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The principle of superposition for waves: The amplitude and phase modulation phenomena

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Abstract: In this paper we will argue that the superposition of waves can be calculated in a simple way. We show, using the Gauss's method to sum an arithmetic sequence, how it is possible to construct the superposition of waves - with different frequencies - in a simple conceptual way. By this method we arrive to the usual result where we can express the superposition of waves as the product of factors, one of them with a cosine function where the cosine's argument is the average frequency. Most important, we will show that the superposition of waves with slightly different frequencies produces the *phase modulation* phenomenon as well as the amplitude modulation phenomenon, where we named as **phase modulation** the phenomenon where there is a phase delay each time that there exits a complete destructive interference. Although this could be a know fact for some physicist (specially those working on the theory of sound), it is important to emphasize this result because, to the best of our knowledge, when studying the superposition of waves with nearly equal frequencies almost all research papers and textbooks only mention that there exist an amplitude modulation.

Keywords: Phase Modulation, Amplitude Modulation, Superposition of Waves.

1. Introduction

Superposition of waves is one of the fundamental concepts in the subject matters of Waves [1–4] and Optics [5, 6]. It allows the study and explanation of interference patterns in Optics, which is one of the most beautiful physical phenomenon. Furthermore, it allows physicists to explain many interesting sound phenomena like beat phenomenon, which is a slow variation in the intensity of sound when we add two waves of slightly different frequencies. The beat effect is used by musicians in tuning theirs musical instruments [7], it is caused by the amplitude modulation phenomenon produced by superposing waves of slightly different frequencies. The conceptual foundation in which is based the study of these phenomena is the Principle of Superposition, which roughly speaking state, in the case of waves, that the resultant of adding N waves is the sum of the individual waves.

Usually, when this subject is studied [8–10], the analysis is commonly carried out by summing only two waves as follows:

$$y(x,t) = A_1 \cos(\omega_1 t - k_1 x + \phi_1) + A_2 \cos(\omega_2 t - k_2 x + \phi_2). \quad (1)$$

If the waves starts out with the same phase constant (i. e. $\phi_1 = \phi_2$) and with the same amplitude, i.e. $A_1 = A_2 = A$ (and observed at some fixed point), we can express the result as the multiplication of two sinusoidal waves as follows [1]:

$$y(t) = 2A\cos\left(\frac{(\omega_2+\omega_1)t}{2}\right)\cos\left(\frac{(\omega_2-\omega_1)t}{2}\right). \quad (2)$$

If the two frequencies are rather similar, that is when:

$$\omega_2 \approx \omega_1, \quad (3)$$

then, it is stated in many research papers [12] that equation (2) represents a wave that **oscillates at frequency** $(\omega_2+\omega_1)/2$ **and whose intensity increase and decrease at the beating frequency** $(\omega_2-\omega_1)$. In fact, it is common to write Equation (2) in the following way:

$$y(t) = A(t)\cos\left(\frac{(\omega_2+\omega_1)t}{2}\right), \quad (4)$$

where $A(t) = 2A\cos((\omega_2-\omega_1)t/2)$. Equation (4) is interpreted as a wave that oscillates at the high frequency

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$(\omega_2 + \omega_1)/2$ multiplied by a slowly variable amplitude $A(t)$. Therefore, most physicists and engineers agree that Equation (4) clearly shows that the sum of two waves of almost equal frequency produces the phenomenon of amplitude modulation. Accordingly, it is usual to reproduce the plot of Equation (2) when $\omega_2 \approx \omega_1$. However, almost nothing is mentioned about the case when equation (3) is not fulfilled.

The next step in the study of superposition of waves (i. e. summing N waves for $N > 2$) is carried out using complex number. Then, representing waves as complex functions (i.e. $e^{i\omega t}$) it is possible to obtain the following analytical expression for the sum of N waves of equal successive phase difference $\delta\omega$ [2,1]:

$$y(t) = A\cos\left(\frac{\omega_N + \omega_1}{2}t\right)\frac{\sin(N\delta\omega t/2)}{\sin(\delta\omega t/2)}, \tag{5}$$

where ω_N is the last frequency of the sequence and ω_1 is the first. In general, the $N-th$ frequency can be obtaining by summing $N-1$ times $\delta\omega$ to the wave with frequency ω_1, that is [1,2]:

$$\omega_N = \omega_1 + (N-1)\delta\omega, \tag{6}$$

To deduce Equation (5) the calculation is done by adding the geometric progression $S(z) = 1 + z + z^2 + \cdots + z^N$, where $z = e^{i\delta\omega}$, see pages 29 and 30 in reference [2] or page 287 in reference [1]. It is worth to mention that another way to arrive to Equation (5) is by using the phasor method [1,2]. Also, it is important to highly that in the same way that it happens with Equation (2) it is common to plot Equation (5) when

$$\delta\omega \ll \omega_1, \tag{7}$$

which implies (for few waves, i. e. N not so high):

$$\omega_N \approx \omega_1. \tag{8}$$

Equation (8) is pretty similar approximation than Equation (3). Both plots, that of Equations (2) and (5), are practically the same when conditions given by Equations (3) and (8) are fulfilled. The oscillations in both situations are similar, and it is taken for granted that the plot of Equation (5) **oscillates at frequency $(\omega_N + \omega_1)/2$ with the envelope oscillating at frequency $(\omega_N - \omega_1)/2$.**

It is worth to mention that the superposition of waves also includes the superposition of Gaussian waves, that could be thought as superposition of waves on a range of frequencies $\Delta\omega$. For and interesting experimental demonstration see reference [11].

