sum_{t=2}^{l_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b}) \times\left(e^{\Sigma_{j=1}^{t-1}\binom{t-1}{j}\nu^{t-2}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)^2 \times\left(\frac{\nu^{t-1}\|S(\xi,q(\xi))-S(\xi,z_n(\xi))\|}{1+\eta^{t-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}-\phi^{t-1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,z_n(\xi))\|}{1+\eta^{t-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\right)^2$$

$$+\prod_{b=1}^{l_2}(1-\gamma_{n,b})\times \left(e^{\Sigma_{j=1}^{l_2}\binom{l_2}{j}\nu^{l_2-1}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)^2 \times\left(\frac{\nu^{l_2}\|S(\xi,q(\xi))-S(\xi,z_n(\xi))\|}{1+\eta^{l_2}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}-\phi^{l_2}\left(\frac{\|S(\xi,q(\xi))-S(\xi,z_n(\xi))\|}{1+\eta^{l_2}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\right)^2,$$

$$\|S(\xi,y_n(\xi))-q(\xi)\|^2 \leq \gamma_{n,1}(\nu^i)^2\|S(\xi,x_n(\xi))-q(\xi)\|^2+\sum_{t=2}^{l_2}\gamma_{n,t}\times \prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^{t-1})^2\|S(\xi,z_n(\xi))-q(\xi)\| +\prod_{b=1}^{l_2}(1-\gamma_{n,b})(\nu^{l_2})^2\|S(\xi,z_n(\xi))-q(\xi)\|^2. \quad (19)$$

Also, we compute the last estimate of (19) by using (8) and (10) as follows:

$$\|S(\xi,z_n(\xi)-q(\xi)\|^2 = \|\delta_{n,1}S(\xi,x_n(\xi)) +\sum_{s=2}^{l_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})\Gamma^{s-1}(\xi,x_n(\xi)) +\prod_{c=1}^{l_3}(1-\delta_{n,c})\Gamma^{l_3}(\xi,x_n(\xi))-q(\xi)\|^2$$

$$\leq \delta_{n,1}\|S(\xi,x_n(\xi))-q(\xi)\|^2+\sum_{s=2}^{l_3}\delta_{n,s}\times \prod_{c=1}^{s-1}(1-\delta_{n,c})\|\Gamma^{s-1}(\xi,x_n(\xi))-q(\xi)\|^2 +\prod_{c=1}^{l_3}(1-\delta_{n,c})\|\Gamma^{l_3}(\xi,x_n(\xi))-q(\xi)\|^2,$$

it follows that

$$\|S(\xi,z_n(\xi)-q(\xi)\|^2 \leq \delta_{n,1}\|S(\xi,x_n(\xi))-q(\xi)\|^2+\sum_{s=2}^{l_3}\delta_{n,s}\times \prod_{c=1}^{s-1}(1-\delta_{n,c}) \times\left(e^{\Sigma_{j=1}^{s-1}\binom{s-1}{j}\nu^{s-2}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)^2 \times\left(\frac{\nu^{s-1}\|S(\xi,q(\xi))-S(\xi,x_n(\xi))\|}{1+\eta^{s-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}-\phi^{s-1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,x_n(\xi))\|}{1+\eta^{s-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\right)^2$$

$$+\prod_{c=1}^{l_3}(1-\delta_{n,c})\times \left(e^{\Sigma_{j=1}^{l_3}\binom{l_3}{j}\nu^{l_3-1}L^j(\xi)\|\Gamma^j(\xi,q(\xi))-q(\xi)\|}\right)^2 \times\left(\frac{\nu^{l_3}\|S(\xi,x_n(\xi))-q(\xi)\|}{1+\eta^{l_3}\|\Gamma(\xi,q(\xi))-q(\xi)\|}-\phi^{l_3}\left(\frac{\|S(\xi,q(\xi))-S(\xi,x_n(\xi))\|}{1+\eta^{l_3}\|\Gamma(\xi,q(\xi))-q(\xi)\|}\right)\right)^2,$$

hence

$$\|S(\xi,z_n(\xi)-q(\xi)\|^2 \leq \delta_{n,1}\|S(\xi,x_n(\xi))-q(\xi)\|^2 +\sum_{s=2}^{l_3}\delta_{n,c}\prod_{c=1}^{s-1}(1-\delta_{n,c})\left(e^{\Sigma_{j=1}^{s-1}\binom{s-1}{j}\nu^{s-2}L^j(\xi)\|0\|}\right)^2 \times\left(\frac{\nu^{s-1}\|S(\xi,q(\xi))-S(\xi,x_n(\xi))\|}{1+\eta^{s-1}\|0\|}\right.$$

$$-\phi^{s-1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,x_n(\xi))\|}{1+\eta^{s-1}\|0\|}\right)\Bigg)^2$$
$$+\prod_{c=1}^{l_3}(1-\delta_{n,c})\left(e^{\sum_{j=1}^{l_3}\binom{l_3}{j}\nu^{l_3-1}L^j(\xi)\|0\|}\right)^2$$
$$\times\left(\frac{\nu^{l_3}\|S(\xi,x_n(\xi))-q(\xi)\|}{1+\eta^{l_3}\|0\|}\right.$$
$$\left.-\phi^{l_3}\left(\frac{\|S(\xi,x_n(\xi))-q(\xi)\|}{1+\eta^{l_3}\|0\|}\right)\right)^2,$$

