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Prior Preferences for the Inverse Power Lomax Distribution: Bayesian Method

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Abstract: In this study, we focus at Bayesian analysis of the unknown shape parameter of the Inverse Power Lomax distribution using various loss functions and different priors, such as Gamma, Power density, and Uniform. LINEX loss function, Entropy loss function, Weighted Balance loss function, and Minimum Expected loss function are the four loss functions considered. A simulation study is carried out using R to assess the behavior of Bayes estimates based on their MSE, and ultimately, a real-world data set is used to test the efficiency of the suggested estimators.

Keywords: Bayesian Estimation, LINEX loss function, Entropy loss function, Weighted Balance loss function and Minimum Expected loss function.

1 Introduction

In life testing, the Lomax distribution is one of the most often used probability distributions. A large number of problems in econometrics, biological sciences, survey sampling, engineering sciences, medical research, and life testing can be solved using the inverse distributions. Some researchers have explored the statistical inference of inverse distribution in recent years. Harris [1] models company failure data using the Lomax distribution for income and wealth. Because of its broad applicability, the Lomax distribution is widely employed in biological investigations and the distribution of computer file sizes. According to Rady et al. [2], the power Lomax (PL) distribution can be derived from the Lomax distribution by using the power transformation $Z = Y^{\frac{1}{\beta}}$, where Y follows the Lomax distribution.The PL distribution pdf is defined as follows:

$$f(z;\alpha,\lambda,\beta) = \alpha\beta\lambda^{\alpha}z^{\beta-1}\left(\lambda+z^{\beta}\right)^{-\alpha-1}; z,\alpha,\lambda,\beta > 0 \tag{1}$$

The pdf of inverse power Lomax (IPL) distribution can be obtained with the help of transformed variable Z by making use of inverse transformation $X = \frac{1}{z}$, which is defined as

$$f(x;\alpha,\lambda,\beta) = \frac{\alpha\beta}{\lambda}x^{-(\beta+1)}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha-1}; x,\alpha,\lambda,\beta > 0 \tag{2}$$

Where α, λ, β are shape, location and scale parameter respectively.

The cdf of the inverse power Lomax (IPL) distribution is defined as

$$F(x;\alpha,\lambda,\beta) = \left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha}; x,\alpha,\lambda,\beta > 0 \tag{3}$$

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The IPL distribution's reliability function is as follows:

$$R(x) = 1 - \left(1 + \frac{x^{-\beta}}{\lambda}\right)^{-\alpha} ; x, \alpha, \lambda, \beta > 0 \quad (4)$$

Prior knowledge about a model parameter is represented by a probability function in Bayesian inference. As a result, we must proceed cautiously when gathering past information. Prior information, in a broader sense, is a method of summarising the existing data. Because there is no single method for selecting a prior distribution, the effect may be minor, and there is always the potential of finding the optimal solution using distorted prior data. We utilise non-informative priors when there is very little explanatory information about the unknown parameter. However, if enough data on the parameter(s) is available, it is preferable to employ informative priors.

When studying scenarios with an actual non-monotonic failure rate, the IPL is particularly adaptable. As a result, the IPL model may be applied to many real-world data modelling and analysis tasks, as seen in Reference [3]. Hassan and Abd-Allah [3] investigated a IPL distribution's statistical properties to facilitate engineering applications. Hassan et al. [4] demonstrate the potential applicability of the shortened power Lomax model using flood data. The truncated power Lomax distribution can provide better fits than certain other similar distributions, according to this application. Schabe [5] studied Bayes estimates under asymmetric loss. Yadav et.al [6] studied Reliability estimation for inverse Lomax distribution under type-II censored data using Markov chain Monte Carlo method. E-Bayesian estimation of two-component mixture of inverse Lomax distribution based on type-I censoring scheme was studied by Othman and Reyad [7]. Dey and Sudhanu [8] presented a Bayesian estimate of the parameter of the Maxwell distribution under various loss functions. For Bayesian analysis, Kazmi et al. [9] compared the class of life time distributions.They used a variety of loss functions to examine the properties of Bayes parameter estimators. The priors for the expotentiated expotential distribution under several loss functions are compared by Afaq et al. [10].

In this study, we examine the Bayesian estimators of the parameter of the inverse power Lomax distribution using three different priors (Gamma, Power density, and Uniform) under four different loss functions (LINEX, Entropy, Weighted balance and Minimum Expected loss function). The likelihood function for the inverse power Lomax distribution and posterior distributions derived under various priors, are presented in section 2. In section 3, we developed Bayes estimates using different variety of loss functions and priors. Section 4 contains a simulation study. In section 5, we showed how real-life data can be used. Finally, the paper is concluded in section 6.

2 Priors and Posterior distributions

Let $x_1, x_2, ..., x_n$ be a random sample of size n taken from the Inverse Power Lomax distribution. Then the likelihood function for the given sample observations is

$$L(x; \alpha, \lambda, \beta) = \frac{\alpha^n \beta^n}{\lambda^n} \prod_{i=1}^{n} \frac{x_i^{(\beta+1)}}{\left(1 + \frac{x_i^{-\beta}}{\lambda}\right)} e^{-\alpha \sum_{i=1}^{n} log\left(1 + \frac{x_i^{-\beta}}{\lambda}\right)} \quad (5)$$

Bayesian estimation necessitates the selection of suitable priors for the parameters. To obtain the related posterior distributions, we use two non-informative priors and one informative prior in this study.

