Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/160416

On Product-Type Operators between H^∞ and Zygmund Spaces

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Received: 12 May 2022, Revised: 2 Jun. 2022, Accepted: 24 Jun. 2022
Published online: 1 Jul. 2022

Abstract: In this paper, we give a complete picture of the boundedness and compactness of the product operator $T_{\Psi_1,\Psi_2,\phi}$ from H^∞ to Zygmund spaces. Specifically, we give the necessary and sufficient conditions for the product operator $T_{\Psi_1,\Psi_2,\phi}$ from H^∞ to Zygmund spaces to be bounded and compact.

Keywords: Products of multiplication, composition, and differentiation operators, compactness, product operator $T_{\Psi_1,\Psi_2,\phi}$.

1 Introduction

The open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, where $\mathbb{C}$ is the complex plane.

Let $H(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$.

The space H^∞ denotes the space of all analytic functions f on the unit disk $\mathbb{D}$ such that

$$||f||_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)| < \infty. \tag{1}$$

The Bloch space $\mathscr{B}$ is defined as

$$||f||_{\mathscr{B}} = \sup_{z\in\mathbb{D}} (\nu(z))|f'(z)| < \infty.$$

By the Zygmund theorem and the closed graph theorem (see [1], Theorem 5.3), we see that $f \in \mathscr{Z}$ if and only if

$$\sup_{z\in\mathbb{D}} (\nu(z))|f''(z)| < \infty.$$

Under the norm

$$||f||_{\mathscr{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \nu(z)|f''(z)|, \tag{2}$$

$\mathscr{Z}$ is a Banach space. This space is called a Zygmund-type space when $\nu(z) = 1 - |z|^2$. Zygmund-type spaces on the unit disk have been well studied [2–5].

Li and Stević introduced a small Zygmund space $\mathscr{Z}_0$ [6] in the following way:

$$f \in \mathscr{Z}_0 \Leftrightarrow \lim_{|z|\to 1^-} \nu(z)|f''(z)| = 0.$$

For any analytic self-mapping ϕ of $\mathbb{D}$, the linear composition operator $C_\phi(f) := f \circ \phi = f(\phi(z))$ [7]. The composition operator has been extensively studied in Banach spaces of analytic functions [8–14]

For $\Psi, f \in H(\mathbb{D})$, let the multiplication operator M_Ψ be defined as follows:

$$M_\Psi(z) = \psi(z).f(z).$$

The differentiation operator D is defined as

$$Df(z) = f^{'}(z).$$

The products of composition and differentiation operators DC_ϕ and $C_\phi D$ are defined, respectively as follows :
are defined, respectively, as follows:

$$DC_\phi f(z) = f^{'}(\phi(z)).\phi^{'}(z), \quad f \in H(\mathbb{D}),$$

$$C_\phi Df(z) = (f^{'} \circ \phi)(z), \quad f \in H(\mathbb{D}).$$

The product of the differentiation and multiplication operators, denoted by DM_Ψ, is defined as

$$DM_\Psi f(z) = \psi'(z).f(z) + \psi(z).f'(z), \quad f \in H(\mathbb{D}).$$

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The weighted composition operator is

$$W_{\psi,\phi}f(z) = (\psi C_\phi)f(z) = \psi(z)f(\phi(z)), \quad f \in H(\mathbb{D}).$$

For $\Psi_1, \Psi_2 \in H(\mathbb{D})$ and ϕ denotes an analytic self-mapping of $\mathbb{D}$. The products of the multiplication, composition, and differentiation operators are defined as follows:

$$T_{\Psi_1,\Psi_2,\phi}f(z) = \Psi_1(z)f(\phi(z)) + \Psi_2(z)f'(\phi(z)), f \in H(\mathbb{D}) \quad (3)$$

Over the past several years, the operator $T_{\Psi_1,\Psi_2,\phi}$ has been studied by many people, and has been a hot topic of research [15–27] . However, Stević et al. were the first to introduce the operator $T_{\Psi_1,\Psi_2,\phi}$ [23].

The following lemmas can be proven in a standard manner (see, e.g., Proposition 3.11 in [28]). These lemmas give the definitions of the boundedness and compactness of the operators $T_{\Psi_1,\Psi_2,\phi} : H^\infty \to \mathscr{Z}$.

Lemma 1. *The operator $T_{\Psi_1,\Psi_2,\phi} : H^\infty \to \mathscr{Z}$ is said to be bounded if there is a positive constant C such that $||T_{\Psi_1,\Psi_2,\phi}f||_{\mathscr{Z}} \le C||f||_\infty$ for all $f \in H^\infty$.*

Lemma 2. *The operator $T_{\Psi_1,\Psi_2,\phi} : H^\infty \to \mathscr{Z}$ is said to be compact if it maps any function in the unit disk in H^∞ onto a precompact set in $\mathscr{Z}$.*

This paper uses the term C to denote a positive constant that is independent of the essential variables.

2 The boundedness of $T_{\Psi_1,\Psi_2,\phi} : H^\infty \to \mathscr{Z}$

In this section, we characterize the operator $T_{\Psi_1,\Psi_2,\phi} : H^\infty \to \mathscr{Z}$. Moreover, we give the conditions that prove the boundedness of the operator $T_{\Psi_1,\Psi_2,\phi}$. We therefore cite the following two necessary lemmas.

Lemma 3. *[20] Suppose $f \in H^\infty$. Then, for each $n \in \mathbb{N}$,*

$$\sup_{z\in\mathbb{D}} (1-|z|)^n |f^{(n)}(z)| \le C \| f \|_\infty .$$

The next lemma is introduced in [29].

Lemma 4. *Suppose $f \in \mathscr{B}$. Then, for each $n \in \mathbb{N}$,*

$$\| f \|_{\mathscr{B}} \asymp \sum_{j=0}^{n-1} |f^{(j)}(0)| + \sup_{z\in\mathbb{D}} (\nu(z))^n |f^{(n)}(z)|.$$

We now introduce the main boundedness results.

Lemma 5. *Suppose a test function in the following form:*

$$(f_i)_\zeta(z) = \frac{-(1-|\zeta|^2)}{1-\overline{\zeta}z} + \frac{a(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^2} + \frac{b(1-|\zeta|^2)^3}{(1-\overline{\zeta}z)^3} + \frac{c(1-|\zeta|^2)^4}{(1-\overline{\zeta}z)^4}, \quad i = 1,2,3. \quad (4)$$

Then $(f_i)_\zeta \in H^\infty$ and

$$(f_1)''_\zeta(\zeta) = (f_1)'''_\zeta(\zeta) = 0, \ (f_1)'_\zeta(\zeta) = \frac{C_1\overline{\zeta}}{1-|\zeta|^2},$$

where $C_1 = 2a+3b+4c-1 \neq 0$;

$$(f_2)'_\zeta(\zeta) = (f_2)'''_\zeta(\zeta) = 0, \ (f_2)''_\zeta(\zeta) = \frac{C_2\overline{\zeta}^2}{(1-|\zeta|^2)^2},$$

where $C_2 = 6a+12b+20c-2 \neq 0$;

$$(f_3)'_\zeta(\zeta) = (f_3)''_\zeta(\zeta) = 0, \ (f_3)'''_\zeta(\zeta) = \frac{C_3\overline{\zeta}^3}{(1-|\zeta|^2)^3},$$

where $C_3 = 24a+60b+120c-6 \neq 0$.

