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Some Properties of Mobius Function Graph $M^{(-1)}$ in Graphs

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Abstract: In this work, a new graph is called the Mobius Function Graph is introduced. Three ways of determining the prime-counting function by using this graph are presented. Also, some properties of this function are proved. Moreover, the domination, independence, chromatic, and clique number of this graph are determined. Finally, the relationship between the domination number and the independence number is discussed.

Keywords: Mobius Function Graph $M^{(-1)}$, domination number, independence number, and chromatic number.

1 Introduction

Graph theory G is a modern language to deal with most of the previous sciences as chemistry, engineering, medicine, physics, and others,by finding alternative or new solutions to the problems of the previous sciences[1, 2,3], after transforming the main elements of the problem into double sets, The first is called the vertex set and denoted by $V(G)$ and the second is edge set and denoted by $E(G)$. Also, graph theory deals with multiple subjects in mathematics as as in general graph [4,5,6,7,8], topological graph[9,10,11], fuzzy graph[12,13,14], labeled graph[15,16],and topological indices [17,18,19, 20,21,22],and others. The important concepts that share to solve some problems in mathematics are the domination, chromatic, clique, and independence number. A set $D \subseteq V$ is called dominating if $N[D] = V$. The minimum cardinality of all dominating sets is called the and is domination number and denoted by $\gamma(G)$[25].A set $I \subseteq V$ of order m is called independent if $(I) \equiv N_m$, where N_m is a null graph of order m. The maximum independent set is called the independence number and denoted by $\beta(G)$ [25]. A vertex-coloring of G is giving any two adjacent vertices different colors. The smallest number of colors can be colored all vertices is called the chromatic number and denoted by $X(G)$[25]. The order of largest complete .each vertex in it is adjacent to all other vertices in it. subgraph of a graph G is called the clique number. In this paper, the relation between the number theory [26, 28] and graph theory is discussed, especially with Mobius function (M) which is defined as follows, If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$
$M(n)=\begin{cases} (-1)^r, \text{ if } \alpha_1 = \alpha_2 = \dots \alpha_r; \\ 0, \ otherwise. \end{cases}$ The graph that deal within this work is $M^{((-1))}$ and construct by this way, the vertex set is $v_1, v_2, \dots, v_s$ and the labeled these vertices by the function $f(v_i) = i$. The vertex v_i is adjacent if v_j if $M(f(v_i)f(v_j)) = -1$. Al-Asadi, Omran, and Al-Maamori, study the graph $M^{((1))}$ where the condition of adjacency is $M(f(v_i)f(v_j)) = 1$,and get important results. function. In this paper, a new method to calculate the prime-counting function (which is counting the number of prime numbers less than or equal to the order of a graph and denoted by $\pi(n)$) is introduced. Additionally, some properties of this graph are discussed. Also, the domination, chromatic, and independence number are determined. Finally, the relationship between the independence with the domination number is obtained. The reader can be founded all concepts not founded here in [29,30] and[**?**].

Theorem 1.. *If G is a Mobius function graph $M^{((-1))}$, then $\pi(n) = deg(v_1) - u : u = \prod_{(i=1)}^{k} P_j, k$ is odd; $k > 1$ where P_j are distinct primes numbers and labeled of $f(v_i) = i$*

If the vertices of a graph G take labeled $f(v_i) = i$, then the vertex v_1 is adjacent to all vertices that have labeled primes number, since $f(v_1.v_i) = f(v_i) = i = P_j$, and P_j is prime number $\forall j$, so $M(P_j) = -1$,. Moreover, $M(\prod_{(j=1)}^{k} P_j) =$

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-1, where k is odd , so the vertex v_1 is adjacent to all vertices that are labeled with the form $\prod_{(j=1)}^{k} P_j$; k is odd. Thus, $\prod(n) = deg(v_1) - u : u = \prod_{(j=1)}^{k} P_j, k \text{ is odd } ; k > 1$ (for example, see Figure 1.1).