2.1 Posterior distribution under Gamma prior

The gamma distribution as a prior distribution of α with parameters a and b is given by

$$g_1(\alpha|a, b) = \frac{b^a \alpha^{a-1}}{\Gamma(a)} e^{-b\alpha}; \alpha, a, b > 0 \quad (6)$$

The posterior distribution for the parameter α is calculated using the likelihood function (5) and the prior (6) and is given by

$$P_1(\alpha|x) = \frac{L(x; \alpha, \lambda, \beta) * g_1(\alpha|a, b)}{\int_0^\infty L(x; \alpha, \lambda, \beta) * g_1(\alpha|a, b) d\alpha}$$

$$P_1(\alpha|x) = \frac{\frac{\alpha^n\beta^n}{\lambda^n}\prod_{i=1}^{n}\frac{x_i^{(\beta+1)}}{\left(1+\frac{x_i^{-\beta}}{\lambda}\right)}\exp\left[-\alpha\sum_{i=1}^{n}log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right]*\frac{b^a\alpha^{a-1}}{\Gamma(a)}\exp(-b\alpha)}{\int_0^\infty\frac{\alpha^n\beta^n}{\lambda^n}\prod_{i=1}^{n}\frac{x_i^{(\beta+1)}}{\left(1+\frac{x_i^{-\beta}}{\lambda}\right)}\exp\left[-\alpha\sum_{i=1}^{n}log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right]*\frac{b^a\alpha^{a-1}}{\Gamma(a)}\exp(-b\alpha)d\alpha}$$

$$= \frac{\alpha^{(n+a)-1}\exp\left[-\alpha\left(b+\sum_{i=1}^{n}log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right)\right]}{\int_0^\infty\alpha^{(n+a)-1}\exp\left[-\alpha\left(b+\sum_{i=1}^{n}log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right)\right]d\alpha}$$

$$= \frac{\alpha^{(n+a)-1}\exp\left[-\alpha\left(b+\sum_{i=1}^{n}log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right)\right]}{\frac{\Gamma n+a}{\left(b+\sum_{i=1}^{n}log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right)}^{n+a}}$$

$$P_1(\alpha|x) = \frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)}, \tag{7}$$

where $S = b+\sum_{i=1}^{n}log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)$

2.2 Posterior distribution under Power density function prior

The Power density function as a prior distribution of α with the rate parameter a. Then its pdf is given by:

$$g_2(\alpha|a) = a\alpha^{a-1}; \alpha > 0, a > 0 \tag{8}$$

The posterior distribution for the parameter α is calculated using the likelihood function (5) and the prior (8) and is given by

$$P_2(\alpha|x) = \frac{L(x;\alpha,\lambda,\beta)*g_2(\alpha|a)}{\int_0^\infty L(x;\alpha,\lambda,\beta)*g_2(\alpha|a)d\alpha}$$

$$P_2(\alpha|x) = \frac{\frac{\alpha^n\beta^n}{\lambda^n}\prod_{i=1}^{n}\frac{x_i^{(\beta+1)}}{\left(1+\frac{x_i^{-\beta}}{\lambda}\right)}\exp\left[-\alpha\sum_{i=1}^{n}log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right]*a\alpha^{a-1}}{\int_0^\infty\frac{\alpha^n\beta^n}{\lambda^n}\prod_{i=1}^{n}\frac{x_i^{(\beta+1)}}{\left(1+\frac{x_i^{-\beta}}{\lambda}\right)}\exp\left[-\alpha\sum_{i=1}^{n}log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right]*a\alpha^{a-1}}$$

$$= \frac{\alpha^{(n+a)-1}\exp\left[-\alpha\left(\sum_{i=1}^{n}log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right)\right]}{\int_0^\infty\alpha^{(n+a)-1}\exp\left[-\alpha\left(\sum_{i=1}^{n}log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right)\right]d\alpha}$$

$$= \frac{\alpha^{(n+a)-1}\exp\left[-\alpha\left(\sum_{i=1}^{n}log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right)\right]}{\frac{\Gamma n+a}{\left(\sum_{i=1}^{n}log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right)}^{n+a}}$$

$$P_2(\alpha|x) = \frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)}, \tag{9}$$

where $R = \sum_{i=1}^{n}log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)$

2.3 Posterior distribution under Uniform prior

The Uniform distribution as a prior distribution of α is given by:

$$g_3(\alpha) \propto 1; 0 < \alpha < 1 \tag{10}$$

The posterior distribution for the parameter α is calculated using the likelihood function (5) and the prior (10) and is given by

$$P_3(\alpha|x) = \frac{L(x;\alpha,\lambda,\beta) * g_3(\alpha)}{\int_0^\infty L(x;\alpha,\lambda,\beta) * g_3(\alpha) d\alpha}$$

$$P_3(\alpha|x) = \frac{\frac{\alpha^n \beta^n}{\lambda^n} \prod_{i=1}^n \frac{x_i^{(\beta+1)}}{\left(1+\frac{x_i^{-\beta}}{\lambda}\right)} \exp\left[-\alpha \sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right] * 1}{\int_0^\infty \frac{\alpha^n \beta^n}{\lambda^n} \prod_{i=1}^n \frac{x_i^{(\beta+1)}}{\left(1+\frac{x_i^{-\beta}}{\lambda}\right)} \exp\left[-\alpha \sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right] * 1 d\alpha}$$

$$= \frac{\alpha^n \exp\left[-\alpha\left(\sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right)\right]}{\int_0^\infty \alpha^n \exp\left[-\alpha\left(\sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right)\right] d\alpha}$$

$$= \frac{\alpha^n \exp\left[-\alpha\left(\sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right)\right]}{\frac{\Gamma n+1}{\left(\sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right)}^{n+1}}$$

$$P_3(\alpha|x) = \frac{T^{n+1} \alpha^n \exp(-\alpha T)}{\Gamma(n+1)}, \tag{11}$$

where $T = \sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)$

3 Bayesian estimation under Different Loss Functions

In this part, we used several priors to derive Bayesian estimates for various loss functions. Four loss functions are examined while estimating Bayes estimates: LINEX loss function, Entropy loss function, Weighted balanced loss function, and Minimum expected loss function.