this leads to

$$\begin{aligned}
&\|S(\xi,y_n(\xi))-q(\xi)\|^2\\
&\le \delta_{n,1}\|S(\xi,x_n(\xi))-q(\xi)\|^2\\
&\quad+\sum_{s=2}^{l_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^{s-1})^2\|S(\xi,x_n(\xi))-q(\xi)\|^2\\
&\quad+\prod_{c=1}^{l_3}(1-\delta_{n,c})(\nu^{l_3})^2\|S(\xi,x_n(\xi))-q(\xi)\|^2\\
&=\left(\delta_{n,1}+\sum_{s=2}^{l_3}\delta_{n,c}\prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^{s-1})^2\right.\\
&\quad\left.+\prod_{c=1}^{l_3}(1-\delta_{n,c})(\nu^{l_3})^2\right)\\
&\quad\times\|S(\xi,x_n(\xi))-q(\xi)\|^2. \qquad (20)
\end{aligned}$$

Since $\nu^{s-1},\nu^{l_3}\in(0,1]$, we have by Proposition 1

$$\begin{aligned}
&\delta_{n,1}+\sum_{s=2}^{l_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^{s-1})^2+\prod_{c=1}^{l_3}(1-\delta_{n,c})(\nu^{l_3})^2\\
&<\delta_{n,1}+\sum_{s=2}^{l_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})+\prod_{c=1}^{l_3}(1-\delta_{n,c})=1,
\end{aligned}$$

so, we have

$$\begin{aligned}
&\|S(\xi,z_n(\xi))-q(\xi)\|^2\\
&\le\left(\delta_{n,1}+\sum_{s=2}^{l_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^{s-1})^2\right.\\
&\quad\left.+\prod_{c=1}^{l_3}(1-\delta_{n,c})(\nu^{l_3})^2\right)\\
&\quad\times\|S(\xi,x_n(\xi))-q(\xi)\|^2\\
&<\|S(\xi,x_n(\xi))-q(\xi)\|^2.
\end{aligned}$$

Applying the interesting above result in (19), we obtain

$$\begin{aligned}
&\|S(\xi,y_n(\xi))-q(\xi)\|^2\\
&\le\gamma_{n,1}\|S(\xi,x_n(\xi))-q(\xi)\|^2+\sum_{t=2}^{l_2}\gamma_{n,t}\\
&\quad\times\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^{t-1})^2\|S(\xi,q(\xi))-S(\xi,z_n(\xi))\|^2\\
&\quad+\prod_{b=1}^{l_2}(1-\gamma_{n,b})(\nu^{l_2})^2\|S(\xi,q(\xi))-S(\xi,z_n(\xi))\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\|S(\xi,y_n(\xi))-q(\xi)\|^2\\
&\le\gamma_{n,1}\|S(\xi,x_n(\xi))-q(\xi)\|^2+\sum_{t=2}^{l_2}\gamma_{n,t}\\
&\quad\times\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^{t-1})^2\|S(\xi,x_n(\xi))-q(\xi)\|^2\\
&\quad+\prod_{b=1}^{l_2}(1-\gamma_{n,b})(\nu^{l_2})^2\|S(\xi,x_n(\xi))-q(\xi)\|^2.\\
&=\left(\gamma_{n,1}+\sum_{t=2}^{l_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^{t-1})^2\right.\\
&\quad\left.+\prod_{b=1}^{l_2}(1-\gamma_{n,b})(\nu^{l_2})^2\right)\times\|S(\xi,x_n(\xi))-q(\xi)\|^2.\\
&<\gamma_{n,1}+\sum_{t=2}^{l_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})+\prod_{b=1}^{l_2}(1-\gamma_{n,b})\\
&\quad\times\|S(\xi,x_n(\xi))-q(\xi)\|^2\\
&=\|S(\xi,x_n(\xi))-q(\xi)\|^2.
\end{aligned}$$

Applying the interesting above result in (18)

$$\begin{aligned}
&\|S(\xi,x_{n+1}(\xi))-q(\xi)\|^2\\
&\le\alpha_{n,1}(\nu^i)^2\|S(\xi,x_n(\xi))-q(\xi)\|^2\\
&\quad+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^{i-1})^2\times\\
&\quad\|S(\xi,y_n(\xi))-q(\xi)\|^2\\
&\quad+\prod_{a=1}^{l_1}(1-\alpha_{n,a})(\nu^{l_1})^2\|S(\xi,y_n(\xi))-q(\xi)\|^2,
\end{aligned}$$

hence

$$\begin{aligned}
&\|S(\xi,x_{n+1}(\xi))-q(\xi)\|^2\\
&\le\alpha_{n,1}(\nu^i)^2\|S(\xi,x_n(\xi))-q(\xi)\|^2+\sum_{i=2}^{l_1}\alpha_{n,i}\\
&\quad\times\prod_{a=1}^{i-2}(1-\alpha_{n,a})(\nu^{i-1})^2\|S(\xi,x_n(\xi))-q(\xi)\|^2\\
&\quad+\prod_{a=1}^{l_1}(1-\alpha_{n,a})(\nu^{l_1})^2\|S(\xi,x_n(\xi))-q(\xi)\|^2\\
&=\left(\alpha_{n,1}(\nu^i)^2+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^{i-1})^2\right.\\
&\quad\left.+\prod_{a=1}^{l_1}(1-\alpha_{n,a})(\nu^{l_1})^2\right)\\
&\quad\times\|S(\xi,x_n(\xi))-q(\xi)\|^2\\
&<\|S(\xi,x_n(\xi))-q(\xi)\|^2.
\end{aligned}$$

Using Lemma 1, we obtain that $\lim_{n\to\infty}\|S(\xi,x_n(\xi))-q(\xi)\|=0$. This completes the proof.