Proof. By the triangle inequality, we have

$$\begin{aligned}|(f_i)_\zeta(z)| &\le \frac{|-1|(1-|\zeta|^2)}{1-|\zeta z|} + \frac{|a|(1-|\zeta|^2)^2}{(1-|\zeta z|)^2} + \frac{|b|(1-|\zeta|^2)^3}{(1-|\zeta z|)^3} + \frac{|c|(1-|\zeta|^2)^4}{(1-|\zeta z|)^4} \\ &\le \frac{(1-|\zeta|^2)}{1-|\zeta|} + \frac{a|(1-|\zeta|^2)^2}{(1-|\zeta|)^2} + \frac{b(1-|\zeta|^2)^3}{(1-|\zeta|)^3} + \frac{c(1-|\zeta|^2)^4}{(1-|\zeta|)^4} \\ &\le 2+4|a|+8|b|+16|c|.\end{aligned}$$

It is therefore clear that, for all $(f_i)_\zeta \in H^\infty$ and

$$\sup_{\zeta\in\mathbb{D}} \| (f_i)_\zeta \|_\infty \le 2+4|a|+8|b|+16|c|. \quad (5)$$

Then

$$(f_i)'_\zeta(z) = \Big(\frac{-(1-|\zeta|^2)}{(1-\overline{\zeta}z)^2} + \frac{2a(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^3} + \frac{3b(1-|\zeta|^2)^3}{(1-\overline{\zeta}z)^4} + \frac{4c(1-|\zeta|^2)^4}{(1-\overline{\zeta}z)^5}\Big)\overline{\zeta}, \quad (6)$$

$$(f_i)''_\zeta(z) = \Big(\frac{-2(1-|\zeta|^2)}{(1-\overline{\zeta}z)^3} + \frac{6a(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^4} + \frac{12b(1-|\zeta|^2)^3}{(1-\overline{\zeta}z)^5} + \frac{20c(1-|\zeta|^2)^4}{(1-\overline{\zeta}z)^6}\Big)\overline{\zeta}^2,$$

$$(f_i)'''_\zeta(z) = \Big(\frac{-6(1-|\zeta|^2)}{(1-\overline{\zeta}z)^3} + \frac{24a(1-|\zeta|^2)^2}{(1-\overline{\zeta}z)^4} + \frac{60b(1-|\zeta|^2)^3}{(1-\overline{\zeta}z)^5} + \frac{120c(1-|\zeta|^2)^4}{(1-\overline{\zeta}z)^6}\Big)\overline{\zeta}^3.$$

We choose the values for the constants a,b,c in (4) such that, when $i=1$,

$$(f_1)''_{\zeta}(\zeta)=(f_1)'''_{\zeta}(\zeta)=0,\ \ (f_1)'_{\zeta}(\zeta)=\frac{C_1\overline{\zeta}}{1-|\zeta|^2},$$

where $C_1=2a+3b+4c-1\neq 0$;

when $i=2$, then

$$(f_2)'_{\zeta}(\zeta)=(f_2)'''_{\zeta}(\zeta)=0,\ \ (f_2)''_{\zeta}(\zeta)=\frac{C_2\overline{\zeta}^2}{(1-|\zeta|^2)^2},$$

where $C_2=6a+12b+20c-2\neq 0$;

and when $i=3$, then

$$(f_3)'_{\zeta}(\zeta)=(f_3)''_{\zeta}(\zeta)=0,\ \ (f_3)'''_{\zeta}(\zeta)=\frac{C_3\overline{\zeta}^3}{(1-|\zeta|^2)^3},$$

where $C_3=24a+60b+120c-6\neq 0$.

Proposition:

Let

$$A_1=\sup_{z\in\mathbb{D}}\frac{\nu(z)\,|\,2\Psi_1'(z)\phi'(z)+\Psi_1(z)\phi''(z)+\Psi_2''(z)\,|}{(1-|\phi(z)|^2)}, \tag{7}$$

$$A_2=\sup_{z\in\mathbb{D}}\frac{\nu(z)\,|\,\Psi_1(z)\phi'^2(z)+2\Psi_2'(z)\phi'(z)+\Psi_2(z)\phi''(z)\,|}{(1-|\phi(z)|^2)^2} \tag{8}$$

and

$$A_3=\sup_{z\in\mathbb{D}}\frac{\nu(z)\,|\,\Psi_2(z)\phi'^2(z)\,|}{(1-|\phi(z)|^2)^3}. \tag{9}$$

Theorem 1. *Let $\Psi_1,\Psi_2\in H(\mathbb{D})$. Then, the following statements are equivalent:*

(a) *$T_{\Psi_1,\Psi_2,\phi}:H^\infty\to\mathscr{Z}$ is a bounded operator,*
(b) *$\Psi_1\in\mathscr{Z}$, where A_1, A_2, and A_3 are finite.*

Proof. $(b)\Rightarrow(a)$. First, assume that $\Psi_1\in\mathscr{Z}$ and (7) to (9) hold. Then, by Lemma 4, we obtain

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}(\nu(z))\,|\,(T_{\Psi_1,\Psi_2,\phi}f)''(z)\,|\\
&=\sup_{z\in\mathbb{D}}(\nu(z))\,|\,\Psi_1''(z)f(\phi(z))+\Psi_1'(z)\phi'(z)f'(\phi)\\
&+\Psi_1'(z)\phi'(z)+\Psi_1(z)\phi''(z)+\Psi_2''(z))f'(\phi(z))\\
&+(\Psi_1(z)\phi'(z)+\Psi_2'(z))\phi'(z)f''(\phi(z))\\
&+\Psi_2'(z)\phi'(z)f''(\phi(z))+\Psi_2(z)\phi''(z)f''(\phi(z))\\
&+\Psi_2(z)\phi'^2(z)f'''(\phi(z))\,|\\
&=\sup_{z\in\mathbb{D}}(\nu(z))\,|\,\Psi_1''(z)f(\phi(z))+(2\Psi_1'(z)\phi'(z)\\
&+\Psi_1(z)\phi''(z)+\Psi_2''(z))f'(\phi(z))+(\Psi_1(z)\phi'^2(z)\\
&+2\Psi_2'(z)\phi'(z)+\Psi_2(z)\phi''(z))f''(\phi(z))\\
&+\Psi_2(z)\phi'^2(z)f'''(\phi(z))\,|\\
&\leq(\nu(z))\,|\,\Psi_1''(z)f(\phi(z))\,|+(\nu(z))\,|\,(2\Psi_1'(z)\phi'(z)\\
&+\Psi_1(z)\phi''(z)+\Psi_2''(z))f'(\phi(z))\,|\\
&+(\nu(z))\,|\,(\Psi_1(z)\phi'^2(z)\\
&+(\nu(z))\,|\,\Psi_2(z)\phi'^2(z)f'''(\phi(z))\,|\\
&\leq C(\nu(z))\Bigg[\,|\,\Psi''(z)\,|\\
&+\frac{|\,2\Psi_1'(z)\phi'(z)+\Psi_1(z)\phi''(z)+\Psi_2''(z)\,|}{(1-|\phi(z)|^2)}\\
&+\frac{|\,\Psi_1(z)\phi'^2(z)+2\Psi_2'(z)\phi'(z)+\Psi_2(z)\phi''(z)\,|}{(1-|\phi(z)|^2)^2}\\
&+\frac{|\,\Psi_2(z)\phi'^2(z)\,|}{(1-|\phi(z)|^2)^3}\Bigg]\,\|\,f\,\|_\infty\\
&\leq C\,\|\,f\,\|_\infty\,.
\end{aligned} \tag{10}$$

Moreover, by using Lemma 4, we obtain

$$\begin{aligned}
&|\,(T_{\Psi_1,\Psi_2,\phi}f)(0)\,|=|\,\Psi_1(0)f(\phi(0))+\Psi_2(0)f'(\phi(0))\,|\\
&\leq C\left(\,|\,\Psi_1(0)\,|+\frac{\Psi_2(0)}{(1-|\phi(0)|^2)}\right)\,\|\,f\,\|_{H^\infty},
\end{aligned} \tag{11}$$

$$\begin{aligned}
&|\,(T_{\Psi_1,\Psi_2,\phi}f)'(0)\,|=|\,(\Psi_1(0)f(\phi(0))+\Psi_2(0)f'(\phi(0)))'\,|\\
&=|\,(\Psi_1'(0)f(\phi(0))+(\Psi_1(0)\phi'(0)\\
&+\Psi_2'(0))f'(\phi(0))+\Psi_2(z)\phi'(0)f''(\phi(0)))\,|\\
&\leq C\left(\,|\,\Psi_1'(0)\,|+\frac{(|\,\Psi_1(0)\phi'(0)+\Psi_2'(0))\,|}{(1-|\phi(0)|^2)}\right.\\
&\left.+\frac{|\,\Psi_2(z)\phi'(0)\,|}{(1-|\phi(0)|^2)^2}\right)\,\|\,f\,\|_{H^\infty}\,.
\end{aligned} \tag{12}$$