3.1 Bayesian Estimation by using Gamma prior under different loss functions

3.1.1 Bayes Estimator under LINEX Loss Function

The Linex loss function which is asymmetric, was introduced by Varian [11] and is given by

$$L(\hat{\alpha}, \alpha) = \exp(q_1(\hat{\alpha} - \alpha)) - c(\hat{\alpha} - \alpha) - 1; q_1, c \neq 0 \tag{12}$$

By using Linex loss function as given in (12), the risk function is given by

$$R(\hat{\alpha}, \alpha) = E[L(\hat{\alpha}, \alpha)] = \int_0^\infty \exp(q_1(\hat{\alpha} - \alpha) - c(\hat{\alpha} - \alpha) - 1).P_1(\alpha|x)$$

$$= \int_0^\infty \left[\exp(q_1 \hat{\alpha}).\exp(-q_1 \alpha) - c\hat{\alpha} + c\alpha - 1\right].\frac{S^{n+a} \alpha^{(n+a)-1} \exp(-\alpha S)}{\Gamma(n+a)} d\alpha$$

$$= \int_0^\infty \exp(q_1 \hat{\alpha}).\exp(-q_1\alpha) \frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)} d\alpha + \int_0^\infty c\alpha \frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)} d\alpha - (c\hat{\alpha}+1)\int_0^\infty \frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)} d\alpha$$

$$= \int_0^\infty \exp(q_1 \hat{\alpha}) \frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S + q_1)}{\Gamma(n+a)} d\alpha - (c\hat{\alpha}+1) + c\int_0^\infty \frac{S^{n+a}}{\Gamma(n+a)} \alpha^{(n+a+1)-1} \exp(-\alpha S) d\alpha$$

$$= \exp(q_1 \hat{\alpha}) \frac{S^{n+a}}{[S+q_1]^{n+a}} + \frac{c}{S}(n+a) - (c\hat{\alpha}+1)$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}} = 0$, we get

$$\exp(q_1 \hat{\alpha}) = \frac{c}{q_1}\left(\frac{S+q_1}{S}\right)^{n+a}$$

Taking log on both sides, we obtain the Bayes estimator as

$$\hat{\alpha}_{BL} = \frac{1}{q_1}\left[log\left(\frac{c}{q_1}\right) + (n+a)log\left(\frac{S+q_1}{S}\right)\right] \tag{13}$$

3.1.2 Bayes Estimator under Entropy Loss Function

Dey et al. [12] defined the Entropy loss function (ELF) of the form:

$$L(\alpha) = \left[\left(\frac{\hat{\alpha}}{\alpha}\right) - \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right] \tag{14}$$

where $\hat{\alpha}$ is an estimator of α.
By using Entropy loss function as given in (14), the risk function is given by

$$R(\hat{\alpha},\alpha) = E[L(\hat{\alpha},\alpha)] = \int_0^\infty \left[\left(\frac{\hat{\alpha}}{\alpha}\right) - \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right].P_1(\alpha|x)$$

$$= \int_0^\infty \left[\left(\frac{\hat{\alpha}}{\alpha}\right) - \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right] \frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)} d\alpha$$

$$= \int_0^\infty \left(\frac{\hat{\alpha}}{\alpha}\right) \frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)} d\alpha - \int_0^\infty \log(\hat{\alpha}) \frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)} d\alpha + \int_0^\infty \log(\alpha) \frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)} - \int_0^\infty \frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)} d\alpha$$

$$= \hat{\alpha}.\frac{S}{n+a-1} - \log(\hat{\alpha}) + \frac{\psi(n+a)}{\Gamma n+a} - 1$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}} = 0$, we get

$$\Longrightarrow \frac{S}{n+a-1} - \frac{1}{\hat{\alpha}} = 0$$

$$\Longrightarrow \hat{\alpha_{ELF}} = \frac{n+a-1}{S} \tag{15}$$

3.1.3 Bayes Estimator under Weighted Balance Loss Function

The weighted Balanced loss function (WBLF) used by Nasir and Aslam [13] is given as

$$L(\alpha)=\left(\frac{\alpha-\hat{\alpha}}{\hat{\alpha}}\right)^2 \tag{16}$$

where $\hat{\alpha}$ is an estimator of α .
By using Weighted balance loss function as given in (16), the risk function is given by

$$R(\hat{\alpha},\alpha)=E[L(\hat{\alpha},\alpha)]=\int_0^\infty\left(\frac{\alpha-\hat{\alpha}}{\hat{\alpha}}\right)^2.P_1(\alpha|x)$$

$$=\int_0^\infty\left[\left(\frac{\alpha^2}{\hat{\alpha}^2}\right)+1-2\left(\frac{\alpha}{\hat{\alpha}}\right)\right].\frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)}d\alpha$$

