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Conservation laws of the Bretherton Equation

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Abstract: This paper obtains the conservation laws of the Bretherton equation that is considered with dual-power law nonlinearity. The multiplier approach is used to extract several conserved densities of this equation. Finally, the conserved quantities are computed by using the 1-soliton solution that has been obtained earlier.

Keywords: Conservation laws, Bretherton equation.

1 Introduction

The study of nonlinear evolution equations (NLEEs) is vital in the areas of applied mathematics and theoretical physics [1-11]. Particularly, in theoretical physics NLEEs serve as a backbone to further research enhancement. The main focus of NLEE appears in nonlinear optics, fluid dynamics, nuclear physics, plasma physics, mathematical biosciences, mathematical chemistry and several other areas [2,11]. The research results, in such applied areas, resonate the scientific development in the remaining branches of science and technology.

One of the primary focus of the NLEEs in nonlinear dynamics is its conservation laws. These conservation laws determine physical quantities that stay invariant with the wave dynamics. These conserved quantities also describe the dynamics of the waves and helps to comprehend it better. Some of these conserved quantities are typically the mass, linear and/or angular momentum, energy, Hamiltonian and many other physical features. This paper will approach the study of the conservation laws of the Bretherton equation (BE) that appears in the context of resonant interaction of water waves.

2 Governing equation and soliton solution

The dimensionless form of the BE is given by

$$q_{tt} - k^2 q_{xx} + a q_{xxxx} + b q^m + c q^n = 0, \tag{1}$$

where k, a, b and c are constants and the exponents m, n represent the power law nonlinearities. Here $q(x,t)$ represents the wave profile. The 1-soliton solution to (1) is given by [10]

$$\begin{aligned} q(x,t) &= A\,\text{sech}^{\frac{2}{m-1}}[B(x-vt)] \\ &= A\,\text{sech}^{\frac{4}{n-1}}[B(x-vt)] \end{aligned} \tag{2}$$

where the amplitude (A) of the solitary wave is given by

$$A = \left\{-\frac{b\,(3m-1)}{2mc}\right\}^{\frac{1}{n-m}} \tag{3}$$

and the inverse width (B) is given by

$$B = \frac{m-1}{2}\left[-\frac{2b}{a\,(m+1)\,m^2}\left\{-\frac{b\,(3m-1)}{2mc}\right\}^{\frac{m-1}{n-m}}\right]^{\frac{1}{4}} \tag{4}$$

while the velocity is

$$v = \left[k^2 - a\sqrt{-\frac{2b}{a\,(m+1)\,m^2}\left\{-\frac{b\,(3m-1)}{2mc}\right\}^{\frac{m-1}{(n-m)}}}\right]^{\frac{1}{2}} \tag{5}$$

These lead to the constraint conditions

$$ab < 0 \tag{6}$$

$$bc < 0 \tag{7}$$

$$k > \sqrt{a} \tag{8}$$

$$2m = n+1 \tag{9}$$

$$m > 1, \quad n > 1 \tag{10}$$

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3 Conservation laws

In order to determine conserved densities and fluxes, we resort to the invariance and multiplier approach based on the well known result that the Euler-Lagrange operator annihilates a total divergence (see [6]). Firstly, if (T^t, T^x) is a conserved vector corresponding to a conservation law, then

$$D_t T^t + D_x T^x = 0 \tag{11}$$

along the solutions of the differential equation ($de = 0$).

Moreover, if there exists a nontrivial differential function Q, called a 'multiplier', such that

$$E_q[Q(q_{tt} - k^2 q_{xx} + a q_{xxxx} + b q^m + c q^n)] = 0, \tag{12}$$

then $Q(q_{tt} - k^2 q_{xx} + a q_{xxxx} + b q^m + c q^n)$ is a total divergence, i.e.,

$$\begin{aligned} &Q(q_{tt} - k^2 q_{xx} + a q_{xxxx} + b q^m + c q^n) \\ &= D_t T^t + D_x T^x, \end{aligned} \tag{13}$$

for some (conserved) vector (T^t, T^x) and E_q is the respective Euler-Lagrange operator. Thus, a knowledge of each multiplier Q leads to a conserved vector determined by, inter alia, a Homotopy operator. See details and references in [6, 7]. If t is the time variable, T^t is the *conserved density* from which the conserved quantity is determined. When Q is chosen to be up to second order in derivatives, i.e., $Q = Q(t, x, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})$, we obtain two solutions leading to two nontrivial conserved vectors given below.