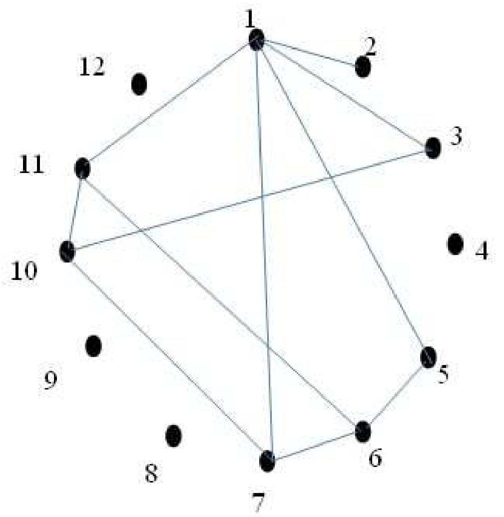

Fig. 1: The Mobius function graph $M^{((-1)}$) of order 12. .

Theorem 2. *The Mobius function graph $M^{((-1))}$ is a free triangle*

Proof. Let v_1, v_2,and v_3 any three vertices, then two cases depend on the greater common divisor are discussed as follows. **Case 1**. If $g.c.d(f(v_i), f(v_j)) \neq 1$, then there is a prime factor common with labeled of different two vertices v_i and p_i, so p_i^2 is divided $f(v_i)f(v_j)$. Thus, the vertex v_i is not adjacent to the vertex v_j, so there is no triangle between the three vertices mentioned above.
Case 2. If $g.c.d(v_i, v_j) = 1, \forall\ i = j$, so $f(v_1) = \prod_{(i=1)}^{k} p_i, f(v_2) = \prod_{(j=1)}^{r} p_j$,and $f(v_3) = \prod_{(w=1)}^{s} p_w$. Thus, each one of k, s or w even, so there are at least two of them that are odd or even. In each case, the product of these numbers is even length and the Mobius of this number is 1. Therefore, these vertices are not adjacent and then there is no triangle. From the two cases above, the required is obtained.

Corollary 3 *The clique number of the Mobius function graph $M^{((-1))}$ of order $n; n \geq 2$ is 2.*

Proof. It is obvious that the Mobius function graph $M^{((-1))}$ is not isomorphic to the null graph, so the clique number is greater than or equal to two. According to Theorem 2.2, this graph is a free triangle. Thus, the clique number is equal to two.

Proposition 4 *Assume that a nontrivial G is a Mobius function graph $M^{((-1))}$, then $X(G) = 2$.*

Proof. According to Corollary 2.3, the clique number is equal to 2, so every two adjacent vertices can be two different colors.

Proposition 5 *Consider G is a Mobius function graph $M^{((-1))}$, then*

1. *All vertices have labeled not free square prime are isolated vertices.*
2. *If G be a Mobius function graph $M^{((-1))}$ of order $n \geq 4$, then graph G is disconnected, otherwise, the graph is connected.*

Proof. Let $f(v_i) = i$, so

1. One can be concluded that each vertex that is labeled as a square prime is not adjacent to all other vertices by definition of Mobius function. Thus, the result is getting.
2. If $n = 4$, then $f(v_4) = 4 = 2^2$, thus the vertex v_4 is not adjacent to all other vertices in the Mobius function graph $M^{((-1))}$. Thus, the vertex v_4 is isolated and so the Mobius function graph $M^{((-1))}$ is disconnected. Now, if $n \leq 3$, it is obvious that the Mobius function graph $M^{((-1))}$ is connected. Therefore, the result is obtained.