By using the conditions (10) to (12), we can deduce $T_{\Psi_1,\Psi_2,\phi}:H^\infty\to\mathscr{Z}$ is bounded.
$(a)\Rightarrow(b)$. Now suppose that $T_{\Psi_1,\Psi_2,\phi}:H^\infty\to\mathscr{Z}$ is bounded. Then,

$$\|\,T_{\Psi_1,\Psi_2,\phi}f(z)\,\|_{\mathscr{Z}}\leq C\,\|\,f\,\|_\infty$$

for all $f \in H^\infty$.
Assume that $f(z) = z^j, j = 0,1,2,3 \in H^\infty$. For $j = 0$, we have $f(z) = 1 \in H^\infty$, and we obtain

$$\| (T_{\Psi_1,\Psi_2,\phi} f)(1) \| = \| \Psi_1(z) \| .$$

Then

$$K_1 := \| \Psi_1 \|_{\mathscr{Z}} = \sup_{z\in\mathbb{D}} (\nu(z)) \mid \Psi_1''(z) \mid < \infty. \quad (13)$$

For $j = 1$, we have $f(z) = z \in H^\infty$, and

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \nu(z) \mid (T_{\Psi_1,\Psi_2,\phi} f)''(z) \mid \\
&= \sup_{z\in\mathbb{D}} (\nu(z)) \mid \Psi_1''(z)\phi(z) + \Psi_1'(z)\phi'(z) \\
&+ \Psi_1'(z)\phi'(z) + \Psi_1(z)\phi''(z) + \Psi_2''(z) \mid \\
&= \sup_{z\in\mathbb{D}} (\nu(z)) \mid \Psi_1''(z)\phi(z) + 2\Psi_1'(z)\phi'(z) \\
&+ \Psi_1(z)\phi''(z) + \Psi_2''(z) \mid < \infty. \quad (14)
\end{aligned}$$

From (13), (14), and the boundedness of the function $\phi(z)$, we obtain

$$K_2 := \sup_{z\in\mathbb{D}} (\nu(z)) \mid 2\Psi_1'(z)\phi'(z) + \Psi_1(z)\phi''(z) + \Psi_2''(z) \mid < \infty. \quad (15)$$

For $j = 2$, we have $f(z) = z^2 \in H^\infty$, and

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} (\nu(z)) \mid (T_{\Psi_1,\Psi_2,\phi} f)''(z) \mid \\
&= \sup_{z\in\mathbb{D}} (\nu(z)) \mid \Psi_1''(z)(\phi(z))^2 + 2\Psi_1'(z)\phi(z)\phi'(z) \\
&+ 2\Psi_1'(z)\phi(z)\phi'(z) + 2\Psi_1(z)(\phi'(z))^2 + 2\Psi_1(z)\phi(z)\phi''(z) \\
&+ 2\Psi_2''(z)\phi(z) + 2\Psi_2'(z)\phi'(z) \\
&+ 2\Psi_2'(z)\phi'(z) + 2\Psi_2(z)\phi''(z) \mid \\
&= \sup_{z\in\mathbb{D}} (\nu(z)) \mid \Psi_1''(z)(\phi(z))^2 + 4\Psi_1'(z)\phi(z)\phi'(z) \\
&+ 2\Psi_1(z)(\phi'(z))^2 + 2\Psi_1(z)\phi(z)\phi''(z) + 2\Psi_2''(z)\phi(z) \\
&+ 4\Psi_2'(z)\phi'(z) + 2\Psi_2(z)\phi''(z) \mid < \infty. \quad (16)
\end{aligned}$$

From (13), (15), (16), and the boundedness of the function $\phi(z)$, we have

$$\begin{aligned}
K_3 : &= \sup_{z\in\mathbb{D}} (\nu(z)) \mid \Psi_1(z)(\phi'(z))^2 + 2\Psi_2'(z)\phi'(z) \\
&+ \Psi_2(z)\phi''(z) \mid < \infty. \quad (17)
\end{aligned}$$

For $j = 3$, we have $f(z) = z^3 \in H^\infty$, and

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} (\nu(z)) \mid (T_{\Psi_1,\Psi_2,\phi} f)''(z) \mid \\
&= \sup_{z\in\mathbb{D}} (\nu(z)) \mid \Psi_1''(z)(\phi(z))^3 + 3\Psi_1'(z)(\phi(z))^2\phi'(z) \\
&+ 3\Psi_1'(z)(\phi(z))^2\phi'(z) + 6\Psi_1(z)\phi(z)(\phi'(z))^2 \\
&+ 3\Psi_1(z)(\phi(z))^2\phi''(z) + 3\Psi_2''(z)(\phi(z))^2 \\
&+ 6\Psi_2'(z)\phi(z)\phi'(z) + 6\Psi_2'(z)\phi(z)\phi'(z) + 6\Psi_2(z)(\phi'(z))^2 \\
&+ 6\Psi_2(z)\phi(z)\phi''(z) \mid \\
&= \sup_{z\in\mathbb{D}} (\nu(z)) \mid \Psi_1''(z)(\phi(z))^3 + 6\Psi_1'(z)(\phi(z))^2\phi'(z) \\
&+ 6\Psi_1(z)\phi(z)(\phi'(z))^2 + 3\Psi_1(z)(\phi(z))^2\phi''(z) \\
&+ 3\Psi_2''(z)(\phi(z))^2 + 12\Psi_2'(z)\phi(z)\phi'(z) \\
&+ 6\Psi_2(z)(\phi'(z))^2 + 6\Psi_2(z)\phi(z)\phi''(z) \mid < \infty. \quad (18)
\end{aligned}$$

From (13),(15), (17), (18), and the boundedness of the function $\phi(z)$, we have

$$K_4 := \sup_{z\in\mathbb{D}} (\nu(z)) \mid \Psi_2(z)(\phi'(z))^2 \mid < \infty. \quad (19)$$

For a fixed $\zeta \in \mathbb{D}$ and using Lemma 5, we obtain

$$\begin{aligned}
C &\geq \sup_{\zeta\in\mathbb{D}} (1- \mid \zeta \mid^2) \mid (T_{\Psi_1,\Psi_2,\phi}(f_1)_\phi)''(\zeta) \mid \\
&= \sup_{\zeta\in\mathbb{D}} (1- \mid \zeta \mid^2) \mid (\Psi_1'(\zeta)(f_1)_{\phi(\zeta)}(\phi(\zeta)) \\
&+ \Psi_1(\zeta)\phi'(\zeta)(f_1)'_{\phi(\zeta)}(\phi(\zeta)) + \Psi_2'(\zeta)(f_1)'_{\phi(\zeta)}(\phi(\zeta)) \\
&+ \Psi_2(\zeta)\phi'(\zeta)(f_1)''_{\phi(\zeta)}(\phi(\zeta)))' \mid \\
&= \sup_{\zeta\in\mathbb{D}} (1- \mid \zeta \mid^2) \mid \Psi_1''(\zeta)(f_1)_{\phi(\zeta)}(\phi(\zeta)) \\
&+ \Psi_1'(\zeta)\phi'(\zeta)(f_1)_{\phi(\zeta)}\phi(\zeta)'(\phi(\zeta)) + (\Psi_1'(\zeta)\phi'(\zeta) \\
&+ \Psi_1(\zeta)\phi''(\zeta) + \Psi_2''(\zeta))(f_1)'_{\phi(\zeta)}(\phi(\zeta)) \\
&+ (\Psi_1(\zeta)\phi'(\zeta) + \Psi_2'(\zeta))\phi'(\zeta)(f_1)''_{\phi(\zeta)}(\phi(\zeta)) \\
&+ \Psi_2'(\zeta)\phi'(\zeta)(f_1)''_{\phi(\zeta)}(\phi(\zeta)) \\
&+ \Psi_2(\zeta)\phi''(\zeta)(f_1)''_{\phi(\zeta)}(\phi(\zeta)) \\
&+ \Psi_2(\zeta)\phi'^2(\zeta)(f_1)'''_{\phi(\zeta)}(\phi(\zeta)) \mid \\
&= \sup_{\zeta\in\mathbb{D}} (1- \mid \zeta \mid^2) \mid \Psi_1''(\zeta)(f_1)_{\phi(\zeta)}(\phi(\zeta)) \\
&+ (2\Psi_1'(\zeta)\phi'(\zeta) + \Psi_1(\zeta)\phi''(\zeta) \\
&+ \Psi_2''(\zeta))(f_1)'_{\phi(\zeta)}(\phi(\zeta)) \\
&+ (\Psi_1(\zeta)\phi'^2(\zeta) + 2\Psi_2'(\zeta)\phi'(\zeta) \\
&+ \Psi_2(\zeta)\phi''(\zeta))(f_1)''_{\phi(\zeta)}(\phi(\zeta)) \\
&+ \Psi_2(\zeta)\phi'^2(\zeta)(f_1)'''_{\phi(\zeta)}(\phi(\zeta)) \mid \\
&\geq \sup_{\zeta\in\mathbb{D}} \frac{C_1(1- \mid \zeta \mid^2) \mid (2\Psi_1'(\zeta)\phi'(\zeta)}{1- \mid \phi(\zeta) \mid^2} \\
&+ \frac{\Psi_1(\zeta)\phi''(\zeta) + \Psi_2''(\zeta)) \mid \overline{\phi(\zeta)}}{1- \mid \phi(\zeta) \mid^2} \quad (20)
\end{aligned}$$