(i) $Q_1 = q_x$:

$$\begin{aligned} T_1^x &= \tfrac{bq^{1+m}}{1+m} + \tfrac{cq^{1+n}}{1+n} + \tfrac{1}{2} q q_{tt} - \tfrac{1}{2} k^2 {q_x}^2 - \tfrac{{q_{xx}}^2}{2} + q_x q_{xxx}, \\ T_1^t &= \tfrac{1}{2}\left(q_t q_x - q q_{xt}\right). \end{aligned}$$

and
(ii) $Q_2 = q_t$:

$$\begin{aligned} T_2^x &= \tfrac{1}{2}(-q_{xt} q_{xx} + q_x q_{xxt} + q_t\left(-k^2 q_x + q_{xxx}\right) \\ &\quad + q\left(k^2 q_{xt} - q_{xxxt}\right)), \\ T_2^t &= \tfrac{1}{2}\left({q_t}^2 + q\left(\tfrac{2bq^m}{1+m} + \tfrac{2cq^n}{1+n} - k^2 q_{xx} + q_{xxxx}\right)\right). \end{aligned}$$

Therefore the conserved quantities are given by

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} T_1^t dx = \frac{1}{2}\int_{-\infty}^{\infty} \left(q_t q_x - q q_{xt}\right) dx \\ &= \int_{-\infty}^{\infty} q_t q_x dx \\ &= \frac{16vA^2B}{(n-1)(n+7)} \frac{\Gamma\left(\frac{4}{n-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{4}{n-1}+\frac{1}{2}\right)} \end{aligned} \tag{14}$$

and

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} T_2^t dx = \frac{1}{2}\int_{-\infty}^{\infty} \\ &\left\{q_t^2 + \frac{2bq^{m+1}}{m+1} + \frac{2cq^{n+1}}{n+1} - k^2 q q_{xx} + q q_{xxxx}\right\} dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\left\{\left(v^2+k^2\right) q_x^2 + \frac{2bq^{m+1}}{m+1}\right. \\ &\quad \left. + \frac{2cq^{n+1}}{n+1} + c q_{xx}^2\right\} dx \\ &= \frac{8A^2B}{(n-1)^2(n+7)(3n+5)}\{(n-1)(3n+5)\left(v^2+k^2\right) \\ &\quad + 16(n+2)cB^2\} \frac{\Gamma\left(\frac{4}{n-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{4}{n-1}+\frac{1}{2}\right)} \\ &\quad + \frac{2bA^{\frac{n+3}{2}}}{(n+3)B} \frac{\Gamma\left(\frac{n+3}{n-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+3}{n-1}+\frac{1}{2}\right)} \\ &\quad + \frac{cA^{n+1}}{(n+1)B} \frac{\Gamma\left(\frac{2n+2}{n-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2n+2}{n-1}+\frac{1}{2}\right)} \end{aligned} \tag{15}$$

These conserved quantities are computed using the 1-soliton solution that is given by (2) and additionally, the relation (9) is used to write these conserved quantities in terms of the single power law parameter n.

4 Conclusion

This paper obtains the conservation laws of the Bretherton equation with dual power law nonlinearity. The multiplier method using the Lie symmetry approach is utilized to compute these conserved densities and finally the 1-soliton solution, obtained earlier, leads to the values of the conserved quantities. These conserved quantities are very precious results that will, in future, lead to further research. The modified conserved quantities will be computed and subsequently the soliton perturbation theory will be adopted to obtain the adiabatic parameter dynamics of the soliton parameters [2]. The stochastic perturbation terms will also be taken into account and the mean free value of the soliton parameter will be determined. These results will be reported in future publications.

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