To summing up, almost all research papers [12] and textbooks on the subject matter of Waves [1–4] and Optics [5,6] agree that when two (or N) waves of slightly different frequencies are superposed the most prominent observed effect is the *amplitude modulation*. Almost all of them does not mention any thing about *phase variation or modulation* [13].

Because of being such an important and fundamental concept, the study of the superposition of waves is of high importance in many branches of engineering and physics. Therefore, it is important to extract all the relevant physical predictions that could be produced by this phenomena.

On the other hand, from the point of view of Physical Education Research (PER) [14–16] there is the goal to present new conceptual ways to approach the solving of relevant problems [17] and the teaching of fundamental physical and mathematical concepts. Then, reaching this goal is one of the most important learning steps that a researcher can focuses on. It is worth to emphasize that one of the fundamental goals for both the PER's field and Mathematical Education is to discover new, imaginative and tested ways for teaching physics and mathematics. Part of this goal could be accomplished by finding easier conceptual and mathematical ways to solve model problems, see reference [17–19] where there is a proposal for an easier conceptual approach to solving quantum physical problems to avoid certain misconceptions.

In this work, we present a method to sum N waves in a simple mathematical and conceptual way; just by using the trick, usually attributed to Gauss, of arranging numbers by pairs to sum them. This is a quite simple conceptual method that could be taught to any student with only basic knowledge on trigonometric identities and, of course, it could be taught to more advanced students. Also, we analyze this superposition in other settings different to the usual restriction of $\delta\omega \ll \omega_1$. We have found out that the sum of N waves is more richer and complex phenomenon than it is usually though. The most important point[1] of this work is that *we found that together with an amplitude modulation there exist a phase modulation when we superpose two or more waves of slightly different frequencies*. It is worth to highly this result because viewing this phenomenon from a different perspective it could have important practical implications.

Furthermore, this approach to study the superposition of waves allows to introduce an historical account in teaching, both to increase the student interest and to show the problems associated with the history of scientific facts, in this case the problem associated with who (and how) discovered the procedure to sum an arithmetic sequence. Also, there is an interesting question in the work done by Rayleigh that we will draw at the end of *sub-subsection 3.1.1*. There have been many advocates about the convenience of treating problems of historical science facts in the classroom to enhance the grasp of something that is complicated or difficult to understand, see for instance the

[1] After we have found that there was something like a phase modulation in the phenomenon of superposition of waves with slightly different frequencies, we made a bibliographic review and our surprise was that this is a quite know fact for people working in the theory of sound, even since the time of John William Strutt, 3rd Baron Rayleigh. In fact, Raleigh's book [28] was the first one were we found it. The surprise comes because we were working in optics for many years and because we teach, also for many years, undergraduate physics students with references [8–10] and neither in books of vibration and waves neither in undergraduate textbooks there is any mention of this fact.

recent published review by Teixeira, et. al. [23], the illuminating works of M. A. B. Whitaker [21,22] and see, also, reference [24].

2. The Mathematics of Arithmetic Sequence

Whether apocryphal or not it is well known the story of Gauss's method to sum an arithmetic sequence of numbers [25] (or Alcuin of York method [25] or Archimedes [26]). The story, its weakness and drawbacks are well summarized in the article by Brian Hayes [25,26]. When Gauss was a child his teacher gave all the students the task of summing an arithmetic progression [25]; in order to be explicit we suppose here that the arithmetic sequence was the first one hundred numbers. It is said that using his great talent Gauss didn't make the procedure of adding one number to the other, instead he grouped together the numbers in pairs and found that the sum of a pair of numbers was the same for numbers located at equidistant distance from the extreme, that is $100+1$=$99+2$=$98+3$=$\cdots$=$52+49$=$51+50=101$. Then, the sum is 50 times 101. It is worth to mention that the difference between the grouped numbers is lowered by two, given a succession of odd numbers, that is $100-1=99$, $99-2=97$, $98-3=95$, $\cdots$, $52-49=3$, $51-50=1$.

More generally, an arithmetic sequence is a set of numbers such that each of them is obtained by summing to the previous number a constant, say $\delta\omega$. For example, the set $(1, 11, 21, 31, \cdots, 101)$ is an arithmetic progression with constant $\delta\omega = 10$, which has as a first number 1 and as a last number 101.

An arbitrary term of the arithmetic sequence can be calculated as follows: If a_1 is the first term then the second term $a_2 = a_1 + \delta\omega$ is:

$$a_1 + (2-1)\delta\omega \tag{9}$$

the third term can be written as:

$$a_1 + (3-1)\delta\omega, \tag{10}$$

hence, the $i-th$ term of the arithmetic sequence is given by:

$$a_i = a_1 + (i-1)\delta\omega. \tag{11}$$

On the other hand, it is worth to notice that Equation (11) is the same Equation (6) (and is the same that Equation (14) below), this will allow us to think about the sum of N waves as representing a *superposition of an arithmetic sequence of frequencies* and to use the properties of this kind of sequences to sum them. This way of thinking contrast with that of textbook's authors who used to think about the sum of N waves in terms of *geometric sequences*, as was stated in the introduction section.