From Theorem 1, we can present the following corollaries.

Corollary 1. *Let C be a non-empty closed and convex subset of a separable Banach space X, and let $\Gamma,S:\Omega\times C\leftrightarrow C$ be two random operators satisfying (8) with $\Gamma(\xi,X)\subseteq S(\xi,X)$. Let $q(\xi)$ be a common random fixed point of (S,Γ,S^i,Γ^i)(i.e., $S(\xi,q(\xi))=\Gamma(\xi,q(\xi))=S^i(\xi,q(\xi))=\Gamma^i(\xi,q(\xi))=q(\xi)$, and for $x_0\in C$, the sequence $\{S(\xi,x_n(\xi))\}_{n=0}^{\infty}$ is the random iterative scheme defined by (11). Then the random common fixed point $q(\xi)$ is Bochner integrable.*

Corollary 2. *Let C be a non-empty closed and convex subset of a separable Banach space X, and let $\Gamma,S:\Omega\times C\leftrightarrow C$ be two random operators satisfying (8) with $\Gamma(\xi,X)\subseteq S(\xi,X)$. Let $q(\xi)$ be a common random fixed point of (S,Γ,S^i,Γ^i) (i.e., $S(\xi,q(\xi))=\Gamma(\xi,q(\xi))=S^i(\xi,q(\xi))=\Gamma^i(\xi,q(\xi))=q(\xi)$, and for $x_0\in C$, the sequence $\{S(\xi,x_n(\xi))\}_{n=0}^{\infty}$ is the random iterative scheme defined by (12). Then the random common fixed point $q(\xi)$ is Bochner integrable.*

4 Stability results

In this section, we establish some stability results in separable Banach space for our new random iterative schemes defined in (9) and (10) under new generalized ϕ-weakly contraction defined in (8).

First, we will prove that the Jungck-DI-CR random iterative scheme $\{S(\xi,x_n(\xi))\}_{n=0}^{\infty}$ defined in (9) is $(S,\Gamma)-$ stable in the following theorem:

Theorem 3. *Let C be a non-empty closed and convex subset of a separable Banach space X, and let $\Gamma,S:\Omega\times C\leftrightarrow C$ be two random operators satisfying (8) with $\Gamma(\xi,X)\subseteq S(\xi,X)$. Let $q(\xi)$ be a common random fixed point of (S,Γ,S^i,Γ^i)(i.e., $S(\xi,q(\xi))=\Gamma(\xi,q(\xi))=S^i(\xi,q(\xi))=\Gamma^i(\xi,q(\xi))=q(\xi)$, and for $x_0\in C$, if the random Jungck-DI-CR random iterative scheme defined by (9) converges to $q(\xi)$. Then the sequence $\{S(\xi,x_n(\xi))\}_{n=0}^{\infty}$ is $(S,\Gamma)-$ stable.*

Proof. Suppose that $\{S(\xi,t_n(\xi))\}_{n=0}^{\infty}$ be arbitrary sequence of random variable in X, and

$$\varepsilon_n=\|S(\xi,t_{n+1}(\xi))-\alpha_{n,1}S(\xi,t_n(\xi))-\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,g_n(\xi))-\prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,g_n(\xi))\|^2, \tag{21}$$

where for every $\xi\in\Omega$,

$$S(\xi,g_n(\xi))=\gamma_{n,1}S(\xi,t_n(\xi))+\sum_{t=2}^{l_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,f_n(\xi))+\prod_{b=1}^{l_2}(1-\gamma_{n,b})\Gamma^{l_2}(\xi,f_n(\xi)), \tag{22}$$

and

$$S(\xi,f_n(\xi))=\delta_{n,1}S(\xi,t_n(\xi))+\sum_{s=2}^{l_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})\Gamma^{s-1}(\xi,t_n(\xi))+\prod_{c=1}^{l_3}(1-\delta_{n,c})\Gamma^{l_3}(\xi,t_n(\xi)). \tag{23}$$

We will prove that $q(\xi)$ is Bochner integrable with respect to the sequence $S(\xi,t_n(\xi))$. Let $\varepsilon_n\to 0$, as $n\to\infty$, then by 21, we get

$$\begin{aligned}&\|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2\\ =&\|\alpha_{n,1}S(\xi,t_n(\xi))+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,g_n(\xi))+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,g_n(\xi))-q(\xi)\\ &-[\alpha_{n,1}S(\xi,t_n(\xi))+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,g_n(\xi))+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,g_n(\xi))-S(\xi,t_{n+1}(\xi))]\|^2,\end{aligned}$$