For $\delta \in (0,1)$, by using (20) and (15), we obtain

$$\sup_{\zeta\in\mathbb{D}} \frac{(1-|\zeta|^2)\,|\,2\Psi_1'(\zeta)\phi'(\phi)+\Psi_1(\zeta)\phi''(\zeta)+\Psi_2''(\zeta)\,|}{(1-|\phi(\zeta)|^2)}$$
$$\leq \sup_{|\zeta|>\delta} \frac{(1-|\zeta|^2)\,|\,2\Psi_1'(\zeta)\phi'(\phi)+\Psi_1(\zeta)\phi''(\zeta)+\Psi_2''(\zeta)\,|}{(1-|\phi(\zeta)|^2)}$$
$$+ \sup_{|\zeta|\leq\delta} \frac{(1-|\zeta|^2)\,|\,2\Psi_1'(\zeta)\phi'(\phi)+\Psi_1(\zeta)\phi''(\zeta)+\Psi_2''(\zeta)\,|}{(1-|\phi(\zeta)|^2)}$$
$$\leq \frac{1}{\delta}\sup_{|\zeta|>\delta} \frac{(1-|\zeta|^2)\,|\,2\Psi_1'(\zeta)\phi'(\phi)+\Psi_1(\zeta)\phi''(\zeta)}{(1-|\phi(\zeta)|^2)}$$
$$+ \frac{\Psi_2''(\zeta)\,|\,\overline{\phi(\zeta)}}{(1-|\phi(\zeta)|^2)}$$
$$+ \frac{\delta}{(1-\delta^2)}\sup_{|\zeta|\leq\delta}(1-|\zeta|^2)\,|\,(2\Psi_1'(\zeta)\phi'(\zeta)$$
$$+\Psi_1(\zeta)\phi''(\zeta)+\Psi_2''(\zeta))\,|\leq C. \tag{21}$$

It follows that condition (7) holds, as desired.

For a fixed $\zeta \in \mathbb{D}$ and by using Lemma 5, we obtain

$$C \geq \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\,|\,(T_{\Psi_1,\Psi_2,\phi}(f_2)_\phi)''(\zeta)\,|$$
$$= \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\,|\,(\Psi_1'(\zeta)(f_2)_{\phi(\zeta)}(\phi(\zeta))$$
$$+\Psi_1(\zeta)\phi'(\zeta)(f_2)'_{\phi(\zeta)}(\phi(\zeta))+\Psi_2'(\zeta)(f_2)'_{\phi(\zeta)}(\phi(\zeta))$$
$$+\Psi_2(\zeta)\phi'(\zeta)(f_2)''_{\phi(\zeta)}(\phi(\zeta)))'\,|$$
$$= \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\,|\,(\Psi_1'(\zeta)(f_2)_{\phi(\zeta)}(\phi(\zeta))$$
$$+(\Psi_1(\zeta)\phi'(\zeta)+\Psi_2'(\zeta))(f_2)'_{\phi(\zeta)}(\phi(\zeta))$$
$$+\Psi_2(\zeta)\phi'(\zeta)(f_2)''_{\phi(\zeta)}(\phi(\zeta)))'\,|$$
$$= \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\,|\,\Psi_1''(\zeta)(f_2)_{\phi(\zeta)}(\phi(\zeta))$$
$$+\Psi_1'(\zeta)\phi'(\zeta)(f_2)_{\phi(\zeta)}\phi(\zeta)'(\phi(\zeta))+(\Psi_1'(\zeta)\phi'(\zeta)$$
$$+\Psi_1(\zeta)\phi''(\zeta)+\Psi_2''(\zeta))(f_2)'_{\phi(\zeta)}(\phi(\zeta))+(\Psi_1(\zeta)\phi'(\zeta)$$
$$+\Psi_2'(\zeta))\phi'(\zeta)(f_2)''_{\phi(\zeta)}(\phi(\zeta))$$
$$+\Psi_2'(\zeta)\phi'(\zeta)(f_2)''_{\phi(\zeta)}(\phi(\zeta))$$
$$+\Psi_2(\zeta)\phi''(\zeta)(f_2)''_{\phi(\zeta)}(\phi(\zeta))$$
$$+\Psi_2(\zeta)\phi'^2(\zeta)(f_2)'''_{\phi(\zeta)}(\phi(\zeta))\,|$$
$$= \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\,|\,\Psi_1''(\zeta)(f_2)_{\phi(\zeta)}(\phi(\zeta))+(2\Psi_1'(\zeta)\phi'(\zeta)$$
$$+\Psi_1(\zeta)\phi''(\zeta)+\Psi_2''(\zeta))(f_2)'_{\phi(\zeta)}(\phi(\zeta))+(\Psi_1(\zeta)\phi'^2(\zeta)$$
$$+2\Psi_2'(\zeta)\phi'(\zeta)+\Psi_2(\zeta)\phi''(\zeta))(f_2)''_{\phi(\zeta)}(\phi(\zeta))$$
$$+\Psi_2(\zeta)\phi'^2(\zeta)(f_2)'''_{\phi(\zeta)}(\phi(\zeta))\,|$$

$$\geq \sup_{\zeta\in\mathbb{D}} \frac{C_2(1-|\zeta|^2)\,|\,(\Psi_1(\zeta)\phi'^2(\zeta)+2\Psi_2'(\zeta)\phi'(\zeta)}{(1-|\phi(\zeta)|^2)^2}$$
$$+\frac{\Psi_2(\zeta)\phi''(\zeta))\overline{\phi(\zeta)}^2}{(1-|\phi(\zeta)|^2)^2}. \tag{22}$$

For $\delta \in (0,1)$, by using (22) and (17), we obtain

$$\sup_{\zeta\in\mathbb{D}} \frac{(1-|\zeta|^2)\,|\,(\Psi_1(\zeta)\phi'^2(\zeta)+2\Psi_2'(\zeta)\phi'(\zeta)}{(1-|\phi(\zeta)|^2)^2}$$
$$+\frac{\Psi_2(\zeta)\phi''(\zeta))\,|}{(1-|\phi(\zeta)|^2)^2}$$
$$\leq \sup_{|\zeta|>\delta} \frac{(1-|\zeta|^2)\,|\,(\Psi_1(\zeta)\phi'^2(\zeta)+2\Psi_2'(\zeta)\phi'(\zeta)}{(1-|\phi(\zeta)|^2)^2}$$
$$+\frac{\Psi_2(\zeta)\phi''(\zeta))\,|}{(1-|\phi(\zeta)|^2)^2}$$
$$+\sup_{|\zeta|\leq\delta} \frac{(1-|\zeta|^2)\,|\,(\Psi_1(\zeta)\phi'^2(\zeta)}{(1-|\phi(\zeta)|^2)^2}$$
$$+\frac{2\Psi_2'(\zeta)\phi'(\zeta)+\Psi_2(\zeta)\phi''(\zeta))\,|}{(1-|\phi(\zeta)|^2)^2}$$
$$\leq \frac{1}{\delta}\sup_{|\zeta|>\delta} \frac{(1-|\zeta|^2)\,|\,(\Psi_1(\zeta)\phi'^2(\zeta)+2\Psi_2'(\zeta)\phi'(\zeta)}{(1-|\phi(\zeta)|^2)^2}$$
$$+\frac{\Psi_2(\zeta)\phi''(\zeta))\,|\,\overline{\phi(\zeta)}^2}{(1-|\phi(\zeta)|^2)^2}$$
$$+\frac{\delta^2}{(1-\delta^2)^2}\sup_{|\zeta|\leq\delta}(1-|\zeta|^2)\,|\,(\Psi_1(\zeta)\phi'^2(\zeta)$$
$$+2\Psi_2'(\zeta)\phi'(\zeta)+\Psi_2(\zeta)\phi''(\zeta))\,|\leq C. \tag{23}$$

It follows that condition (8) holds, as desired.