$$=\int_0^\infty\left(\frac{\alpha^2}{\hat{\alpha}^2}\right)\frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)}d\alpha-\frac{2}{\hat{\alpha}}\int_0^\infty(\alpha)\frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)}d\alpha+\int_0^\infty\frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)}d\alpha$$

$$=\frac{1}{\hat{\alpha}^2}\frac{(n+a)(n+a+1)}{S^2}-\frac{2}{\hat{\alpha}}\frac{(n+a)}{S}+1$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}}=0$, we get

$$\Longrightarrow\frac{-2}{\hat{\alpha}^3}\frac{(n+a)(n+a+1)}{S^2}+\frac{2}{\hat{\alpha}^2}\frac{(n+a)}{S}=0$$

$$\Longrightarrow\hat{\alpha}_{WBlF}=\frac{(n+a+1)}{S} \tag{17}$$

3.1.4 Bayes Estimator under Minimum Expected Loss Function

Tummala and Sathe [14] defined the minimum expected loss function (MELF) as follows:

$$L(\alpha)=\frac{(\hat{\alpha}-\alpha)^2}{\alpha^2} \tag{18}$$

where $\hat{\alpha}$ is an estimator of α .
By using Minimum expected loss function as given in (18), the risk function is given by

$$R(\hat{\alpha},\alpha)=E[L(\hat{\alpha},\alpha)]=\int_0^\infty\frac{(\hat{\alpha}-\alpha)^2}{\alpha^2}.P_1(\alpha|x)$$

$$=\int_0^\infty\left[\left(\frac{\hat{\alpha}^2}{\alpha^2}\right)-2\left(\frac{\hat{\alpha}}{\alpha}\right)+1\right].\frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)}d\alpha$$

$$=\int_0^\infty\left(\frac{\hat{\alpha}^2}{\alpha^2}\right)\frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)}d\alpha-2\hat{\alpha}\int_0^\infty\left(\frac{1}{\alpha}\right)\frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)}d\alpha+\int_0^\infty\frac{S^{n+a}\alpha^{(n+a)-1}\exp(-\alpha S)}{\Gamma(n+a)}d\alpha$$

$$=\hat{\alpha}^2\frac{S^2}{(n+a-1)(n+a-2)}-2\hat{\alpha}\frac{S}{(n+a-1)}+1$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}}=0$, we get

$$\Longrightarrow \frac{2\hat{\alpha}S^2}{(n+a-1)(n+a-2)}-\frac{2S}{(n+a-1)}=0$$

$$\Longrightarrow \hat{\alpha}_{MELF}=\frac{(n+a-2)}{S} \tag{19}$$

3.2 Bayesian Estimation by using Power density function prior under different loss functions

3.2.1 Bayes Estimator under LINEX Loss Function

By using Linex loss function as given in (12) and the Posterior distribution as given in (9), the risk function is given by

$$R(\hat{\alpha},\alpha)=E[L(\hat{\alpha},\alpha)]=\int_0^\infty \exp(q_1(\hat{\alpha}-\alpha)-c(\hat{\alpha}-\alpha)-1).P_2(\alpha|x)$$

$$=\int_0^\infty \left[\exp(q_1\hat{\alpha}).\exp(-q_1\alpha)-c\hat{\alpha}+c\alpha-1\right].\frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)}d\alpha$$

$$=\exp(q_1\hat{\alpha})\int_0^\infty .\exp(-q_1\alpha)\frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)}d\alpha+\int_0^\infty c\alpha\frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)} d\alpha-(c\hat{\alpha}+1)\int_0^\infty \frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)}d\alpha$$

$$=\exp(q_1\hat{\alpha})\frac{R^{n+a}}{[R+q_1]^{n+a}}+\frac{c}{R}(n+a)-(c\hat{\alpha}+1)$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}}=0$, we get

$$\exp(q_1\hat{\alpha})=\frac{c}{q_1}\left(\frac{R+q_1}{R}\right)^{n+a}$$

Taking log on both sides, we obtain the Bayes estimator as

$$\hat{\alpha}_{BL}=\frac{1}{q_1}\left[log\left(\frac{c}{q_1}\right)+(n+a)log\left(\frac{R+q_1}{R}\right)\right] \tag{20}$$

3.2.2 Bayes Estimator under Entropy Loss Function

By using LINEX loss function as given in (14) and the Posterior distribution as given in (9), the risk function is given by

$$R(\hat{\alpha},\alpha)=E[L(\hat{\alpha},\alpha)]=\int_0^\infty \left[\left(\frac{\hat{\alpha}}{\alpha}\right)-\log\left(\frac{\hat{\alpha}}{\alpha}\right)-1\right].P_2(\alpha|x)$$

$$=\int_0^\infty \left[\left(\frac{\hat{\alpha}}{\alpha}\right)-\log\left(\frac{\hat{\alpha}}{\alpha}\right)-1\right]\frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)}d\alpha$$

$$=\int_0^\infty \left(\frac{\hat{\alpha}}{\alpha}\right)\frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)}d\alpha-\int_0^\infty \log(\hat{\alpha})\frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)}d\alpha +\int_0^\infty \log(\alpha)\frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)}d\alpha-\int_0^\infty \frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)}d\alpha$$

$$= \int_0^\infty (\hat{\alpha}) \frac{R^{n+a}\alpha^{(n+a-1)-1}\exp(-\alpha R)}{\Gamma(n+a)} d\alpha - \log\hat{\alpha} \int_0^\infty \frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)} d\alpha + \frac{R^{n+a}}{\Gamma(n+a)} \int_0^\infty \log(\alpha).\exp(-\alpha R).\alpha^{(n+a)-1} d\alpha - 1$$