it follows that

$$\begin{aligned}&\|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2\\ \le&\|\alpha_{n,1}S(\xi,t_n(\xi))+\sum_{i=2}^{l_1}\alpha_{n,i}\times\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,g_n(\xi))+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,g_n(\xi))-q(\xi)\|^2\\ &+\|-[\alpha_{n,1}S(\xi,t_n(\xi))+\sum_{i=2}^{l_1}\alpha_{n,i}\times\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,g_n(\xi))+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,g_n(\xi))-S(\xi,t_{n+1}(\xi))]\|^2,\end{aligned}$$

this leads to

$$\begin{aligned}&\|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2\\ &\leq \|\alpha_{n,1}S(\xi,t_n(\xi))+\sum_{i=2}^{l_1}\alpha_{n,i}\times\\ &\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,g_n(\xi))\\ +&\prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,g_n(\xi))-q(\xi)\|^2+\varepsilon_n,\end{aligned}$$

hence

$$\begin{aligned}&\|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2\\ &\leq \varepsilon_n+\alpha_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2+\sum_{i=2}^{l_1}\alpha_{n,i}\times\\ &\prod_{a=1}^{i-1}(1-\alpha_{n,a})\|\Gamma^{i-1}(\xi,g_n(\xi))-q(\xi)\|^2\\ &+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\|\Gamma^{l_1}(\xi,g_n(\xi))-q(\xi)\|^2,\end{aligned}$$

it follows that

$$\begin{aligned}&\|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2\\ &\leq \varepsilon_n+\alpha_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2\\ &+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\times\\ &\left(e^{\sum_{j=1}^{i-1}\binom{i-1}{j}\nu^{i-2}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)^2\\ &\times\left(\frac{\nu^{i-1}\|S(\xi,q(\xi))-S(\xi,g_n(\xi))\|}{1+\eta^{i-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right.\\ &\left.-\phi^{i-1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,g_n(\xi))\|}{1+\eta^{i-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\right)^2\\ &+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\times\\ &\left(e^{\sum_{j=1}^{l_1}\binom{l_1}{j}\nu^{l_1-1}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)\\ &\times\left(\frac{\nu^{l_1}\|S(\xi,q(\xi))-S(\xi,g_n(\xi))\|}{1+\eta^{l_1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right.\\ &\left.-\phi^{l_1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,g_n(\xi))\|}{1+\eta^{l_1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\right)^2,\end{aligned}$$

Now, we have

$$\begin{aligned}&\|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2\\ &\leq \varepsilon_n+\alpha_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2+\sum_{i=2}^{l_1}\alpha_{n,i}\\ &\times\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^{i-1})^2\|S(\xi,q(\xi))-S(\xi,g_n(\xi))\|^2\\ &+\prod_{a=1}^{l_1}(1-\alpha_{n,a})(\nu^{l_1})^2\|S(\xi,q(\xi))-S(\xi,g_n(\xi))\|^2. \quad (24)\end{aligned}$$

Again, using (8) and (22) with Preposition 2 to compute the following:

$$\begin{aligned}&\|S(\xi,g_n(\xi))-q(\xi)\|^2\\ &=\|\gamma_{n,1}S(\xi,t_n(\xi))\\ &+\sum_{t=2}^{l_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\Gamma^{t-1}(\xi,f_n(\xi))\\ &+\prod_{b=1}^{l_2}(1-\gamma_{n,b})\Gamma^{l_1}(\xi,f_n(\xi))-q(\xi)\|^2\\ &\leq \gamma_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2+\sum_{t=2}^{l_2}\gamma_{n,t}\times\\ &\prod_{b=1}^{t-1}(1-\gamma_{n,b})\|\Gamma^{t-1}(\xi,f_n(\xi))-q(\xi)\|^2\\ &+\prod_{b=1}^{l_2}(1-\gamma_{n,b})\|\Gamma^{l_2}(\xi,f_n(\xi))-q(\xi)\|^2,\end{aligned}$$

it follows that

$$\begin{aligned}&\|S(\xi,g_n(\xi))-q(\xi)\|^2\\ &\leq \gamma_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2+\sum_{t=2}^{l_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})\\ &\times\left(e^{\sum_{j=1}^{t-1}\binom{t-1}{j}\nu^{t-2}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)^2\\ &\times\left(\frac{\nu^{t-1}\|S(\xi,q(\xi))-S(\xi,f_n(\xi))\|}{1+\eta^{t-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right.\\ &\left.-\phi^{t-1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,f_n(\xi))\|}{1+\eta^{t-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\right)^2\\ &+\prod_{b=1}^{l_2}(1-\gamma_{n,b})\times\\ &\left(e^{\sum_{j=1}^{l_2}\binom{l_2}{j}\nu^{l_2-1}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)^2\\ &\times\left(\frac{\nu^{l_2}\|S(\xi,q(\xi))-S(\xi,f_n(\xi))\|}{1+\eta^{l_2}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right.\\ &\left.-\phi^{l_2}\left(\frac{\|S(\xi,q(\xi))-S(\xi,f_n(\xi))\|}{1+\eta^{l_2}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\right)^2,\end{aligned}$$

this leads to

$$\begin{aligned}&\|S(\xi,g_n(\xi))-q(\xi)\|^2\\ &\leq \gamma_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2+\sum_{j=2}^{l_2}\gamma_{n,t}\\ &\prod_{b=1}^{t-1}(1-\gamma_{n,b})\left(e^{\sum_{j=1}^{t-1}\binom{t-1}{j}\nu^{t-2}L^j(\xi)\|0\|}\right)^2\\ &\times\left(\frac{\nu^{t-1}\|S(\xi,q(\xi))-S(\xi,f_n(\xi))\|}{1+\eta^{t-1}\|0\|}\right.\end{aligned}$$