The historical problem about who was the first person who discovered the trick to sum an arithmetic sequence is explained by Haynes in references [25] and [26]. This historical event can be used to show students the problems to exactly determine some historical facts and to increase their understanding of the nature of science [23]. It is notable that many people really know that the trick was found by Gauss, but few people know that such trick was published before (in books) by at least two people: Alcuin of York and Archimedes [25,26].

3. Superposition of waves with different frequencies at a fixed point

3.1. The sum of frequencies as a constant

As was stated above, when we add two waves which have slightly different frequencies their superposition produces an amplitude-modulated wave. Based on the superposition principle, the sum of two or more waves is a wave too, which fulfills the wave equation. The magnitude of the wave sum at any space point and at some time depends on the phase value of every wave component. That is, the wave's phase is a function of both the space and time.

In this section we are going to study the case where we superpose N waves (with N an even number) with the same initial phase constant, same amplitude and different frequencies at a fixed point x. We consider an interval of frequencies $\Delta\omega$ and make a partition of the interval in an even number between, say, ω_1 to ω_{100}, i. e. $\Delta\omega = \omega_{100} - \omega_1$. That is, we have:

$$y_T(x,t) = A \sum_{i=1}^{N=100} \cos(k_i x - \omega_i t + \phi_i), \tag{12}$$

where ω_i is the angular frequency of the $i-th$ wave, k_i is the wave number, and ϕ_i is the initial phase constant. For the sake of simplicity we are going to consider the case where the waves have the same initial phase constant, i. e. $\phi_i = 0$. Also, as we are interested in analyzing the case where the waves are superposed at some fixed point, without loss of generality we can set $x = 0$. Therefore Equation (12) reduces to:

$$y_T(t) = A \sum_{i=1}^{N=100} \cos(\omega_i t). \tag{13}$$

We consider the case when the upper limit in the sum N is an even number. Also, we make the partition of $\Delta\omega$ by adding a $\delta\omega$ to the next frequency in the sequence, that is, we have:

$$\omega_N = \omega_1 + (N-1)\delta\omega. \tag{14}$$

Therefore, if we compare Equation (14) with Equation (11) we conclude that the set of frequencies form an arithmetic sequence. Hence, in order to be able to use Gauss method, we group the sum given in Equation (13) by pairs as follows:

$$y_T(t) = A \left\{ \left[\cos(\omega_{100} t) + \cos(\omega_1 t) \right] + \left[\cos(\omega_{99} t) + \cos(\omega_2 t) \right] + \cdots + \left[\cos(\omega_{51} t) + \cos(\omega_{50} t) \right] \right\}. \quad (15)$$

Now, using the following trigonometric identity

$$\cos(\omega_i t) + \cos(\omega_j t) = 2\cos\left(\frac{(\omega_i + \omega_j)t}{2}\right) \times \cos\left(\frac{(\omega_i - \omega_j)t}{2}\right), \quad (16)$$

we can rewrite the sum of the grouped cosine functions in Equation (15) as the product of two cosines functions. Then, by substituting Equation (16) in Equation (15) we have:

$$y_T(t) = A\left\{ \left[2\cos\left(\frac{(\omega_{100}+\omega_1)t}{2}\right)\cos\left(\frac{(\omega_{100}-\omega_1)t}{2}\right)\right] + \left[2\cos\left(\frac{(\omega_{99}+\omega_2)t}{2}\right)\cos\left(\frac{(\omega_{99}-\omega_2)t}{2}\right)\right] + \cdots + \left[2\cos\left(\frac{(\omega_{51}+\omega_{50})t}{2}\right)\cos\left(\frac{(\omega_{51}-\omega_{50})t}{2}\right)\right]\right\} \quad (17)$$

Now, as by Equation (14) $\omega_{100} = \omega_1 + 99\delta\omega$, $\omega_{99} = \omega_1 + 98\delta\omega$, and so on. Therefore, $\omega_{100} + \omega_1 = \omega_{99} + \omega_2 = \cdots = \omega_{51} + \omega_{50} = 2\omega_1 + 99\delta\omega$, then

$$\cos\left(\frac{(\omega_{100}+\omega_1)t}{2}\right) = \cos\left(\frac{(\omega_{99}+\omega_2)t}{2}\right) = \cdots = \cos\left(\frac{(\omega_{51}+\omega_1)t}{2}\right) = \cos\left(\frac{(2\omega_1+99\delta\omega)t}{2}\right), \quad (18)$$

so, we can factorize the cosine of the frequencies' sum in equation (17) as follows:

$$y_T = 2A \cos\left(\frac{(2\omega_1+99\delta\omega)t}{2}\right)\left\{\cos\left(\frac{(\omega_{100}-\omega_1)t}{2}\right) + \cos\left(\frac{(\omega_{99}-\omega_2)t}{2}\right) + \cdots + \cos\left(\frac{(\omega_{51}-\omega_{50})t}{2}\right)\right\}, \quad (19)$$