$$= \hat{\alpha}.\frac{R}{n+a-1} - \log\hat{\alpha} + \frac{\psi(n+a)}{\Gamma n+a} - 1$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we get

$$\frac{R}{n+a-1} - \frac{1}{\hat{\alpha}} = 0$$

$$\alpha_{\hat{ELF}} = \frac{n+a-1}{R} \tag{21}$$

3.2.3 Bayes Estimator under Weighted Balance Loss Function

By using Weighted Balance loss function as given in (16) and the Posterior distribution as given in (9), the risk function is given by

$$R(\hat{\alpha},\alpha) = E[L(\hat{\alpha},\alpha)] = \int_0^\infty \left(\frac{\alpha-\hat{\alpha}}{\hat{\alpha}}\right)^2 .P_2(\alpha|x)$$

$$= \int_0^\infty \left[\left(\frac{\alpha^2}{\hat{\alpha}^2}\right) + 1 - 2\left(\frac{\alpha}{\hat{\alpha}}\right)\right] . \frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)} d\alpha$$

$$= \int_0^\infty \left(\frac{\alpha^2}{\hat{\alpha}^2}\right) \frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)} d\alpha - \frac{2}{\hat{\alpha}} \int_0^\infty (\alpha) \frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)} d\alpha + \int_0^\infty \frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)} d\alpha$$

$$= \frac{1}{\hat{\alpha}^2} \int_0^\infty \frac{R^{n+a}\alpha^{(n+a+2)-1}\exp(-\alpha R)}{\Gamma(n+a)} d\alpha - \frac{2}{\hat{\alpha}} \int_0^\infty \frac{R^{n+a}\alpha^{(n+a+1)-1}\exp(-\alpha R)}{\Gamma(n+a)} d\alpha + 1$$

$$= \frac{1}{\hat{\alpha}^2} \frac{(n+a)(n+a+1)}{R^2} - \frac{2}{\hat{\alpha}} \frac{(n+a)}{R} + 1$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial\hat{\alpha}} = 0$, we get

$$\implies \frac{-2}{\hat{\alpha}^3} \frac{(n+a)(n+a+1)}{R^2} + \frac{2}{\hat{\alpha}^2} \frac{(n+a)}{R} = 0$$

$$\implies \hat{\alpha}_{WBlF} = \frac{(n+a+1)}{R} \tag{22}$$

3.2.4 Bayes Estimator under Minimum Expected Loss Function

By using Minimum Expected loss function as given in (18) and the posterior distribution as given by (9), the risk function is given by

$$R(\hat{\alpha},\alpha) = E[L(\hat{\alpha},\alpha)] = \int_0^\infty \frac{(\hat{\alpha}-\alpha)^2}{\alpha^2} .P_2(\alpha|x)$$

$$= \int_0^\infty \left[\left(\frac{\hat{\alpha}^2}{\alpha^2}\right) - 2\left(\frac{\hat{\alpha}}{\alpha}\right) + 1\right] . \frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)} d\alpha$$

$$= \int_0^\infty \left(\frac{\hat{\alpha}^2}{\alpha^2}\right) \frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)} d\alpha - 2\hat{\alpha} \int_0^\infty \left(\frac{1}{\alpha}\right) \frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)} d\alpha + \int_0^\infty \frac{R^{n+a}\alpha^{(n+a)-1}\exp(-\alpha R)}{\Gamma(n+a)} d\alpha$$

$$= \hat{\alpha}^2 \frac{R^2}{(n+a-1)(n+a-2)} - 2\hat{\alpha} \frac{R}{(n+a-1)} + 1$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}} = 0$, we get

$$\implies \frac{2\hat{\alpha}R^2}{(n+a-1)(n+a-2)} - \frac{2R}{(n+a-1)} = 0$$

$$\implies \hat{\alpha}_{MELF} = \frac{(n+a-2)}{R} \tag{23}$$

3.3 Bayesian Estimation by using Uniform prior under different loss functions

3.3.1 Bayes Estimator under LINEX Loss Function

By using LINEX loss function as given in (12) and the Posterior distribution as given in (10), the risk function is given by

$$R(\hat{\alpha},\alpha) = E[L(\hat{\alpha},\alpha)] = \int_0^\infty \exp(q_1(\hat{\alpha}-\alpha) - c(\hat{\alpha}-\alpha) - 1).P_3(\alpha|x)$$

$$= \int_0^\infty \left[\exp(q_1\hat{\alpha}).\exp(-q_1\alpha) - c\hat{\alpha} + c\alpha - 1\right]. \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha$$

$$= \exp(q_1\hat{\alpha}) \int_0^\infty .\exp(-q_1\alpha) \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha + \int_0^\infty c\alpha \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha - (c\hat{\alpha}+1) \int_0^\infty \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha$$

$$= \exp(q_1\hat{\alpha}) \left[\frac{T}{T+q_1}\right]^{n+1} + c\frac{(n+1)}{T} - (c\hat{\alpha}+1)$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}} = 0$, we get

$$\exp(q_1\hat{\alpha}) = \frac{c}{q_1}\left(\frac{T+q_1}{T}\right)^{n+1}$$

Taking log on both sides, we obtain the Bayes estimator as

$$\hat{\alpha}_{BL} = \frac{1}{q_1}\left[log\left(\frac{c}{q_1}\right) + (n+1)log\left(\frac{T+q_1}{T}\right)\right] \tag{24}$$