$$-\phi^{t-1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,f_n(\xi))\|}{1+\eta^{t-1}\|0\|}\right)\Bigg)^2$$
$$+\prod_{b=1}^{l_2}(1-\gamma_{n,b})\left(e^{\sum_{j=1}^{l_2}\binom{l_2}{j}\nu^{l_2-1}L^j(\xi)\|0\|}\right)^2$$
$$\times\Bigg(\frac{\nu^{l_2}\|S(\xi,q(\xi))-S(\xi,f_n(\xi))\|}{1+\eta^{l_2}\|0\|}$$
$$-\phi^{l_2}\left(\frac{\|S(\xi,q(\xi))-S(\xi,f_n(\xi))\|}{1+\eta^{l_2}\|0\|}\right)\Bigg)^2,$$

it follows that

$$\|S(\xi,f_n(\xi))-q(\xi)\|^2$$
$$\leq \gamma_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2+\sum_{t=2}^{l_2}\gamma_{n,t}\times$$
$$\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^{t-1})^2\|S(\xi,q(\xi))-S(\xi,f_n(\xi))\|^2$$
$$+\prod_{b=1}^{l_2}(1-\gamma_{n,b})(\nu^{l_2})^2\|S(\xi,q(\xi))-S(\xi,f_n(\xi))\|^2. \quad (25)$$

Finally, we compute the following:

$$\|S(\xi,f_n(\xi))-q(\xi)\|^2$$
$$=\|\delta_{n,1}S(\xi,t_n(\xi))+\sum_{s=2}^{l_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,s})\Gamma^{s-1}(\xi,t_n(\xi))$$
$$+\prod_{c=1}^{l_3}(1-\delta_{n,c})\Gamma^{l_3}(\xi,t_n(\xi))-q(\xi)\|^2$$

$$\leq \delta_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2$$
$$+\sum_{s=2}^{l_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,s})\|\Gamma^{s-1}(\xi,t_n(\xi))-q(\xi)\|^2$$
$$+\prod_{c=1}^{l_3}(1-\delta_{n,c})\|\Gamma^{l_3}(\xi,t_n(\xi))-q(\xi)\|^2,$$

hence

$$\|S(\xi,f_n(\xi))-q(\xi)\|^2$$
$$\leq \delta_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2+\sum_{s=2}^{l_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})$$
$$\times\left(e^{\sum_{j=1}^{s-1}\binom{s-1}{j}\nu^{s-2}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)^2$$
$$\times\Bigg(\frac{\nu^{s-1}\|S(\xi,q(\xi))-S(\xi,t_n(\xi))\|}{1+\eta^{s-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}$$
$$-\phi^{s-1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,t_n(\xi))\|}{1+\eta^{s-1}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\Bigg)^2$$
$$+\prod_{c=1}^{l_3}(1-\delta_{n,c})\times$$
$$\left(e^{\sum_{j=1}^{l_3}\binom{l_3}{j}\nu^{l_3-1}L^j(\xi)\|S^j(\xi,q(\xi))-\Gamma^j(\xi,q(\xi))\|}\right)^2$$
$$\times\Bigg(\frac{\nu^{l_3}\|S(\xi,q(\xi))-S(\xi,t_n(\xi))\|}{1+\eta^{l_3}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}$$
$$-\phi^{l_3}\left(\frac{\|S(\xi,q(\xi))-S(\xi,t_n(\xi))\|}{1+\eta^{l_3}\|S(\xi,q(\xi))-\Gamma(\xi,q(\xi))\|}\right)\Bigg)^2.$$

Hence, we get

$$\delta_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2$$
$$\leq \delta_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2+\sum_{s=2}^{l_3}\delta_{n,s}$$
$$\prod_{c=1}^{s-1}(1-\delta_{n,c})\left(e^{\sum_{j=1}^{s-1}\binom{s-1}{j}\nu^{s-2}L^j(\xi)\|0\|}\right)^2$$
$$\times\Bigg(\frac{\nu^{s-1}\|S(\xi,q(\xi))-S(\xi,t_n(\xi))\|}{1+\eta^{s-1}\|0\|}$$

$$-\phi^{s-1}\left(\frac{\|S(\xi,q(\xi))-S(\xi,t_n(\xi))\|}{1+\eta^{s-1}\|0\|}\right)\Bigg)^2$$
$$+\prod_{c=1}^{l_3}(1-\delta_{n,c})e^{\sum_{j=1}^{l_3}\binom{l_3}{j}\nu^{l_3-1}L^j(\xi)\|0\|}$$
$$\times\Bigg(\frac{\nu^{l_3}\|S(\xi,q(\xi))-S(\xi,t_n(\xi))\|}{1+\eta^{l_3}\|0\|}$$
$$-\phi^{l_3}\frac{\|S(\xi,q(\xi))-S(\xi,t_n(\xi))\|}{1+\eta^{l_3}\|0\|}\Bigg)^2,$$

this implies that

$$\delta_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2$$
$$\leq \delta_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2+\sum_{s=2}^{l_3}\delta_{n,s}$$
$$\prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^{s-1})^2\|S(\xi,q(\xi))-S(\xi,t_n(\xi))\|^2$$