Corollary 6 *The Mobius function graph $M^{((-1))}$ of order n is a tree if and only if $n \leq 3$.*

Proposition 7 *Consider G is a Mobius function graph $M^{((-1))}$ of order n, then* $\beta = \begin{cases} 1, \text{ if } n = 2 \\ \lfloor \frac{n}{2} \rfloor + 1, n = 1,3,4 \\ \lceil \frac{n}{2} \rceil + 1 \text{ if } n\text{=5,6} \\ |S \cup S_1|, \forall n \geq 7. \end{cases}$. *where S is the isolated vertices set in G and $S_1 = \prod_{(i=1)}^{k} p_i$,k is odd.*

Proof. Two cases depend on the order n are classification as follows. **Case 1.** One can conclude easily the result when $n = 1, 2, \ldots, 6$.
Case 2. If $n \geq 7$, then let v_r is an isolated vertex, so this vertex belongs to all independent set (I) of G, thus $S \subseteq I$so, $\beta(G) \geq |S|$.
Now, if a vertex v_r is not an isolated vertex, so this vertex is free square prime. Let $S_1 = \prod_{(i=1)}^{k} p_i$,$kisodd$,
let u and v be any two vertices of the set S_1, so two subcases are classification as follows.
Subcase 1. If $g.c.d(f(v), f(u)) \neq 1$, $M(f(u)f(v)) = p_r^2 \prod_{(i=1)}^{k} p_i$, Therefore, u and v are not adjacent.
Subcase 1. If $g.c.d(f(v), f(u)) = 1$, $M(f(u)f(v)) = \prod_{(i=1)}^{k} p_i$;$k$ is even, so $M(f(u)f(v)) = 1$. Therefore, u and v are not adjacent. Thus, the set S_1 is an independent set depending on the above two subcases. Note that $S \cap S_1 = \Phi$, then $\beta(G) \geq S_1 \cup S$.
Now, if there is a vertex $w; w \notin S_1 \cap S$, so

$M(f(w)) = \prod_{(i=1)}^{k} p_i$;$k$ is even then $\forall z \in S_1 \cup S$;$g.c.d(f(w), f(z)) = 1$, $M(f(u)f(v)) = -1$, so the vertex w is adjacent to the vertex z. Thus, the set $S_1 \cup S$ is the maximum independent and the result is obtained.

Theorem 8. *let G be a Mobius function graph $M^{((-1))}$ of order n, then*

$$\gamma(M^{(}-1))=\begin{cases} 1, \; if\, n = 1,2,3 \\ 2, n = 4,5 \; where\ S\ set\ of\ isolated\ vertices. \\ 2 + |S|, if\ n \geq 6 \end{cases}$$

Proof. Depending on the number of vertices four cases are discussed as bellow.

Case 1. If $n = 1,2,3$ then it is obvious that $\gamma(M^{((-1))}) = 1$. **Case 2.** If $n = 4,5$, then the vertex v_1 dominates all other vertices except the vertex v_4 since this vertex is isolated. Thus, $\gamma(M^{((-1))}) = 2$.

Case 3. If $n \geq 6$, then three subcases depend on the factorization of the number $a \leq n$ as follows.

Subcase 1. If there is $p_i^{\alpha}; \alpha > 1$ as a factor of a, then the vertex of labeled a is isolated. Let S be the set of all isolated vertices. Thus, $\gamma(M^{((-1))}) \geq S$. **Subcase 2.** If there is no $p_i^{\alpha}; \alpha > 1$ as a factor of a, then a can be written as the form $\prod_{(i=1)}^{k} p_i$again there are two subcases as follows.

1. If k is odd, then all vertices of this form are adjacent to the vertex v_1, since $f(v_1)f(v_a)$=$\prod_{(i=1)}^{k} p_i$.
2. If k is even, then let p_s be the largest prime number less than or equal n, so all vertices of this form is adjacent to the vertex of labeled p_s, since $f(v)f(v_a) = p_s \prod_{(i=1)}^{k} p_i$=$\prod_{(i=1)}^{(k+1)} p_i$.
   Thus all vertices in this case are dominated by only two vertices. Therefore, according to case 1 and case 2, $\gamma(M^{((-1))}) = 2 + |s|$ (as an example, see Figure 2.2).