For a fixed $\zeta \in \mathbb{D}$ and by using Lemma 5, we obtain

$$C \geq \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\,|\,(T_{\Psi_1,\Psi_2,\phi}(f_3)_\phi)''(\zeta)\,|$$
$$= \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\,|\,(\Psi_1'(\zeta)(f_3)_{\phi(\zeta)}(\phi(\zeta))$$
$$+\Psi_1(\zeta)\phi'(\zeta)(f_3)'_{\phi(\zeta)}(\phi(\zeta))+\Psi_2'(\zeta)(f_3)'_{\phi(\zeta)}(\phi(\zeta))$$
$$+\Psi_2(\zeta)\phi'(\zeta)(f_3)''_{\phi(\zeta)}(\phi(\zeta)))'\,|$$
$$= \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\,|\,(\Psi_1'(\zeta)(f_3)_{\phi(\zeta)}(\phi(\zeta))$$
$$+(\Psi_1(\zeta)\phi'(\zeta)+\Psi_2'(\zeta))(f_3)'_{\phi(\zeta)}(\phi(\zeta))$$
$$+\Psi_2(\zeta)\phi'(\zeta)(f_3)''_{\phi(\zeta)}(\phi(\zeta)))'\,|$$
$$= \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\,|\,\Psi_1''(\zeta)(f_3)_{\phi(\zeta)}(\phi(\zeta))$$
$$+\Psi_1'(\zeta)\phi'(\zeta)(f_3)_{\phi(\zeta)}\phi(\zeta)'(\phi(\zeta))+(\Psi_1'(\zeta)\phi'(\zeta)$$
$$+\Psi_1(\zeta)\phi''(\zeta)+\Psi_2''(\zeta))(f_3)'_{\phi(\zeta)}(\phi(\zeta))$$

$$
\begin{aligned}
&+ (\Psi_1(\zeta)\phi'(\zeta)+\Psi_2'(\zeta))\phi'(\zeta)(f_3)''_{\phi(\zeta)}(\phi(\zeta))\\
&+ \Psi_2'(\zeta)\phi'(\zeta)(f_3)''_{\phi(\zeta)}(\phi(\zeta))\\
&+ \Psi_2(\zeta)\phi''(\zeta)(f_3)''_{\phi(\zeta)}(\phi(\zeta))\\
&+ \Psi_2(\zeta)\phi'^2(\zeta)(f_3)'''_{\phi(\zeta)}(\phi(\zeta)) \mid\\
&= \sup_{\zeta\in\mathbb{D}} (1-\mid\zeta\mid^2)\mid \Psi_1''(\zeta)(f_3)_{\phi(\zeta)}(\phi(\zeta))\\
&+ (2\Psi_1'(\zeta)\phi'(\zeta)+\Psi_1(\zeta)\phi''(\zeta)\\
&+ \Psi_2''(\zeta))(f_3)'_{\phi(\zeta)}(\phi(\zeta))\\
&+ (\Psi_1(\zeta)\phi'^2(\zeta)+2\Psi_2'(\zeta)\phi'(\zeta)\\
&+ \Psi_2(\zeta)\phi''(\zeta))(f_3)''_{\phi(\zeta)}(\phi(\zeta))\\
&+ \Psi_2(\zeta)\phi'^2(\zeta)(f_3)'''_{\phi(\zeta)}(\phi(\zeta)) \mid\\
&\geq \sup_{\zeta\in\mathbb{D}} \frac{C_3(1-\mid\zeta\mid^2)\mid(\Psi_2(\zeta)\phi'^2(\zeta))\mid \overline{\phi(\zeta)}^3}{(1-\mid\phi(\zeta)\mid^2)^3}. \qquad (24)
\end{aligned}
$$

For $\delta\in(0,1)$, by using (24) and (19), we obtain

$$
\begin{aligned}
&\sup_{\zeta\in\mathbb{D}} \frac{(1-\mid\zeta\mid^2)\mid(\Psi_2(\zeta)\phi'^2(\zeta))\mid}{(1-\mid\phi(\zeta)\mid^2)^3}\\
&\leq \sup_{|\zeta|>\delta} \frac{(1-\mid\zeta\mid^2)\mid(\Psi_2(\zeta)\phi'^2(\zeta))\mid}{(1-\mid\phi(\zeta)\mid^2)^3}\\
&+ \sup_{|\zeta|\leq\delta} \frac{(1-\mid\zeta\mid^2)\mid(\Psi_2(\zeta)\phi'^2(\zeta))\mid}{(1-\mid\phi(\zeta)\mid^2)^3}\\
&\leq \frac{1}{\delta^3}\sup_{|\zeta|>\delta} \frac{(1-\mid\zeta\mid^2)\mid(\Psi_2(\zeta)\phi'^2(\zeta))\mid \overline{\phi(\zeta)}^3}{(1-\mid\phi(\zeta)\mid^2)^3}\\
&+ \frac{1}{(1-\delta^2)^3}\sup_{|\zeta|\leq\delta}(1-\mid\zeta\mid^2)\mid(\Psi_2(\zeta)\phi'^2(\zeta))\mid\leq C. \qquad (25)
\end{aligned}
$$

It follows that condition (9) holds, as desired.
That ends the proof of Theorem 1.

3 The compactness of $T_{\Psi_1,\Psi_2,\phi}: H^\infty\to\mathscr{Z}$

In this section, we give the conditions that prove the compactness of the operator $T_{\Psi_1,\Psi_2,\phi}$.
The following lemma can be proven in a standard manner (see, e.g., Proposition 3.11 in [20]).

Lemma 6. *Suppose $\Psi_1,\Psi_2\in H(\mathbb{D})$. Then $T_{\Psi_1,\Psi_2,\phi}: H^\infty\to\mathscr{Z}$ is compact if and only if $T_{\Psi_1,\Psi_2,\phi}: H^\infty\to\mathscr{Z}$ is bounded, and for any bounded sequence $\{f_n\}$ in H^∞ that converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$, we have $\| T_{\Psi_1,\Psi_2,\phi}f_k\|_{\mathscr{Z}}\to 0$ as $n\to\infty$.*

We now introduce the main compactness results.

Lemma 7. *Suppose we have a test function of the form*

$$
\begin{aligned}
(g_i)_k(z) &= \frac{-(1-\mid(\phi(z_k))\mid^2)}{1-\overline{(\phi(z_k))}z}+\frac{a(1-\mid(\phi(z_k))\mid^2)^2}{(1-\overline{(\phi(z_k))}z)^2}\\
&+\frac{b(1-\mid(\phi(z_k))\mid^2)^3}{(1-\overline{(\phi(z_k))}z)^3}\\
&+\frac{c(1-\mid(\phi(z_k))\mid^2)^4}{(1-\overline{(\phi(z_k))}z)^4},\ i=1,2,3. \qquad (26)
\end{aligned}
$$

Then, $(g_i)_k\in H^\infty$ and

$$(g_1)''_k(\phi(z_k))=(g_1)'''_k(\phi(z_k))=0,$$

$$(g_1)'_k(\phi(z_k))=\frac{C_1\overline{\phi(z_k)}}{1-\mid\phi(z_k)\mid^2},$$

where $C_1=2a+3b+4c-1\neq 0$;

$$(g_2)'_k(\phi(z_k))=(g_2)'''_k(\phi(z_k))=0,$$

$$(g_2)''_k(\phi(z_k))=\frac{C_2\overline{\phi(z_k)}^2}{(1-\mid\phi(z_k)\mid^2)^2},$$

where $C_2=6a+12b+20c-2\neq 0$;

$$(g_3)'_k(\phi(z_k))=(g_3)''_k(\phi(z_k))=0,$$

$$(g_3)'''_k(\phi(z_k))=\frac{C_3\overline{\phi(z_k)}^3}{(1-\mid\phi(z_k)\mid^2)^3},$$

where $C_3=24a+60b+120c-6\neq 0$.