on the other hand, by Equation (14), we have $\omega_{100} - \omega_1 = 99\delta\omega, \omega_{99} - \omega_2 = 97\delta\omega, \cdots, \omega_{51} - \omega_{50} = \delta\omega$. Then, the Equation (19) can be written as:

$$y_T = 2A \cos\left(\frac{(2\omega_1+99\delta\omega)t}{2}\right)\left\{\cos\left(\frac{99\delta\omega t}{2}\right) + \cos\left(\frac{97\delta\omega t}{2}\right) + \cdots + \cos\left(\frac{\delta\omega t}{2}\right)\right\}. \quad (20)$$

We can still carry out a further step and group together by pairs the cosine functions that are inside the braces of Equation (20). By applying Equation (16), this will produce twenty five factors with the common factor $\cos(25\delta\omega t)$ which can be factorized. You can make additional steps by grouping by pairs the rest of the cosine functions, and you can follow this procedure until you arrive to a short expression. We left this as an exercise to the interested reader. To show how the procedure works in full details, in the next subsection we carry out all the steps in the case where we superpose ten waves.

3.1.1. Example 1

In this sub-subsection we are going to study the case of the superposition of ten waves. In this case the arithmetic sequence is ω_1, $\omega_2 = \omega_1 + \delta\omega$, $\omega_3 = \omega_1 + 2\delta\omega$, $\omega_4 = \omega_1 + 3\delta\omega, \omega_5 = \omega_1 + 4\delta\omega, \omega_6 = \omega_1 + 5\delta\omega, \omega_7 = \omega_1 + 6\delta\omega, \omega_8 = \omega_1 + 7\delta\omega, \omega_9 = \omega_1 + 8\delta\omega, \omega_{10} = \omega_1 + 9\delta\omega$. The total superposition will be, where we have grouped together the cosines functions by pairs in a convenient way:

$$y_T(t) = A\left\{ \left[\cos(\omega_{10} t) + \cos(\omega_1 t)\right] + \left[\cos(\omega_9 t) + \cos(\omega_2 t)\right] + \left[\cos(\omega_8 t) + \cos(\omega_3 t)\right] + \left[\cos(\omega_7 t) + \cos(\omega_4 t)\right] + \left[\cos(\omega_6 t) + \cos(\omega_5 t)\right]\right\}. \quad (21)$$

Therefore the total sum, as $\omega_{10} + \omega_1 = \omega_9 + \omega_2 = \omega_8 + \omega_3 = \omega_7 + \omega_4 = \omega_6 + \omega_5 = 2\omega_1 + 9\delta\omega$ and $\omega_{10} - \omega_1 = 9\delta\omega, \omega_9 - \omega_2 = 7\delta\omega, \omega_8 - \omega_3 = 5\delta\omega, \omega_7 - \omega_4 = 3\delta\omega, \omega_6 - \omega_5 = \delta\omega$, is:

$$y_T(t) = 2A \cos\left[\frac{(2\omega_1+9\delta\omega)t}{2}\right]\left\{\cos\left[\frac{9\delta\omega t}{2}\right] + \cos\left[\frac{7\delta\omega t}{2}\right] + \cos\left[\frac{5\delta\omega t}{2}\right] + \cos\left[\frac{3\delta\omega t}{2}\right] + \cos\left[\frac{\delta\omega t}{2}\right]\right\}. \quad (22)$$

The next step is to group together by pairs the cosine functions that are inside the braces in Equation (22) and to use the trigonometric identity given by Equation (16) as follows:

$$\left\{\left[\cos\left(\frac{9\delta\omega t}{2}\right) + \cos\left(\frac{\delta\omega t}{2}\right)\right] + \left[\cos\left(\frac{7\delta\omega t}{2}\right) + \cos\left(\frac{3\delta\omega t}{2}\right)\right] + \cos\left(\frac{5\delta\omega t}{2}\right)\right\} =$$

$$\left\{2\cos\left(\frac{5\delta\omega t}{2}\right)\cos\left(\frac{4\delta\omega t}{2}\right)+\right.$$
$$\left.2\cos\left(\frac{5\delta\omega t}{2}\right)\cos\left(\frac{2\delta\omega t}{2}\right)+\cos\left(\frac{5\delta\omega t}{2}\right)\right\}=$$
$$\cos\left(\frac{5\delta\omega t}{2}\right)\left\{2\left[\cos\left(2\delta\omega t\right)+\cos\left(\delta\omega t\right)\right]+1\right\}=$$
$$\cos\left(\frac{5\delta\omega t}{2}\right)\left\{4\cos\left(\frac{3\delta\omega t}{2}\right)\cos\left(\frac{\delta\omega t}{2}\right)+1\right\}, \quad (23)$$

substituting Equation (23) into Equation (22) we arrive to:

$$y_T(t)=2A\cos\left[\frac{(2\omega_1+9\delta\omega)t}{2}\right]\cos\left(\frac{5\delta\omega t}{2}\right)$$
$$\times\left\{4\cos\left(\frac{3\delta\omega t}{2}\right)\cos\left(\frac{\delta\omega t}{2}\right)+1\right\}. \quad (24)$$