3.3.2 Bayes Estimator under Entropy Loss Function

By using Linex loss function as given in (14) and the Posterior distribution as given in (10), the risk function is given by

$$R(\hat{\alpha},\alpha) = E[L(\hat{\alpha},\alpha)] = \int_0^\infty \left[\left(\frac{\hat{\alpha}}{\alpha}\right) - \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right].P_3(\alpha|x)$$

$$= \int_0^\infty \left[\left(\frac{\hat{\alpha}}{\alpha}\right) - \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right] \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha$$

$$= \int_0^\infty \left(\frac{\hat{\alpha}}{\alpha}\right) \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha - \int_0^\infty \log(\hat{\alpha}) \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha$$
$$+ \int_0^\infty \log(\alpha) \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha - \int_0^\infty \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha$$

$$= (\hat{\alpha}) \frac{T^{n+1}}{\Gamma(n+1)} \int_0^\infty \alpha^{n-1} \exp(-\alpha T) d\alpha - \log\hat{\alpha} \int_0^\infty \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha$$
$$+ \frac{T^{n+1}}{\Gamma(n+1)} \int_0^\infty \log(\alpha). \exp(-\alpha T). \alpha^{(n+1)-1} d\alpha - 1$$

$$= \hat{\alpha}. \frac{T}{n\Gamma n}. \Gamma n - \log(\hat{\alpha}) + \frac{\psi(n+1)}{\Gamma n+1} - 1$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}} = 0$, we get

$$\Longrightarrow \frac{T}{n} - \frac{1}{\hat{\alpha}} = 0$$
$$\Longrightarrow \hat{\alpha_{ELF}} = \frac{n}{T} \tag{25}$$

3.3.3 Bayes Estimator under Weighted Balance Loss Function

By using Weighted Balance loss function as given in (16) and the Posterior distribution as given in (10), the risk function is given by

$$R(\hat{\alpha}, \alpha) = E[L(\hat{\alpha}, \alpha)] = \int_0^\infty \left(\frac{\alpha - \hat{\alpha}}{\hat{\alpha}}\right)^2 .P_3(\alpha|x)$$
$$= \int_0^\infty \left[\left(\frac{\alpha^2}{\hat{\alpha}^2}\right) + 1 - 2\left(\frac{\alpha}{\hat{\alpha}}\right)\right]. \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha$$
$$= \int_0^\infty \left(\frac{\alpha^2}{\hat{\alpha}^2}\right) \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha - \frac{2}{\hat{\alpha}} \int_0^\infty (\alpha) \frac{T^{n+1}\alpha^{n+1} \exp(-\alpha T)}{\Gamma(n+1)} d\alpha + \int_0^\infty \frac{T^{n+1}\alpha^{n+1} \exp(-\alpha T)}{\Gamma(n+1)} d\alpha$$
$$= \frac{1}{\hat{\alpha}^2} \frac{(n+1)(n+1)}{T^2} - \frac{2}{\hat{\alpha}} \frac{(n+1)}{T} + 1$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}} = 0$, we get

$$\Longrightarrow \frac{-2}{\hat{\alpha}^3} \frac{(n+1)(n+2)}{T^2} + \frac{2}{\hat{\alpha}^2} \frac{(n+1)}{T} = 0$$
$$\Longrightarrow \hat{\alpha}_{WBlF} = \frac{(n+2)}{T} \tag{26}$$

3.3.4 Bayes Estimator under Minimum Expected Loss Function

By using Minimum Expected loss function as given in (18) and the posterior distribution as given by (10), the risk function is given by

$$R(\hat{\alpha}, \alpha) = E[L(\hat{\alpha}, \alpha)] = \int_0^\infty \frac{(\hat{\alpha} - \alpha)^2}{\alpha^2} .P_3(\alpha|x)$$
$$= \int_0^\infty \left[\left(\frac{\hat{\alpha}^2}{\alpha^2}\right) - 2\left(\frac{\hat{\alpha}}{\alpha}\right) + 1\right]. \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha$$
$$= \int_0^\infty \left(\frac{\hat{\alpha}^2}{\alpha^2}\right) \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha - 2\hat{\alpha} \int_0^\infty \left(\frac{1}{\alpha}\right) \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha + \int_0^\infty \frac{T^{n+1}\alpha^n \exp(-\alpha T)}{\Gamma(n+1)} d\alpha$$

$$= \hat{\alpha}^2 \frac{T^2}{n(n-1)} - 2\hat{\alpha}\frac{T}{(n)} + 1$$

Now solving $\frac{\partial R(\hat{\alpha},\alpha)}{\partial \hat{\alpha}} = 0$, we get

$$\implies 2\hat{\alpha}\frac{T^2}{n(n-1)} = 2.\frac{T}{n}$$

$$\implies \hat{\alpha}_{MELF} = \frac{(n-1)}{T} \quad (27)$$

4 Simulation Study

A simulation study using R software was carried out to assess and compare the performance of the estimates under various priors and loss functions. The algorithm used is as follows:

1. Choose a sample size of $n = 20, 60, 80, 100, 120, 150, 200$ to represent small, medium and large samples from inverse power Lomax distribution using Quantile function i.e.exp$(\frac{-1}{\alpha}.\log(u) - 1)^{(\frac{-1}{\beta})}$, where $u \sim U(0,1)$.