Proof. By the triangle inequality, we have

$$
\begin{aligned}
&\mid (g_i)_k(z)\mid\\
&\leq \frac{|-1|(1-\mid(\phi(z_k))\mid^2)}{1-|\overline{(\phi(z_k))}z|}+\frac{|a|(1-\mid(\phi(z_k))\mid^2)^2}{(1-|\overline{(\phi(z_k))}z|)^2}\\
&+\frac{|b|(1-\mid(\phi(z_k))\mid^2)^3}{(1-|\overline{(\phi(z_k))}z|)^3}+\frac{|c|(1-\mid(\phi(z_k))\mid^2)^3}{(1-|\overline{(\phi(z_k))}z|)^3}\\
&\leq \frac{(1-\mid(\phi(z_k))\mid^2)}{1-|(\phi(z_k))|}+\frac{a|(1-\mid(\phi(z_k))\mid^2)^2}{(1-|(\phi(z_k))|)^2}\\
&+\frac{b(1-\mid(\phi(z_k))\mid^2)^3}{(1-|(\phi(z_k))|)^3}+\frac{c(1-\mid(\phi(z_k))\mid^2)^4}{(1-|(\phi(z_k))|)^4}\\
&\leq 2+4|a|+8|b|+16|c|.
\end{aligned}
$$

It is therefore clear that, for all $(g_i)_k\in H^\infty$,

$$\sup_{k\in\mathbb{N}}\|(g_i)_k\|_\infty\leq 2+4|a|+8|b|+16|c|. \qquad (27)$$

Then

$$
\begin{aligned}
(g_i)'_k(z) &= (\frac{-(1-\mid(\phi(z_k))\mid^2)}{(1-\overline{(\phi(z_k))}z)^2}+\frac{2a(1-\mid(\phi(z_k))\mid^2)^2}{(1-\overline{(\phi(z_k))}z)^3}\\
&+\frac{3b(1-\mid(\phi(z_k))\mid^2)^3}{(1-\overline{(\phi(z_k))}z)^4}\\
&+\frac{4c(1-\mid(\phi(z_k))\mid^2)^4}{(1-\overline{(\phi(z_k))}z)^5})\overline{(\phi(z_k))}, \qquad (28)
\end{aligned}
$$

$$(g_i)''_k(z) = (\frac{-2(1-|(\phi(z_k))|^2)}{(1-\overline{(\phi(z_k))}z)^3} + \frac{6a(1-|(\phi(z_k))|^2)^2}{(1-\overline{(\phi(z_k))}z)^4} + \frac{12b(1-|(\phi(z_k))|^2)^3}{(1-\overline{(\phi(z_k))}z)^5} + \frac{20c(1-|(\phi(z_k))|^2)^4}{(1-\overline{(\phi(z_k))}z)^6})\overline{(\phi(z_k))}^2,$$

$$(g_i)'''_k(z) = (\frac{-6(1-|(\phi(z_k))|^2)}{(1-\overline{(\phi(z_k))}z)^3} + \frac{24a(1-|(\phi(z_k))|^2)^2}{(1-\overline{(\phi(z_k))}z)^4} + \frac{60b(1-|(\phi(z_k))|^2)^3}{(1-\overline{(\phi(z_k))}z)^5} + \frac{120c(1-|(\phi(z_k))|^2)^4}{(1-\overline{(\phi(z_k))}z)^6})\overline{(\phi(z_k))}^3.$$

We choose values for the constants a,b,c in (26) such that,
when $i=1$,

$$(g_1)''_k(\phi(z_k)) = (g_1)'''_k(\phi(z_k)) = 0,$$
$$(g_1)'_k(\phi(z_k)) = \frac{C_1\overline{\phi(z_k)}}{1-|\phi(z_k)|^2},$$

where $C_1 = 2a+3b+4c-1 \neq 0$;
when $i=2$,

$$(g_2)'_k(\phi(z_k)) = (g_2)'''_k(\phi(z_k)) = 0,$$
$$(g_2)''_k(\phi(z_k)) = \frac{C_2\overline{\phi(z_k)}^2}{(1-|\phi(z_k)|^2)^2},$$

where $C_2 = 6a+12b+20c-2 \neq 0$;
when $i=3$,

$$(g_3)'_k(\phi(z_k)) = (g_3)''_k(\phi(z_k)) = 0,$$
$$(g_3)'''_k(\phi(z_k)) = \frac{C_3\overline{\phi(z_k)}^3}{(1-|\phi(z_k)|^2)^3},$$

where $C_3 = 24a+60b+120c-6 \neq 0$.

Proposition Let

$$B_1 = \lim_{|\phi(z)|\to 1} \frac{(\nu(z))\,|\,2\Psi_1'(z)\phi'(z)}{(1-|\phi(z)|^2)} + \frac{\Psi_1(z)\phi''(z)+\Psi_2''(z)\,|}{(1-|\phi(z)|^2)}, \tag{29}$$

$$B_2 = \lim_{|\phi(z)|\to 1} \frac{(\nu(z))\,|\,\Psi_1(z)\phi'^2(z)}{(1-|\phi(z)|^2)^2} + \frac{2\Psi_2'(z)\phi'(z)+\Psi_2(z)\phi''(z)\,|}{(1-|\phi(z)|^2)^2}, \tag{30}$$

and

$$B_3 = \lim_{|\phi(z)|\to 1} \frac{(\nu(z))\,|\,\Psi_2(z)\phi'^2(z)\,|}{(1-|\phi(z)|^2)^3}. \tag{31}$$

Theorem 2. *Suppose $\Psi_1,\Psi_2 \in H(\mathbb{D})$. Then the following statements are equivalent.*

(a) *$T_{\Psi_1,\Psi_2,\phi} : H^\infty \to \mathscr{Z}$ is a compact operator,*
(b) *$T_{\Psi_1,\Psi_2,\phi} : H^\infty \to \mathscr{Z}$ is a bounded operator,*
where $B_1 = B_2 = B_2 = 0$.

Proof. $(b) \Rightarrow (a)$. Suppose that $T_{\Psi_1,\Psi_2,\phi} : H^\infty \to \mathscr{Z}$ is bounded and (29), (30) and (31) hold. To prove that $T_{\Psi_1,\Psi_2,\phi} : H^\infty \to \mathscr{Z}$ is compact for any bounded sequence $\{f_k\}$ in H^∞ with $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, let $\|f_k\|_{H^\infty} \leq 1$. Then, it suffices, in view of Lemma 6, to show that

$$\|T_{\Psi_1,\Psi_2,\phi} f_k\|_{\mathscr{Z}} \to 0 \quad as \quad k\to\infty.$$

By (29) to (31), for any $\varepsilon > 0$, there exists $\rho \in (0,1)$ such that

$$\frac{(\nu(z))\,|\,2\Psi_1'(z)\phi'(z)+\Psi_1(z)\phi''(z)+\Psi_2''(z)\,|}{(1-|\phi(z)|^2)} < \varepsilon, \tag{32}$$

$$\frac{(\nu(z))\,|\,\Psi_1(z)\phi'^2(z)+2\Psi_2'(z)\phi'(z)+\Psi_2(z)\phi''(z)\,|}{(1-|\phi(z)|^2)^2} < \varepsilon, \tag{33}$$

and

$$\frac{(\nu(z))\,|\,\Psi_2(z)\phi'^2(z)\,|}{(1-|\phi(z)|^2)^3} < \varepsilon. \tag{34}$$

From the proof of Theorem 1 and the boundedness of the operator $T_{\Psi_1,\Psi_2,\phi}$, the conditions (13), (15), (17), and (19) hold.