Equation (24) has, like Equation (5), a cosine term with an argument that oscillates at the mean frequency $\overline{\omega}=(\omega_{10}+\omega_1)/2=(2\omega_1+9\delta\omega)/2$. ***The plot of Equation (24) when*** $\delta\omega\ll\omega_1$ ***is the same as the plot of Equation (5) - with*** $N=10$***, under similar condition***. In this case, that is when Equation (7) is fulfilled, the function seem effectively to oscillate at the mean frequency $\overline{\omega}=(2\omega_1+9\delta\omega)/2$. However, ***when we plot Equation (24) together with a wave that oscillates at the mean frequency*** $\bar{\omega}$ ***we find that there is a phase delay each time that there is a destructive interference***. We show this in Figure 1, where we plot Equation (24) and also we have plotted the cosine function that oscillates at the mean frequency, i. e. $\cos\left(\overline{\omega}t\right)$. The plot clearly shows that there is a phase delay between the superposition of waves given by Equation (24) and the $\cos\left(\overline{\omega}t\right)$ function. To the best of our knowledge, this phase delay was not previously highlighted. Also, *it is worth to mention that the same phase delay is observed if we add two, ten or more waves*. We will give a probable explanation of this physical phenomenon in the last part of this sub-subsection. Previously, we are going to examine Equation (24) in a different setting.

On the other hand, we can explore the effects produced by different conditions on the superposition of waves. In particular, we can focuses on the condition given by:

$$\delta\omega\gg\omega_1. \quad (25)$$

As $\omega_N+\omega_1=2\omega_1+(N-1)\delta\omega$ and $\omega_N-\omega_1=(N-1)\delta\omega$, then condition given by Equation (25) implies:

$$\omega_N+\omega_1\approx\omega_N-\omega_1. \quad (26)$$

In Figure 2 we plot Equation (24) when condition given by Equation (25) is fulfilled. The plot shows an interesting fact: the sum of N waves with different frequencies which fulfills the condition given by equation (25) and starts out with the same phase serves to produce a wave with narrow peak amplitude. That is to say, to experimentally produce a

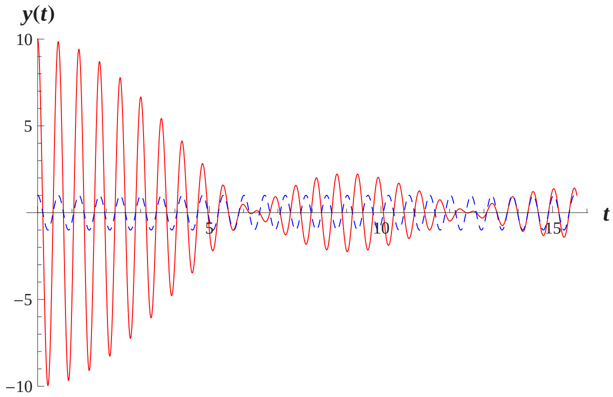

Figure 1 Plot of equation (24) for $\delta\omega\ll\omega_1$. We have set $\omega_1=10$, A=1, and $\delta\omega=\omega_1\times10^{-2}$. The dotted line is the plot of the function $\cos(\bar{\omega}t)$.

wave with narrow width you can sum N waves with difference frequencies. This can be used, for example, in Optics. In fact, this effect is used to produce very short-duration laser pulses, this technique is know as mode-locking [27].

On the other hand, in this case (i.e. when $\delta\omega\gg\omega_1$) it is clear that the wave sum does not oscillates at the mean frequency $\overline{\omega}$, this is show in Figure 2 where we plot also the function $\cos(2\bar{\omega})$. It is easy to see that the wave sum oscillates at a frequency similar to $2\bar{\omega}=\omega_{10}-\omega_1$; however, there are deviations from $\cos(2\bar{\omega})$ which strongly suggest that this plot shows a frequency or phase modulation. This could also explain the phase delay shown in Figure 1 which then could be produced by a phase modulation phenomenon.

To state it clearly: *together with an amplitude modulation, in the superposition of waves with similar frequencies, it is probable that there is also a phase modulation. Which in the case of Figure 1* ***it is expressed by a phase delay each time there is a destructive interference***.

To show that this is effectively the case, let us recall that a phase modulated signal $s_m(t)$ is represented by the following equation, see problem 6.30 in Crawford's book [1]:

$$s_{pm}(t)=A_{pm}\cos\left[\omega_c t+a_m\sin\left(\omega_m t\right)\right]. \quad (27)$$

Therefore, if there exist a phase modulation then there has to be a way to express Equation (1) in the form that Equation (27) has (here we restrict ourselves to the case of summing only two waves). That is to say, we need to be able to write Equation (1) in the following two ways:

$$y(t)=\ 2A\cos\left(\frac{(\omega_2+\omega_1)t}{2}\right)\cos\left(\frac{(\omega_2-\omega_1)t}{2}\right)$$
$$=\ A_{pm}\cos\left[\omega_c t+a_m\sin\left(\omega_m t\right)\right]. \quad (28)$$