$$+\prod_{c=1}^{l_3}(1-\delta_{n,c})(\nu^{l_3})^2\|S(\xi,q(\xi))-S(\xi,t_n(\xi))\|^2$$
$$=\Bigg(\delta_{n,1}+\sum_{s=2}^{l_3}\delta_{n,s}\prod_{c=1}^{s-1}(1-\delta_{n,c})(\nu^{s-1})^2$$
$$+\prod_{c=1}^{l_3}(1-\delta_{n,c})(\nu^{l_3})^2\Bigg)$$
$$\times\|S(\xi,q(\xi))-S(\xi,t_n(\xi))\|^2$$
$$<\|S(\xi,q(\xi))-S(\xi,t_n(\xi))\|^2, \quad (26)$$

by using $\nu^{s-1}, \nu^{l_3} \in (0,1]$ and Proposition 1. Applying (26) in (25), we obtain

$$\begin{aligned}
&\|S(\xi,g_n(\xi))-q(\xi)\|^2\\
&\leq \gamma_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2+\sum_{t=2}^{l_2}\gamma_{n,t}\times\\
&\prod_{b=1}^{t-1}(1-\gamma_{n,c})(\nu^{t-1})^2\|S(\xi,q(\xi))-S(\xi,t_n(\xi))\|^2\\
&+\prod_{b=1}^{l_2}(1-\gamma_{n,b})(\nu^{l_2})^2\|S(\xi,q(\xi))-S(\xi,t_n(\xi))\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\|S(\xi,g_n(\xi))-q(\xi)\|^2\\
&\leq \Bigg(\gamma_{n,1}+\sum_{t=2}^{l_1}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})(\nu^t)^2\\
&+\prod_{c=1}^{l_2}(1-\gamma_{n,b})(\nu^{l_2})^2\Bigg)\\
&\times\|S(\xi,x_n(\xi))-q(\xi)\|^2.\\
&<\left(\gamma_{n,1}+\sum_{t=2}^{l_2}\gamma_{n,t}\prod_{b=1}^{t-1}(1-\gamma_{n,b})+\prod_{b=1}^{l_2}(1-\gamma_{n,b})\right)\\
&\times\|S(\xi,t_n(\xi))-q(\xi)\|^2\\
&=\|S(\xi,t_n(\xi))-q(\xi)\|^2. \qquad (27)
\end{aligned}$$

Applying (27) in (24), we obtain

$$\begin{aligned}
&\|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2\\
&\leq \varepsilon_n+\alpha_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2+\sum_{i=2}^{l_1}\alpha_{n,i}\times\\
&\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2\|S(\xi,q(\xi))-S(\xi,g_n(\xi))\|^2\\
&+\prod_{a=1}^{l_1}(1-\alpha_{n,a})(\nu^i)^2\|S(\xi,q(\xi))-S(\xi,g_n(\xi))\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2\\
&<\varepsilon_n+\Bigg(\alpha_{n,1}+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^i)^2\\
&+\prod_{a=1}^{l_1}(1-\alpha_{n,a})(\nu^i)^2\Bigg)\\
&\times\|S(\xi,t_n(\xi))-q(\xi)\|^2\\
&<\varepsilon_n+\left(\alpha_{n,1}+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\right)\\
&\times\|S(\xi,t_n(\xi))-q(\xi)\|^2\\
&<\varepsilon_n+\|S(\xi,t_n(\xi))-q(\xi)\|^2. \qquad (28)
\end{aligned}$$

Using Lemma 1 and 2, we obtain that $\lim_{n\to\infty}S(\xi,t_n(\xi)) = q(\xi)$. Conversely, let $\lim_{n\to\infty}S(\xi,t_n(\xi))=0$, then, we will show that $\varepsilon_n\to 0$ as $n\to\infty$.

$$\begin{aligned}
\varepsilon_n = \|&S(\xi,t_{n+1}(\xi))-q(\xi)-[\alpha_{n,1}S(\xi,t_n(\xi))\\
&+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,g_n(\xi))\\
&+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,g_n(\xi))-q(\xi)]\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\varepsilon_n\\
&\leq \|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2+\|\alpha_{n,1}S(\xi,t_n(\xi))\\
&\quad+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\Gamma^{i-1}(\xi,g_n(\xi))\\
&\quad+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\Gamma^{l_1}(\xi,g_n(\xi))-q(\xi)\|^2\\
&\leq \|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2+\alpha_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2\\
&\quad+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})\|\Gamma^{i-1}(\xi,g_n(\xi))-q(\xi)\|^2\\
&\quad+\prod_{a=1}^{l_1}(1-\alpha_{n,a})\|\Gamma^{l_1}(\xi,g_n(\xi))-q(\xi)\|^2. \qquad (29)
\end{aligned}$$

By the same way of computing the estimate $\|\Gamma^i(\xi,g_n(\xi))-q(\xi)\|$, we can prove that

$$\begin{aligned}
\|\Gamma^i(\xi,g_n(\xi))-q(\xi)\| &< (\nu^i)^2\|S(\xi,g_n(\xi))-q(\xi)\|\\
&< \|S(\xi,t_n(\xi))-q(\xi)\|.
\end{aligned}$$