Since $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimate shows that f_k', f_k'', and f_k''' converge to zero uniformly on compact subsets of $\mathbb{D}$, and there exists $K_0 \in \mathbb{N}$ such that $k > K_0$ tends to

$$\begin{aligned}
&\sup_{|\phi(z)|\leq\rho} |\,(\nu(z))(T_{\Psi_1,\Psi_2,\phi} f_k)''(z)\,| \\
\leq &\sup_{|\phi(z)|\leq\rho} (\nu(z))\,|\,(\Psi_1(z)f_k(\phi(z))+\Psi_2(z)f_k'(\phi(z)))''\,| \\
= &\sup_{|\phi(z)|\leq\rho} (\nu(z))\,|\,(\Psi_1'(z)f_k(\phi(z))+\Psi_1(z)\phi'(z)f_k'(\phi(z)) \\
&+\Psi_2'(z)f_k'(\phi(z))+\Psi_2(z)\phi'(z)f_k''(\phi(z)))'\,| \\
= &\sup_{|\phi(z)|\leq\rho} (\nu(z))\,|\,(\Psi_1'(z)f_k(\phi(z))+(\Psi_1(z)\phi'(z) \\
&+\Psi_2'(z))f_k'(\phi(z))+\Psi_2(z)\phi'(z)f_k''(\phi(z)))'\,| \\
= &\sup_{|\phi(z)|\leq\rho} (\nu(z))\,|\,\Psi_1''(z)f_k(\phi(z))+\Psi_1'(z)\phi'(z)f_k'(\phi) \\
&+(\Psi_1'(z)\phi'(z)+\Psi_1(z)\phi''(z)+\Psi_2''(z))f_k'(\phi(z)) \\
&+(\Psi_1(z)\phi'(z)+\Psi_2'(z))\phi'(z)f_k''(\phi(z)) \\
&+\Psi_2'(z)\phi'(z)f_k''(\phi(z)) \\
&+\Psi_2(z)\phi''(z)f_k''(\phi(z))+\Psi_2(z)\phi'^2(z)f_k'''(\phi(z))\,| \\
= &\sup_{|\phi(z)|\leq\rho} (\nu(z))\,|\,\Psi_1''(z)f_k(\phi(z))+(2\Psi_1'(z)\phi'(z) \\
&+\Psi_1(z)\phi''(z)+\Psi_2''(z))f_k'(\phi(z))+(\Psi_1(z)\phi'^2(z) \\
&+2\Psi_2'(z)\phi'(z)+\Psi_2(z)\phi''(z))f_k''(\phi(z))
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \Psi_2(z)\phi'^2(z)f_k'''(\phi(z)) \mid \\
&\leq \quad K_1 \sup_{|\phi(z)|\leq\rho} \mid f_k(\phi(z)) \mid + K_2 \sup_{|\phi(z)|\leq\rho} \mid f_k'(\phi(z)) \mid \\
&+ \quad K_3 \sup_{|\phi(z)|\leq\rho} \mid f_k''(\phi(z)) \mid + K_4 \sup_{|\phi(z)|\leq\rho} \mid f_k'''(\phi(z)) \mid \\
&\leq C\varepsilon. \qquad (35)
\end{aligned}
$$

Moreover, by using Lemma 4, we obtain

$$
\begin{aligned}
\mid (T_{\Psi_1,\Psi_2,\phi}f_k)(0) \mid &\leq \mid \Psi_1(0)f_k(\phi(0)) + \Psi_2(0)f_k'(\phi(0)) \\
&+ \mid \Psi_1'(0)f_k(\phi(0)) \mid \leq C\varepsilon, \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
\mid (T_{\Psi_1,\Psi_2,\phi}f_k)'(0) \mid &\leq \mid (\Psi_1(0)\phi'(0) + \Psi_2'(0))f_k'(\phi(0)) \\
&+ \Psi_2(z)\phi'(0)f_k''(\phi(0))) \mid \leq C\varepsilon. \qquad (37)
\end{aligned}
$$

When $k > K_0$, from (32) to (37) and Lemma 4, we obtain

$$
\begin{aligned}
&\mid (T_{\Psi_1,\Psi_2,\phi}f_k)(0) \mid + \mid (T_{\Psi_1,\Psi_2,\phi}f_k)'(0) \mid \\
&+ \sup_{z\in\mathbb{D}} \mid (\nu(z))(T_{\Psi_1,\Psi_2,\phi}f_k)''(z) \mid \\
&\leq \big(\mid (T_{\Psi_1,\Psi_2,\phi}f_k)(0) \mid + \mid (T_{\Psi_1,\Psi_2,\phi}f_k)'(0) \mid \\
&+ \sup_{\phi(z)\leq\rho} \mid (\nu(z))(T_{\Psi_1,\Psi_2,\phi}f_k)''(z) \mid\big) \\
&+ \sup_{\rho<\phi(z)<1} \mid (\nu(z))(T_{\Psi_1,\Psi_2,\phi}f_k)''(z) \mid \\
&= C\varepsilon \\
&+ \sup_{\rho<\phi(z)<1} (\nu(z)) \mid (\Psi_1(z)f_k(\phi(z)) + \Psi_2(z)f_k'(\phi(z)))'' \mid \\
&= C\varepsilon \\
&+ \sup_{\rho<\phi(z)<1} (\nu(z)) \mid \Psi_1''(z)f_k(\phi(z)) + \Psi_1'(z)\phi'(z)f_k'(\phi) \\
&+ (\Psi_1'(z)\phi'(z) \\
&+ \Psi_1(z)\phi''(z) + \Psi_2''(z))f_k'(\phi(z)) + (\Psi_1(z)\phi'(z) \\
&+ \Psi_2'(z))\phi'(z)f_k''(\phi(z)) + \Psi_2'(z)\phi'(z)f_k''(\phi(z)) \\
&+ \Psi_2(z)\phi''(z)f_k''(\phi(z)) + \Psi_2(z)\phi'^2(z)f_k'''(\phi(z)) \mid \\
&= C\varepsilon + \sup_{\rho<\phi(z)<1} (\nu(z)) \mid \Psi_1''(z)f_k(\phi(z)) + (2\Psi_1'(z)\phi'(z) \\
&+ \Psi_1(z)\phi''(z) + \Psi_2''(z))f_k'(\phi(z)) + (\Psi_1(z)\phi'^2(z) \\
&+ 2\Psi_2'(z)\phi'(z) + \Psi_2(z)\phi''(z))f_k''(\phi(z)) \\
&+ \Psi_2(z)\phi'^2(z)f_k'''(\phi(z)) \mid \\
&\leq C\varepsilon + (\nu(z)) \mid \Psi_1''(z)f_k(\phi(z)) \mid + (\nu(z)) \mid (2\Psi_1'(z)\phi'(z) \\
&+ \Psi_1(z)\phi''(z) + \Psi_2''(z))f_k'(\phi(z)) \mid + (\nu(z)) \mid (\Psi_1(z)\phi'^2(z) \\
&+ 2\Psi_2'(z)\phi'(z) + \Psi_2(z)\phi''(z))f_k''(\phi(z)) \mid \\
&+ (\nu(z)) \mid \Psi_2(z)\phi'^2(z)f_k'''(\phi(z)) \mid \\
&\leq C\varepsilon \\
&+ C(\nu(z))\bigg[\mid \Psi''(z) \mid \\
&+ \frac{\mid 2\Psi_1'(z)\phi'(z) + \Psi_1(z)\phi''(z) + \Psi_2''(z) \mid}{(1- \mid \phi(z) \mid^2)} \\
&+ \frac{\mid \Psi_1(z)\phi'^2(z) + 2\Psi_2'(z)\phi'(z) + \Psi_2(z)\phi''(z) \mid}{(1- \mid \phi(z) \mid^2)^2} \\
&+ \frac{\mid \Psi_2(z)\phi'^2(z) \mid}{(1- \mid \phi(z) \mid^2)^3}\bigg] \parallel f_k \parallel_\infty \\
&\leq 5C\varepsilon. \qquad (38)
\end{aligned}
$$

From lemma 6, the operator $T_{\Psi_1,\Psi_2,\phi} : H^\infty \to \mathscr{Z}$ is compact.

$(a) \Rightarrow (b)$. The compactness of the operator $T_{\Psi_1,\Psi_2,\phi} : H^\infty \to \mathscr{Z}$ implies the boundedness of $T_{\Psi_1,\Psi_2,\phi} : H^\infty \to \mathscr{Z}$. If $\parallel \phi \parallel_\infty < 1$, the limit in (29) to (31) equals zero. Hence, let $\parallel \phi \parallel_\infty = 1$ and $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\phi(z_k)| \to 1$ as $k \to \infty$.