Fortunately, this has been done long time ago. At least since the time where John William Strutt, 3rd Baron Rayleigh

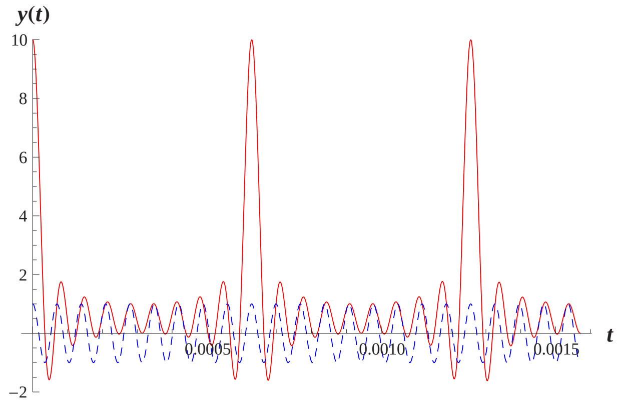

Figure 2 Plot of equation (24) for $\delta\omega \gg \omega_1$. We have set $\omega_1 = 10$, $A = 1$, and $\delta\omega = \omega_1 \times 10^3$. The dotted line is the plot of the function $\cos(2\bar{\omega}t)$..

(a British scientist) lived. At the end of the 19 century Strutt wrote a book titled *The Theory of Sound* [28], in this book (see page 23) he wrote the sum of two waves similar to the last right part of Equation (28). The equation that he wrote is:

$$u = r\cos\left(2\pi m t - \theta(t)\right), \tag{29}$$

where

$$r^2 = a^2 + a'^2 + 2aa'\cos\left(2\pi(m-n)t + \epsilon' - \epsilon\right), \tag{30}$$

and

$$\tan\theta(t) = \frac{a\sin\epsilon + a'\sin\left(2\pi(m-n)t + \epsilon'\right)}{a\cos\epsilon + a'\cos\left(2\pi(m-n)t + \epsilon\right)}, \tag{31}$$

where a and a' are the amplitude of the two waves, ϵ and ϵ' are the initial phase; m and n are the frequencies. This equation can be calculated using the phasor method. So choosing $a = a'$ and $\epsilon = \epsilon' = 0$ we fulfill the initial condition of our problem.

Additionally, we have found a book that has an equation similar to Equation (29) [29]. The authors of this book, Rossing and Fletcher [29], write the superposition of two waves as:

$$\widetilde{x} = A(t)e^{j(\omega_1 t + \phi(t))}, \tag{32}$$

where j is the imaginary number, and $A(t)$ and $\phi(t)$ are similar to Equations (30) and (31) respectively, see page 9 in reference [29]. Notably, these authors, clearly state that the resulting vibration has "both amplitude and phase varying slowly".

Therefore, in conclusion, the sum of two waves can be writing in the usual form that a phase modulation equation has. That is to say, the far right of Equation (28) is true. Hence, Equation (29) (and (32)) allow us to conclude that effectively **there is a phase modulation** - at the same time that *there is an amplitude modulation* - when we sum two (or N) waves of nearly the same frequency. **This phase modulation results in a phase delay by π in the resultant wave each time there is a destructive interference.** It is important to clarify that when we talk a bout a phase modulation we are stating that there is a phase delay each time that there is a destructive interference, that is to say the far right of Eq. (28) , i. e. $A_{pm}\cos\left[\omega_c t + a_m \sin\left(\omega_m t\right)\right]$, implies that when the frequencies are almost equal there is a phase delay each time that there is an interference.

In conclusion, because when Eq. (3) is not fulfilled then there is not a phase delay mainly because there is not such thing as a complete destructive interference, therefore Eq. (28) not always implies the effect that we are calling *phase modulation*. That is to say, although the concept of phase differences in the superposition of waves is inherent in explaining any interference phenomena here we are talking about a different phenomenon, i.e. the phase delay by π each time there is a complete destructive interference, which we call ***phase modulation***.

Now, we are going to argue how this phase modulation is produced and what it actually means. Figure (3) shows a plot of Equation (31) whit $2\pi m = 11$ and $2\pi n = 10$. The dotted line represents the superposition of the two waves. The plot clearly show that there is a slow variation on the phase angle, however, there is also a sudden change in the phase each time the destructive interference happens. Mathematically this sudden changes comes from the change in the sign of the ArcTan function. Physically, this sudden change happens because the two waves are completely out of phase at this time, and after this time the two waves star to develop an increasing phase delay between them, to reach the time (2π times latter) where they are completely out of phase again. In fact, the waves star in phase at $t = 0$, then after a variation of π radiants the waves are out of phase, then after a variation of π radiants the waves are in phase again, this entering and exit of phase and out of phase produces the phase modulation. This in turn produces the destructive and constructive interferences. To state it clearly, *it is the phase modulation which produces the destructive interference*. Probably, it could also be said that it is the phase modulation the responsible for the amplitude modulation.

Also, we can infer from Equations (29) and (30) that there exits a correlation between the modulated amplitude and the phase modulation: in this case both oscillates at the same frequency, i. e. at frequency $m - n$.