Applying this in (29), we get,

$$\begin{aligned}
&\varepsilon_n\\
&< \|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2\\
&\quad+\alpha_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2\\
&\quad+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^{i-2})^2\|S(\xi,g_n(\xi))-q(\xi)\|^2\\
&\quad+\prod_{a=1}^{l_1}(1-\alpha_{n,a})(\nu^{l_1})^2\|S(\xi,g_n(\xi))-q(\xi)\|^2\\
&= \|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2+\alpha_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2\\
&\quad+\left(\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(\nu^{i-2})^2+\prod_{a=1}^{l_1}(1-\alpha_{n,a})(\nu^{l_1})^2\right)\\
&\quad\times\|S(\xi,g_n(\xi))-q(\xi)\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\varepsilon_n \\
&< \|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2+\alpha_{n,1}\|S(\xi,t_n(\xi))-q(\xi)\|^2 \\
&\quad+\left(\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(v^{i-1})^2+\prod_{a=1}^{l_1}(1-\alpha_{n,a})(v^{l_1})^2\right) \\
&\quad\times\|S(\xi,t_n(\xi))-q(\xi)\|^2 \\
&= \|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2 \\
&\quad+\left(\alpha_{n,1}+\sum_{i=2}^{l_1}\alpha_{n,i}\prod_{a=1}^{i-1}(1-\alpha_{n,a})(v^{i-1})^2\right. \\
&\quad\left.+\prod_{a=1}^{l_1}(1-\alpha_{n,a})(v^{l_1})^2\right) \\
&\quad\times\|S(\xi,t_n(\xi))-q(\xi)\|^2 \\
&< \|S(\xi,t_{n+1}(\xi))-q(\xi)\|^2+\|S(\xi,t_n(\xi))-q(\xi)\|^2.
\end{aligned}$$

The right hand side of the above inequality tends to zero as $n\to\infty$. Thus, $\varepsilon_n\to 0$. This completes the proof.

Theorem 4. *Let C be a non-empty closed and convex subset of a separable Banach space X, and let $\Gamma,S:\Omega\times C\leftrightarrow C$ be two random operators satisfying (8) with $\Gamma(\xi,X)\subseteq S(\xi,X)$. Let $q(\xi)$ be a common random fixed point of (S,Γ,S^i,Γ^i) (i.e., $S(\xi,q(\xi))=\Gamma(\xi,q(\xi))=S^i(\xi,q(\xi))=\Gamma^i(\xi,q(\xi))=q(\xi)$), and for $x_0\in C$, if the Jungck-DI-Karahan-Ozdemir random iterative scheme defined by (10) converges to $q(\xi)$. Then the sequence $\{S(\xi,x_n(\xi))\}_{n=0}^{\infty}$ is $(S,T)-$ stable.*

Proof. The proof of Theorem 4 follows similar lines of the proof of Theorem 3.

From Theorem 3, we can present the following corollaries.

Corollary 3. *Let C be a non-empty closed and convex subset of a separable Banach space X, and let $\Gamma,S:\Omega\times C\leftrightarrow C$ be two random operators satisfying (8) with $\Gamma(\xi,X)\subseteq S(\xi,X)$. Let $q(\xi)$ be a common random fixed point of $(S,\Gamma,S^i,\Gamma^i)(i.e.,S(\xi,q(\xi))=\Gamma(\xi,q(\xi))=S^i(\xi,q(\xi))=\Gamma^i(\xi,q(\xi))=q(\xi)$, and for $x_0\in C$, if the random Jungck-DI-CR random iterative scheme defined by (11) converges to $q(\xi)$. Then the sequence $\{S(\xi,x_n(\xi))\}_{n=0}^{\infty}$ is $(S,T)-$ stable.*

Corollary 4. *Let C be a non-empty closed and convex subset of a separable Banach space X, and let $\Gamma,S:\Omega\times C\leftrightarrow C$ be two random operators satisfying (8) with $\Gamma(\xi,X)\subseteq S(\xi,X)$. Let $q(\xi)$ be a common random fixed point of $(S,\Gamma,S^i,\Gamma^i)(i.e.,S(\xi,q(\xi))=\Gamma(\xi,q(\xi))=S^i(\xi,q(\xi))=\Gamma^i(\xi,q(\xi))=q(\xi)$, and for $x_0\in C$, if the random Jungck-DI-CR random iterative scheme defined by (12) converges to $q(\xi)$. Then the sequence $\{S(\xi,x_n(\xi))\}_{n=0}^{\infty}$ is $(S,T)-$ stable.*

5 Conclusion

In this paper, we have introduced new random iterative schemes namely, Jungck-DI-CR random and Jungck-DI-Karahan-Ozdemir random iterative schemes. Also, we have studied the convergence and stability of theses random iterative schemes under new generalized ϕ- weakly contraction. Ultimately, we omit the sum condition of the countably finite family of the control sequences and injectivity condition of the operators.

Competing interests

The author declares that they have no competing interests.

References

[1] I. Beg, M. Abbas, Iterative procedure for solutions of random operator equations in Banach spaces, *J. Math. Appl.* **315,** 181-201 (2006).

[2] F. O. Isiogugu, C. Izuchukwu C. C. Okeke, New iteration scheme for approximating a common fixed point of a finite family of mappings, *Hindawi J. Math.* **2020,** Article ID 3287968 (2020).