2. The parameters of the inverse power Lomax distribution are fixed at $\alpha = 1.5, \beta = 0.5, \lambda = 1$.

3. Generate the values of hyper parameters a and b randomly from uniform and gamma distribution respectively.

4. Iterate 1000 times and value of parameter was calculated under each prior.

The results obtained are represented in the tables below:

Table 1: Mean Square Error of α under different priors and loss functions

n	Gamma prior				Power density prior				Uniform prior			
	$\hat{\alpha}_{BL}$	$\hat{\alpha}_{ELF}$	$\hat{\alpha}_{WBLF}$	$\hat{\alpha}_{MELF}$	$\hat{\alpha}_{BL}$	$\hat{\alpha}_{ELF}$	$\hat{\alpha}_{WBLF}$	$\hat{\alpha}_{MELF}$	$\hat{\alpha}_{BL}$	$\hat{\alpha}_{ELF}$	$\hat{\alpha}_{WBLF}$	$\hat{\alpha}_{MELF}$
20	0.2785	0.2278	0.4626	0.1413	0.5631	0.4906	0.8577	0.3453	0.6485	0.5729	0.9654	0.4148
40	0.1557	0.1366	0.2155	0.1038	0.2377	0.2139	0.3156	0.1704	0.2622	0.2374	0.3442	0.1915
60	0.1320	0.1202	0.1671	0.0997	0.1786	0.1648	0.2209	0.1398	0.1923	0.1780	0.2362	0.1520
80	0.1245	0.1159	0.1495	0.1006	0.1572	0.1475	0.1861	0.1298	0.1667	0.1567	0.1965	0.1385
100	0.1095	0.1030	0.1279	0.0916	0.1332	0.1260	0.1539	0.1131	0.1400	0.1327	0.1613	0.1195
120	0.0862	0.0814	0.0994	0.0731	0.1029	0.0976	0.1175	0.0884	0.1078	0.1024	0.1228	0.0930
150	0.0542	0.0512	0.0622	0.0461	0.0640	0.0607	0.0728	0.0551	0.0670	0.0636	0.0760	0.0579
200	0.0270	0.0254	0.0310	0.0228	0.0318	0.0301	0.0362	0.0272	0.0333	0.0315	0.0378	0.0286