The meaning of the phase modulation is that the phase of the resultant wave changes in time with a modulation given by $\theta(t)$, as can be inferred from Equation (29), in this particular case at two rates, one is a slower change and the other is the sudden change each time the destructive interference happens.

Now, to finish this subsection, we will draw a historical issue about these points. It is interesting to notice that Equation (29) was not noticed before by many textbook's authors, including famous books like that of reference [8]. On the other hand, it seems that people working in the field of Sound uses this equation as a teaching tool. Then, the

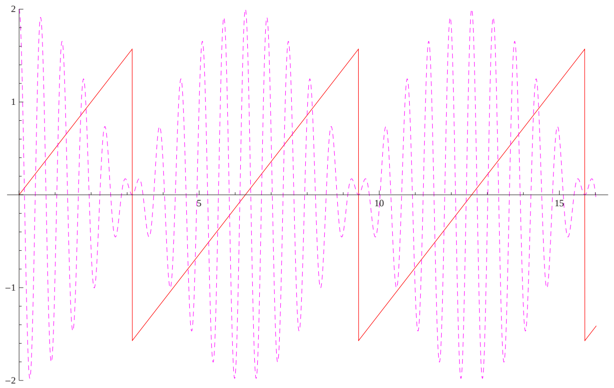

Figure 3 Plot of $\theta(t)$ given by Equation (31) for $a = a'$. We have set $\epsilon = \epsilon' = 0$, $2\pi m = 11$, $2\pi n = 10$. The dotted line is the resulting wave.

historical problem here is: Which book and author first published Equation (29)?, Why scientist working in the subject matters of Waves and Optics have not noticed this equation before?

4. The difference between frequencies as a constant

Instead of grouping the arithmetic sequence as we did in the the previous section, we can group it by pairs as $100 - 50 = 99 - 49 = 98 - 48 = \cdots 52 - 2 = 51 - 1 = 50$. Therefore, **the difference between numbers is a constant**. In this case, their sum descent by two in the interval between $[150, 52]$ as follows: $100 + 50 = 150, 99 + 49 = 148, 98 + 48 = 146, \cdots 52 + 2 = 54, 51 + 1 = 52$.

Then, instead of Equation (15) we can group together by pairs the wave's sum as follows:

$$y_T(t) = \Big\{ \Big[\cos(\omega_{100}t) + \cos(\omega_{50}t)\Big] + \Big[\cos(\omega_{99}t) + \cos(\omega_{49}t)\Big] + \cdots + \Big[\cos(\omega_{51}t) + \cos(\omega_1 t)\Big]\Big\}. \quad (33)$$

Now, using the trigonometric identity given by equation (16) we have:

$$y_T(t) = \Big\{ \Big[\cos\left(\frac{(\omega_{100}+\omega_{50})t}{2}\right)\cos\left(\frac{(\omega_{100}-\omega_{50})t}{2}\right)\Big] + \Big[\cos\left(\frac{(\omega_{99}+\omega_{49})t}{2}\right)\cos\left(\frac{(\omega_{99}-\omega_{49})t}{2}\right)\Big] + \cdots + \Big[\cos\left(\frac{(\omega_{51}+\omega_{1})t}{2}\right)\cos\left(\frac{(\omega_{51}-\omega_{1})t}{2}\right)\Big]\Big\} \quad (34)$$

Now, as $\omega_N = \omega_1 + (N-1)\delta\omega$ then $\omega_{100} - \omega_{50} = \omega_{99} - \omega_{49} = \cdots = \omega_{51} - \omega_1 = 50\delta\omega$; also we have that $\omega_{100} + \omega_{50} = 2\omega_1 + 148\delta\omega$, $\omega_{99} + \omega_{49} = 2\omega_1 + 146\delta\omega$, $\cdots$, $\omega_{51} + \omega_1 = 2\omega_1 + 50\delta\omega$. Therefore,

$$\cos\left(\frac{(\omega_{100}-\omega_{50})t}{2}\right) = \cos\left(\frac{(\omega_{99}-\omega_{49})t}{2}\right) = \cdots = \cos\left(\frac{(\omega_{51}-\omega_{1})t}{2}\right) = \cos\left(\frac{50\delta\omega t}{2}\right), \quad (35)$$

so, in Equation (34) we can factorize the cosines that have the same argument $(50\delta\omega t/2)$:

$$y_T(t) = \cos(25\delta\omega t)\left\{\cos\left(\frac{(2\omega_1+148\delta\omega)t}{2}\right) + \cos\left(\frac{(2\omega_1+146\delta\omega)t}{2}\right) + \cdots + \cos\left(\frac{(2\omega_1+50\delta\omega)t}{2}\right)\right\} \quad (36)$$

In a further step, we can group by pairs cosines functions that are inside the braces in Equation (36) in such a way that the difference between the arguments of the cosine function are equal. This will provide twenty five cosine functions, and the procedure can be followed to the step where you find the shorter expression. However, we will not carried ou this procedure, instead in the next subsection we focuses in the case where we have only ten waves.