Proposition 9 *Assume that G is a Mobius function graph $M^{((-1))}$ and $u,v \in M^{((-1))}$, $g.c.d(u,v) = 1$, then $M(u.v) = M(u).M(v)$.*

Proof. Let G be a Mobius function graph $M^{((-1))}$ and $u,v \in M^{((-1))}$, $g.c.d(u,v) = 1$, then four cases are appeared as bellow.

Case 1. If $M(u) = M(v) = -1$ then $u = \prod_{(i=1)}^{s} P_i$ and $v = \prod_{(j=1)}^{r} P_i$, where r and s are odd number, so $u.v = \prod_{(i=1)}^{s} P_i \prod_{(j=1)}^{r} P_i = \prod_{(k=1)}^{(s+r)} P_i$ and it is obvious that $s + r$ is even, since $g.c.d(u,v) = 1$. Thus, $M(u.v) = 1 = M(u).M(v)$.

Case 2. If $M(u) = 1$ and $M(v) = -1$, then $u = \prod_{(i=1)}^{s} P_i$; s is even and $v = \prod_{(j=1)}^{r} P_i$; r is odd, then $u.v = \prod_{(i=1)}^{s} P_i \prod_{(j=1)}^{r} P_i = \prod_{(k=1)}^{(s+r)} P_i$ and it is obvious that $s + r$ is odd, since $g.c.d(u,v) = 1$. Thus, $M(u.v) = -1 = M(u).M(v)$.

Case 3. If $M(u) = 1$ and $M(v) = 1$, then $u = \prod_{(i=1)}^{s} P_i$;$s$ is even and $v = \prod_{(j=1)}^{r} P_i$;$r$ is even , then $u.v = \prod_{(i=1)}^{s} P_i \prod_{(j=1)}^{r} P_i = \prod_{(k=1)}^{(s+r)} P_i$ and it is obvious that $s + r$ is even, since $g.c.d(u,v) = 1$. Thus, $M(u.v) = 1 = M(u).M(v)$.

Case 4. If there is factor prime $P_i^{\alpha} i; \alpha i > 1$ is divided the labeled of at least one vertex from two vertices u and v say u, then $M(u) = 0$. Now, the factor prime $P_i^{\alpha} i$ is divided the labeled of the product of two vertices u and v. Thus, $M(u.v) = 0 = M(u).M(v)$. According to cases above, the proof is done

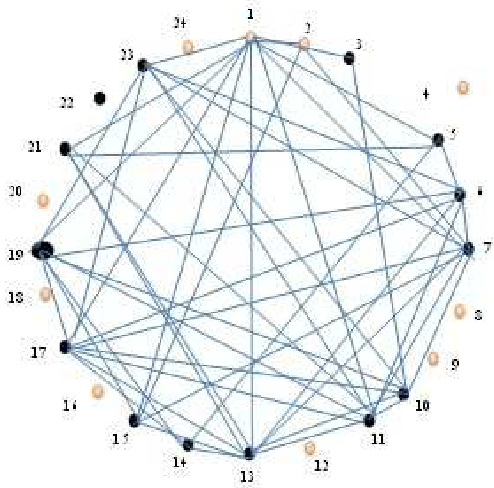

Fig. 2: The domination number of $M^{((-1)}$) of order n

Theorem 10. *Assume that G is a Mobius function graph $M^{((-1))}$ and non-trivial, then*

1. *$\beta(G) + \gamma(G) \geq n$, $\forall n$ except $n = 7$.*
2. *$\gamma(G) \leq \beta(G)$.*

Proof. 1. Two cases depend on n are classifications as follows.
   Case1 If $1 \leq n \leq 6$, then one can easily conclude the result.
   Case 2. If $n \geq 7$, then according to Theorem 2.7, $\gamma(M^{((-1))}) = 2 \cup |S|$ and by Proposition 2.6, $\beta(G) = |S \cup S_{(1)}|$, where S is the isolated vertices set in G and $S_1 = \prod_{(i=1)}^{k} p_i$,k *is odd*. Thus, $\beta(G) + \gamma(G) = 2 + |S| + |S \cup S_{(1)}| = 2 + 2|S| + |S_{(1)}|$, so $\beta(G) + \gamma(G) \geq n$.
2. One can be concluded easily that the inequality is correct when $n = 1,2,\ldots,6$. So, $\forall n \geq 7$, $\gamma(G) = |S| + 2 < |S \cup S_{(1)}|$=$\beta(G)$ Thus, the result is obtained