Thus, for a fixed $z_k \in \mathbb{D}$ and by using Lemma 7, we obtain

$$
\begin{aligned}
C &\geq \sup_{z_k\in\mathbb{D}} \mid (1- \mid z_k \mid^2)(T_{\Psi_1,\Psi_2,\phi}(g_1)_k)''(z_k) \mid \\
&= \sup_{z_k\in\mathbb{D}} (1- \mid z_k \mid^2) \mid (\Psi_1(z_k)(g_1)_k(\phi(z_k)) \\
&+ \Psi_2(z_k)(g_1)_k'(\phi(z_k)))'' \mid \\
&= \sup_{z_k\in\mathbb{D}} (1- \mid z_k \mid^2) \mid (\Psi_1'(z_k)(g_1)_k(\phi(z_k)) \\
&+ \Psi_1(z_k)\phi'(z_k)(g_1)_k'(\phi(z_k)) \\
&+ \Psi_2'(z_k)(g_1)_k'(\phi(z_k)) + \Psi_2(z_k)\phi'(z_k)(g_1)_k''(\phi(z_k)))' \mid \\
&= \sup_{z_k\in\mathbb{D}} (1- \mid z_k \mid^2) \mid (\Psi_1'(z_k)(g_1)_k(\phi(z_k)) \\
&+ (\Psi_1(z_k)\phi'(z_k) + \Psi_2'(z_k))(g_1)_k'(\phi(z_k)) \\
&+ \Psi_2(z_k)\phi'(z_k)(g_1)_k''(\phi(z_k)))' \mid \\
&= \sup_{z_k\in\mathbb{D}} (1- \mid z_k \mid^2) \mid \Psi_1''(z_k)(g_1)_k(\phi(z_k)) \\
&+ \Psi_1'(z_k)\phi'(z_k)(g_1)_k'(\phi(z_k)) + (\Psi_1'(z_k)\phi'(z_k) \\
&+ \Psi_1(z_k)\phi''(z_k) + \Psi_2''(z_k))(g_1)_k'(\phi(z_k)) + (\Psi_1(z_k)\phi'(z_k) \\
&+ \Psi_2'(z_k))\phi'(z_k)(g_1)_k''(\phi(z_k)) \\
&+ \Psi_2'(z_k)\phi'(z_k)(g_1)_k''(\phi(z_k)) \\
&+ \Psi_2(z_k)\phi''(z_k)(g_1)_k''(\phi(z_k)) \\
&+ \Psi_2(z_k)\phi'^2(z_k)(g_1)_k'''(\phi(z_k)) \mid \\
&= \sup_{z_k\in\mathbb{D}} (1- \mid z_k \mid^2) \mid \Psi_1''(z_k)(g_1)_k(\phi(z_k)) \\
&+ (2\Psi_1'(z_k)\phi'(z_k) + \Psi_1(z_k)\phi''(z_k) \\
&+ \Psi_2''(z_k))(g_1)_k'(\phi(z_k)) + (\Psi_1(z_k)\phi'^2(z_k) \\
&+ 2\Psi_2'(z_k)\phi'(z_k) + \Psi_2(z_k)\phi''(z_k))(g_1)_k''(\phi(z_k)) \\
&+ \Psi_2(z_k)\phi'^2(z_k)(g_1)_k'''(\phi(z_k)) \mid \\
&\geq \sup_{z_k\in\mathbb{D}} \bigg| \frac{C_1(1- \mid z_k \mid^2) \mid (2\Psi_1'(z_k)\phi'(z_k) + \Psi_1(z_k)\phi''(z_k)}{1- \mid \phi(z_k) \mid^2} \\
&+ \frac{\Psi_2''(z_k))\overline{\phi(z_k)}}{1- \mid \phi(z_k) \mid^2}\bigg|. \qquad (39)
\end{aligned}
$$

Since $(g_1)_k \to 0$ uniformly on $\mathbb{D}$, $(g_1)_k$ converges to zero uniformly on the compact subsets of $\mathbb{D}$. Therefore $(g_1)_k$ is bounded in H^∞, which converges to zero uniformly on compact subsets of $\mathbb{D}$:

$$\lim_{k\to\infty} \| T_{\Psi_1,\Psi_2,\phi}(g_1)_k \|_{\mathscr{Z}} \to 0, \tag{40}$$

$$\left| \frac{C_1(1-|z_k|^2)\,|(2\Psi_1'(z_k)\phi'(z_k)+\Psi_1(z_k)\phi''(z_k)}{1-|\phi(z_k)|^2} + \frac{\Psi_2''(z_k))\overline{\phi(z_k)}}{1-|\phi(z_k)|^2} \right| \leq \| T_{\Psi_1,\Psi_2,\phi}(g_1)_k \|_{\mathscr{Z}} \to 0 \quad as \quad k \to 0. \tag{41}$$

By (41) and $|\phi(zk)| \to 1$, we have

$$\lim_{k\to\infty} \frac{(\nu(z))\,|\,2\Psi_1'(z_k)\phi'(z_k)+\Psi_1(z_k)\phi''(z_k)+\Psi_2''(z_k)\,|}{(1-|\phi(z_k)|^2)} = 0. \tag{42}$$

It follows that condition (29) holds, as desired.

To prove (30), let a fixed $z_k \in \mathbb{D}$, and, by using Lemma 7, since $(g_2)_k \to 0$ uniformly on $\mathbb{D}$, $(g_2)_k$ converges to zero uniformly on the compact subsets of $\mathbb{D}$. Therefore $(g_2)_k$ is bounded in H^∞, which converges to zero uniformly on compact subsets of $\mathbb{D}$. Using Lemma 6, we obtain

$$\lim_{k\to\infty} \| T_{\Psi_1,\Psi_2,\phi}(g_2)_k \|_{\mathscr{Z}} \to 0, \tag{43}$$

$$\left| \frac{C_2(1-|z_k|^2)(\Psi_1(z_k)\phi'^2(z_k)+2\Psi_2'(z_k)\phi'(z_k))}{(1-|\phi(z_k)|^2)^2} + \frac{\Psi_2(z_k)\phi''(z_k))\overline{\phi(z_k)}^2}{(1-|\phi(z_k)|^2)^2} \right| \leq \| T_{\Psi_1,\Psi_2,\phi} g_k \|_{\mathscr{Z}} \to 0 \quad as \quad k \to 0. \tag{44}$$

By (44) and $|\phi(zk)| \to 1$, we have

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)\,|\,(\Psi_1(z_k)\phi'^2(z_k)+2\Psi_2'(z_k)\phi'(z_k)}{(1-|\phi(z_k)|^2)^2} + \frac{\Psi_2(z_k)\phi''(z_k))\,|}{(1-|\phi(z_k)|^2)^2} = 0. \tag{45}$$

It follows that condition (30) holds, as desired.

To prove (31), let a fixed $z_k \in \mathbb{D}$, and, by using Lemma 7, since $(g_3)_k \to 0$ uniformly on $\mathbb{D}$, $(g_3)_k$ converges to zero uniformly on the compact subsets of $\mathbb{D}$. Therefore $(g_3)_k$ is bounded in H^∞, which converges to zero uniformly on compact subsets of $\mathbb{D}$. Using Lemma 6, we obtain

$$\lim_{k\to\infty} \| T_{\Psi_1,\Psi_2,\phi}(g_3)_k \|_{\mathscr{Z}} \to 0, \tag{46}$$

$$\left| \frac{C_3(1-|z_k|^2)\,|\,(\Psi_2(z_k)\phi'^2(z_k))\,|\,\overline{\phi(z_k)}^3}{(1-|\phi(z_k)|^2)^3} \right| \leq \| T_{\Psi_1,\Psi_2,\phi}(g_3)_k \|_{\mathscr{Z}} \to 0 \quad as \quad k \to 0. \tag{47}$$

By (44) and $|\phi(zk)| \to 1$, we have

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)\,|\,(\Psi_2(z_k)\phi'^2(z_k))\,|}{(1-|\phi(z_k)|^2)^3} = 0. \tag{48}$$

It follows that condition (31) holds, as desired. That ends the proof of Theorem 2.

4 Applications

Operator theory on different spaces of analytic functions have been actively appearing in different areas of mathematical sciences like dynamical systems, theory of semigroups, isometries and quantum mechanics (see [30]).
Our results in this paper can be generalized and applied to some analytic and hyperbolic classes to obtain strong and new characterizations of several classes of functions.

5 Conclusion

In this paper, we characterized the boundedness and compactness of the new product operator $T_{\Psi_1,\Psi_2,\phi}$ from H^∞ to Zygmund spaces. Moreover, we proved that the properties of boundedness and compactness still hold for this operator from H^∞ to Zygmund spaces. In addition, we gave the conditions for the product operator $T_{\Psi_1,\Psi_2,\phi}$ to be bounded and compact.

Acknowledgement

We would like to thank the reviewers for helpful and constructive comments, which have made great contributions to the improvement of the paper.

Conflict of Interest

The authors declare that they have no conflict of interest

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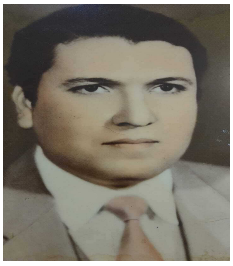

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