4.1. Example 2

In this subsubsection we are going to study the case of the superposition of ten waves. In this case the arithmetic sequence is $\omega_1, \omega_2 = \omega_1 + \delta\omega, \omega_3 = \omega_1 + 2\delta\omega, \omega_4 = \omega_1 + 3\delta\omega, \omega_5 = \omega_1 + 4\delta\omega, \omega_6 = \omega_1 + 5\delta\omega, \omega_7 = \omega_1 + 6\delta\omega, \omega_8 = \omega_1 + 7\delta\omega, \omega_9 = \omega_1 + 8\delta\omega, \omega_{10} = \omega_1 + 9\delta\omega$. The total superposition will be, where we have grouped by pairs the cosines functions in a convenient way:

$$y_T(t) = A\Big\{\Big[\cos(\omega_{10}t) + \cos(\omega_5 t)\Big] + \Big[\cos(\omega_9 t) + \cos(\omega_4 t)\Big] + \Big[\cos(\omega_8 t) + \cos(\omega_3 t)\Big] + \Big[\cos(\omega_7 t) + \cos(\omega_2 t)\Big] + \Big[\cos(\omega_6 t) + \cos(\omega_1 t)\Big]\Big\}. \quad (37)$$

Following the same steps of the Example 1 in subsection 3.1.1, but this time factorizing the factors which have the difference in frequencies, we arrive to the following final result:

$$y_T(t) = 2A\cos\left(\frac{5\delta\omega}{2}t\right)\left\{4\cos^2(\delta\omega t)\cos\left[(w_1 + 5\delta\omega)t\right] + \cos\left[\frac{2\omega_1 + 5\delta\omega}{2}t\right]\right\} \quad (38)$$

A plot of Eq. (38) reproduces the same plot, when using the same values, as the one giving in Fig. 1.

5. A note

Usually it is stated that the condition for the existence of the amplitude modulation phenomenon is that the frequencies are almost equal, this is the condition given in Equation (3), i. e. for the case of two waves where $\omega_2 \approx \omega_1$. Mathematically, this condition implies

$$\omega_2 - \omega_1 \rightarrow 0. \tag{39}$$

However, you can find that the amplitude modulation phenomenon can be produced inclusive when $\omega_2 - \omega_1 \gg 0$. For example, you can use $\omega_2 = 550,000$ and $\omega_1 = 500,000$ to produce the amplitude modulation phenomenon, in this case the frequency difference is more grater than one (i.e $\omega_2 - \omega_1 \gg 1$), then Equation (39) is not fulfilled.

We don't know if the condition given by Equation (3) has been physically motivated or where it comes from. But it seems that the physical condition to produce the amplitude modulation phenomenon is that the resultant wave oscillates at a highly greater frequency than the phase modulation, i. e. the latter has to oscillate slower, that is:

$$\frac{\omega_2 + \omega_1}{2} \gg \frac{\omega_2 - \omega_1}{2}. \tag{40}$$

this condition means that the phase variation between the waves is slow.

To resume, Equation (40) is motivated from the physical condition that the resultant wave oscillates at a high frequency whereas the modulated amplitude oscillates at much slower frequency. It seems that this is a better physical condition than the condition given by Equation (3) and it serves to explain the beating phenomenon in cases where the condition given by Equation (3) is not fulfilled.

Additionally, it is worth to mention a physical condition encountered in the physics for narrow bands, that is $(w_2 - w_1)/w_1 \ll 1$.

6. Conclusions

In this work we present the well known Gauss method to sum an arithmetic sequence, but now as an useful tool to add N waves. This method is simple, easy to apply and avoids to use complex number. This method can be used to teach this subject matter since the beginning of the syllabus.

Most important, results observed in the plots and the equations shows a phase delay in the superposition of waves not previously highlighted. We have shown that together with an amplitude modulation there must exist a **phase modulation**. We are calling as ***phase modulation*** not just the representation of the superposition of waves by the equation $A_{pm} \cos[\omega_c t + a_m \sin(\omega_m t)]$, *but the phenomenon of phase delay each time that there is a complete destructive interference in the superposition of waves of nearly equal frequencies*.

The amplitude modulation produces the phenomenon of beating. We think that the **phase modulation** produces a phenomenon that can not be noticed by human detectors (the ears, which in fact registers just the intensity), but probably this phenomenon could be recorded by other detector system and probably could have new applications.

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L. M. Arévalo Aguilar thanks the support of Consejo Nacional de Ciencia y Técnologia (CONACYT), under grant No. 117319, for a "Retención" program at BUAP. C. Robledo-Sánchez, M. L. Arroyo Carrasco and M. M. Mendez Otero thanks the support from CONACYT. All of us thanks Vicerrectoría de Investigación y Estudios de Postgrado (VIEP-BUAP) for support. Finally, the authors are grateful to the anonymous referee for helpful comments that improved this paper.

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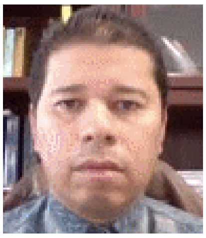

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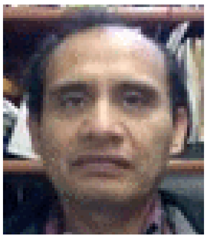

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