Proposition 11*let G be a Mobius function graph $M^{((-1)}$) and non-trivial, then The*

1.*The radius, Rad* $(G(\mu_n^{-1})=\begin{cases} 1, \text{ if } n \leq 3 \\ \infty, n > 3, \end{cases}$ *and the Cent* $(G($ $mu_n^{(}(-1))))$ *is isomorphic to* K_1 *graph, if* $n \leq 3$

2.*The diameter,* $Diam(G)=\begin{cases} n-1, \text{ if } n \leq 3 \\ \infty, n \geq 5, \end{cases}$ *and the* $Per(G)$ *is isomorphic to* K_2, N_2 *graph, if* $n = 2,3$ *respictivily*

Proof. 1.If $n \leq 3$, it is obvious that $Rad(G) = 1$ and $Cent(G)$bis isomorphic to K_1.
Now, if $n > 3$, then graph G contains an isolated vertex, so graph G is disconnected. Thus, $Rad(G) = \infty$.
2.Three cases are classification as follows.
Case 1. If $n = 2$, then $G \equiv K_2$ and $Diam(G) = 1$, so $Per(G) \equiv K_2$.
Case 2. If $n = 3$, then $G \equiv P_3$ and $Diam(G) = 2$, so $Per(G) \equiv N_2$.
Case 3. if $n > 3$, then the graph G contains an isolated vertex, so the graph G is disconnected. Thus, $Daim(G) = \infty$.

2 Conclusion

A new graph is constructed in this work, it is called the Mobius function graph $M^{(}(-1)$). According to the obtained result in this work, the domination, independence, and clique number are determined. Also, some properties of this graph are calculated. Moreover, the relation between the independence number and domination number is discussed.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

[1] G. Chartrand and L. Lesniak, Graphs and Digraphs, Chapman and Hall/CRC, (2005).

[2] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, NY, USA, (1998).

[3] T. A. Ibrahim and A. A. Omran, Upper whole domination in a graph, Journal of Discrete Mathematical Sciences and Cryptography, **25(1)**, 73-81,(2022).

[4] M. N. Al-Harere, A. A. Omran, and A. T. Breesam, Captive domination in graphs, Discrete Mathematics, Algorithms and Applications,**12(6)**, (2020).

[5] A. A. Omran, M. N. Al-Harere,and Sahib Sh. Kahat, Equality co-neighborhood domination in graphs, Discrete Mathematics, Algorithms and Applications, 2021.

[6] A. A. Omran and Haneen H. Oda, Hn-Domination in Graphs, Baghdad Science Journal **16(1)**, (2019), 242-247.

[7] A. Alwan, A. A. Omran, Domination Polynomial of the Composition of Complete Graph and Star Graph, J. Phys.: Conf. Ser. 1591 012048,2020.

[8] J.R Banavar, J. Damuth, A. Maritan, A. Rinaldo, Proceedings of the National Academy of Sciences of the United States of America **99**, 10506-10509 (2002).

[9]] K. S. Al'Dzhabri, A. A. Omran, and M. N. Al-Harere, DG-domination topology in Digraph, Journal of Prime Research in Mathematics, 2021, 17(2), pp. 93-100.

[10] A. A. Jabor., A. A. Omran, Topological domination in graph theory, AIP Conference Proceedings 2334, 020010, (2021)

[11] A. A. Jabor and A. A. Omran, Hausdorff Topological of Path in Graph, 2020 IOPConf. Ser.: Mater. Sci. Eng. 928 042008.

[12] A. A. Omran and T. A. Ibrahim, Fuzzy co-even domination of strong fuzzy graphs, Int. J. Nonlinear Anal. Appl. **12(1)**,(2021), 727-734.