[3] K. S. Kim, Convergence and stabililty of generalized ϕ-weak contraction mapping in CAT(0) Spaces, *Open Math.* **15,** 1063-1074 (2017).

[4] G. Jungck. Commuting mappings and fixed points, *The American Mathematical Monthly*. **83,** 261-263 (1976).

[5] S. L. Singh, C. Bhatnagar and S. N. Mishra, Stability of Jungck-type iterative procedures, *International Journal of Mathematics and Mathematical Sciences.* **19,** 3035-3043 (2005).

[6] M. O. Olatinwo, Some stability and strong convergence results for the Jungck-Ishikawa iteration process, *Creative Mathematics and Informatics.* **17,** 33-42 (2008).

[7] N. Hussain, V. Kumar, MA. Kutbi, On the rate of convergence of Jungck-type iterative schemes, *Abstr. Appl. Anal.* **2013** Article ID 132626 (2013).

[8] M. O. Olutinwo, Some stability results for two hybrid fixed point iterative algorithms in normed linear space. *Mat.vesn.* **61**(4), 247-256 (2009).

[9] R. Chugh and V. Kumar, Stability of Hybrid Fixed Point Iterative Algorithmsof Kirk-Noor Type in Normed Linear Space for Self and Nonself Operators, *Int. J. Contemp. Math. Sciences* **7**(24), 1165-1184 (2012).

[10] I. Karahan, M. Ozdemir, A general iterative method for approximation of fixed points and their applications, *Adv. Fixed Point Theory,* **3**(3), 510-526 (2013).

[11] A. M. Harder, Fixed point theory and stability results for fixed points iteration procedures, *Ph. D. Thesis, University of Missouri- Rolla.* (1987).

[12] A. M. Harder, T. L. Hicks, A stable iteration procedure for nonexpansive mappings, *Math. Japonica.* **33(5),** 687-692 (1988).

[13] A. M. Harder, T. L. Hicks, Stability results for fixed point iteration procedures, *Math. Japonica.* **33(5),** 693-706 (1988).

[14] M. O. Olatinwo, Some stability results for two hybrid fixed point iterative algorithms of Kirk-Ishikawa and Kirk-Mann type, *Journal of Advanced Mathematical Studies.* **1,** 5-14 (2008).
[15] R. A. Rashwan, H. A. Hammad and G. A. Okeke, Convergence and almost sure $(S,T)-$stability for random iterative schemes, *International Journal of Advances in Mathematics*. **2016(1),** 1-16, (2016).
[16] R. A. Rashwan and H. A. Hammad, A coupled random fixed point result with application in polish spaces, Sahand Communications in Mathematical Analysis (SCMA) **11(1)**, 99-113, (2018).
[17] R. A. Rashwan and H. A. Hammad, Stability and strong convergence results for random Jungck-Kirk-Noor iterative scheme, Fasciculi Mathematici, **58**, 165-180.
[18] R. A. Rashwan and H. A. Hammad, Convergence and stability of modified random SP-iteration for a generalized asymptotically quasi-nonexpansive mappings, Mathematics Interdisciplinary Research, **2**, 9-21, (2017).
[19] M. O. Osilike, Stability of the Mann and Ishikawa iteration procedures for $\phi-$ strong pseudo-contractions and nonlinear equations of the $\phi-$strongly accretive type, *J. Math. Anal. Appl.* **227(2)**, 319-334 (1998).
[20] I. K. Ageu, D. I. Igbokwe, Stability and convergence of new random approximation algorithms for random contractive-type operators in separable Hilbert spaces, *Malaya J. Mat.* **9(3),** 148-167 (2021).
[21] G. A. Okeke and M. Abbas, Convergence and almost sure T-stability for a random iterative sequence generated by a generalized random operator, *Journal of Inequalities and Applications.* **146,** 1-11 (2015).
[22] S. S. Zhang, X. R. Wang, M. Liu, Almost sure T-stability and convergence for random iterative algorithms, *Appl. Math. Mech.* **32(6),** 805-810 (2011).
[23] D. M. Albaqeri, R. A. Rashwan, The comparably almost (S,T)- stability for random Jungck-type iterative scheme, *Facta Universities (NIŠ)*. **34(2),** 175-192 (2019)
[24] G. A. Okeke, S. A. Bishop and H. Akewe, Random fixed point theorems in Banach spaces applied to a random nonlinear integral equation of the Hammerstein type. *Fixed Point Theory and Applications,* **15,** 1-24 (2019).
[25] V. Berinde, Iterative Approximation of Fixed Points, *Springer-Verlag* Berlin Heidelberg, (2007).

Dhekra M. Al-Baqeri working as Associate Professorat Sana'a& University, Yemen. She received the Master's& degree in Mathematics (Functional Analysis) from King Abdulaziz University in Saud,i Arabia, and Ph. D from Assiut University in Egypt. Her research interests in the field of functional analysis, especially fixed-point theory and random fixed-point theory and its applications. She published several papers in reputed international journal.

Hasanen. A. Hammad working as Assistant Professor at Sohag University, Egypt. He received Master's and Ph.D. degrees in Mathematics (Functional Analysis) at the University of Sohag. His research interests are in the field of functional analysis especially, fixed point theory and its applications, random fixed points and its applications, Optimization, and variational inequalities. He has published more than 95 papers in reputed international journals of Mathematics. In addition, he is a referee and editor of good mathematical journals.