[13] S. S. Kahat, A. A. Omran, and M. N. Al-Harere, Fuzzy equality co-neighborhood domination of graphs, Int. J. Nonlinear Anal. Appl. **12(1)**,(2021) 537-545.

[14] H. J. Yousif and A. A. Omran, Closed Fuzzy Dominating Set in Fuzzy Graphs, J.Phys.: Conf. Ser. 1879 (2021) 032022 doi:10.1088/1742-6596/1879/3/032022.

[15] M.N. Al-Harere, A. A. Omran, On binary operation graphs, Boletim da Sociedade Paranaense de Matematica, **13(7)**, 59-67,2020.

[16] M.N. Al-Harere, A. A. Omran, Binary operation graphs, AIP conference proceeding 2086, 2019.

[17] A. Alsinai, A. Saleh, H. Ahmed, L. N. Mishra, and N. D. Soner, On fourth leap Zagreb index of graphs. Discrete Mathematics, Algorithms and Applications. (2022), DOI: 10.1142/S179383092250077X

[18] A. Alsinai, H. Ahmed, A. Alwardi, and N.D.Soner *HDR Degree Bassed Indices and Mhr−Polynomial for the Treatment of COVID−19*, Biointerface Research in Applied Chemistry.,**12(6)**,(2021),7214-7225. DOI: 10.33263/BRIAC126.72147225 .

[19] F. Afzal, A. Alsinai, S. Hussain, D. Afzal, F. Chaudhry, and M. Cancan, On topological aspects of silicate network using M-polynomial, Journal of Discrete Mathematical Sciences and Cryptography, 1-11.(2021). DOI: 10.1080/09720529.2021.1977486

[20] A. Alsinai, Hafiz Mutee ur Rehman, Yasir Manzoor, Murat Cancan, Ziyattin Tas and Moahmmad Reza Farahani ,Sharp upper bounds on forgotten and SK indices of cactusgraph, Journal of Discrete Mathematical Sciences andCryptography.,(2022) DOI:10.1080/09720529.2022.2027605

[21] S. Javaraju, A. Alsinai, H. Ahmed, A. Alwardi, and N.D. Soner, Reciprocal leap indices of some wheel related graphs., Journal of Prime Research in Mathematics, **17(2)**, 101-110, (2021).
[22] A. Hasan, M. H. A. Qasmi, A. Alsinai, M. Alaeiyan, M.R. Farahani, and M. Cancan. Distance and Degree Based Topological Polynomial and Indices of X-Level Wheel Graph. Journal of Prime Research in Mathematics., **17(2)**, 39-50, (2021).
[23] S. Javarajua, H. Ahmed, A. Alsinaia, and N.D. Soner, Domination topological properties of carbidopa-levodopa used for treatment Parkinson's disease by using φ_p-polynomial.**3(9)**,614-624, (2021), 10.22034/ECC.2021.295039.1203.
[24] Haneen H. Aljanaby, and A. A. Omran, Odd Neighbors Domination in Graphs, accept to publish in AIP Conference Proceedings, 2021.
[25] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, NY, USA, 1998.
[26] F. Al-Maamori, T. Hilberdink, An Example In Beurling's Theory of Generalised Primes, Acta Arithmetica,168(4), pp. 383-395, 2015.
[27] F.A. Al-Maamori, Examples of beurling prime systems, Mathematica Slovaca, 67(2), pp. 321-344, 2017.
[28] F. AL-Maamori, S.A. AL-Ameedee, and W.G. Atshan, On sandwich results of univalent functions defined by a linear operator, Journal of Interdisciplinary Mathematics, 23(4), pp. 803-809, 2020.
[29] R.L. Brooks, On Coloring the Nodes of a Network, Proc. Camb. Philos. Soc. 37, 194–197, (1941).
[30] G. Chartrand and L. Lesniak, Graphs and Digraphs, Chapman and Hall/CRC, 2005
[31] F. Harary, Graph theory, Addison-Wesley, Reading, Mass., 1969

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