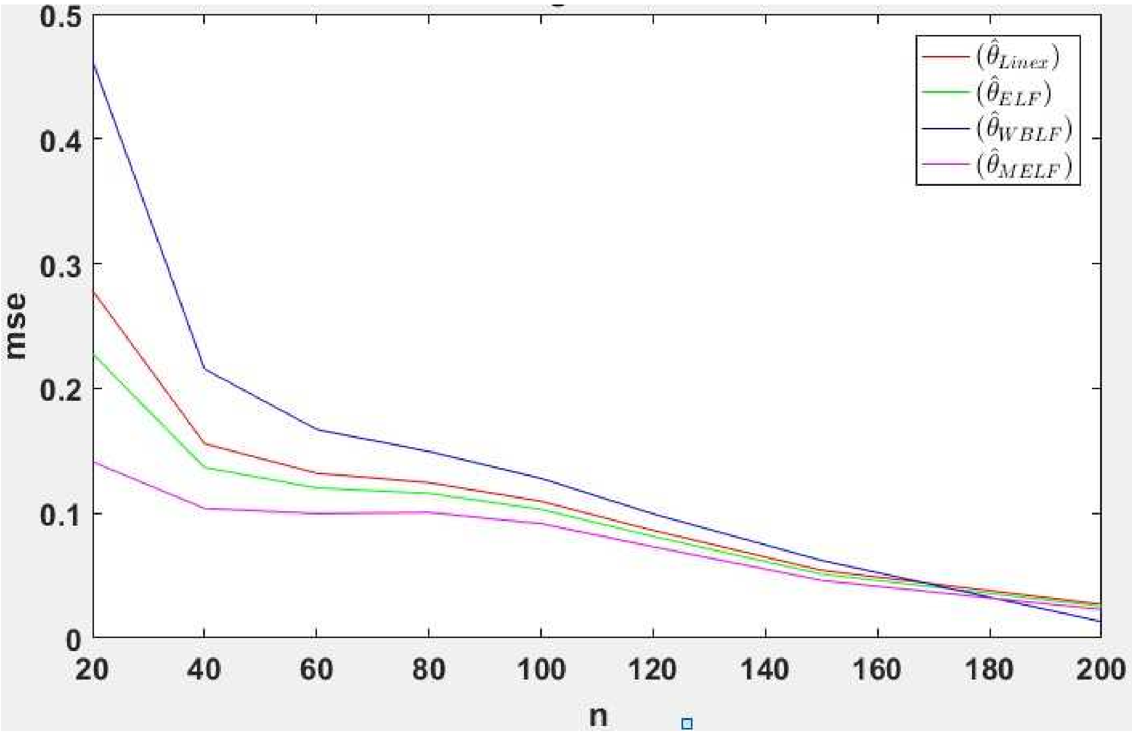

Fig. 1: Mean Square Error of α under Gamma prior

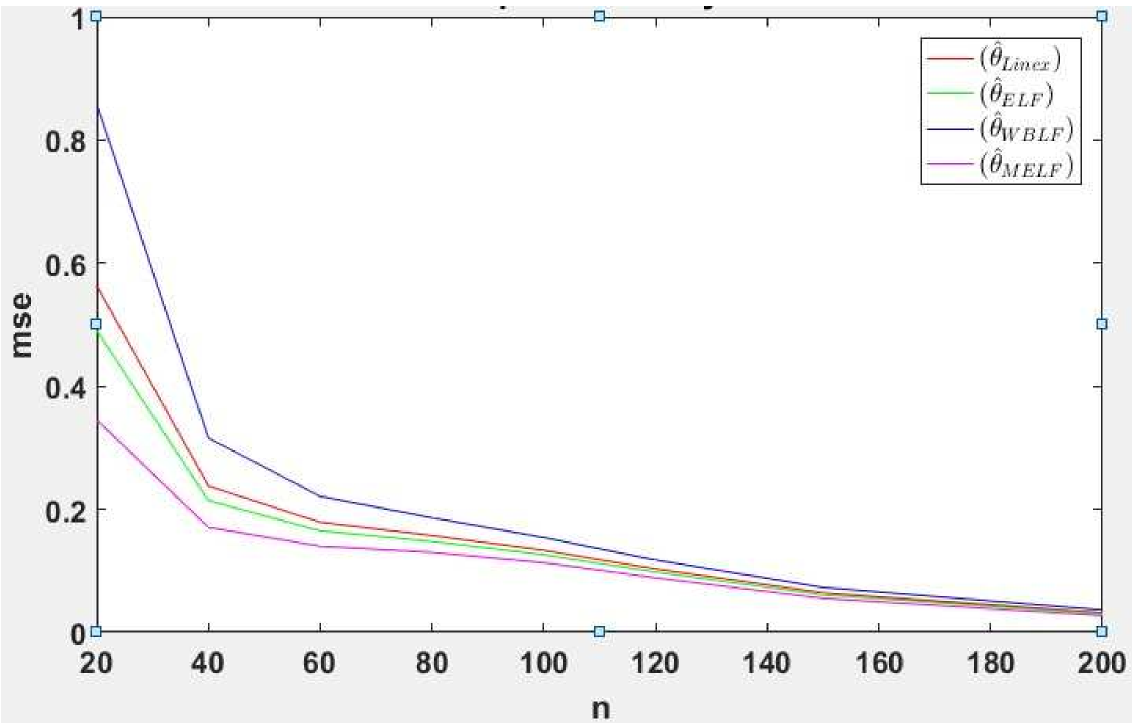

Fig. 2: Mean Square Error of α under Power density prior

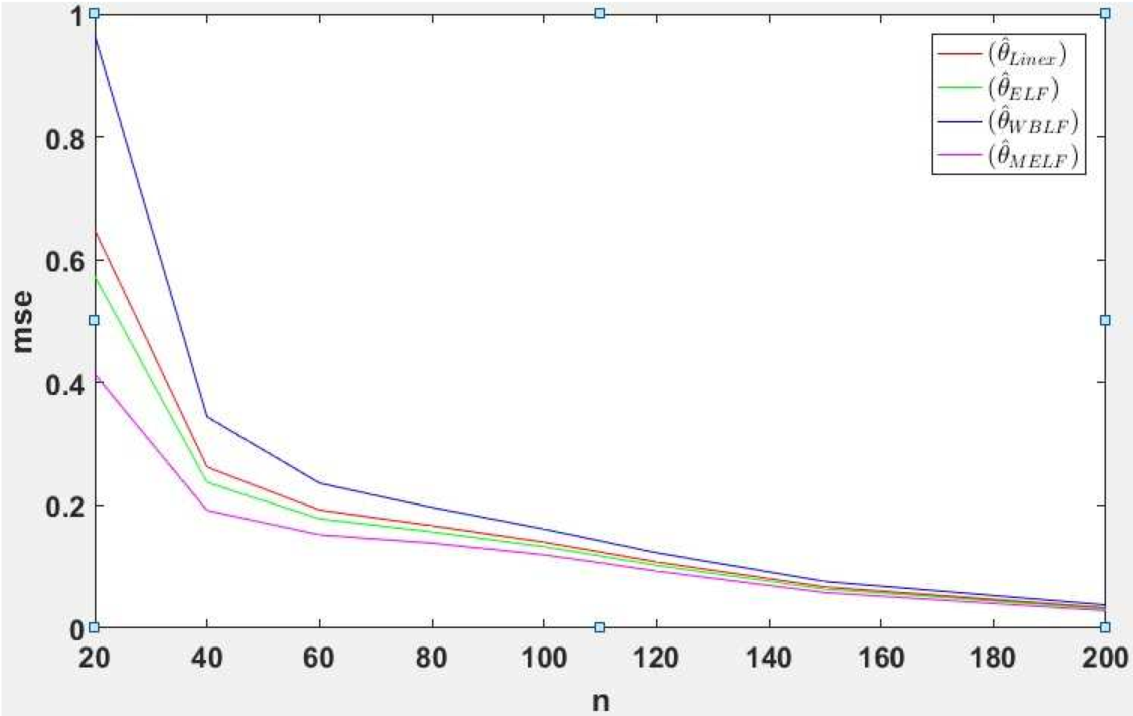

Fig. 3: Mean Square Error of α under Uniform prior

5 Real Data Analysis

A real life data set is used to illustrate the proposed method in this section. The survival periods (in days) of 72 guinea pigs infected with virulent tubercle bacilli are represented in this data set. Bjerkedal [15] observed and reported this data collection.The data are listed as follows:
0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Hassan and Abd-Allah [3] used the above data set to fit the inverse power Lomax distribution. By using non-informative priors i.e. Power density and Uniform prior and informative i.e Gamma prior under different Loss functions i.e. LINEX loss function, Entropy loss function, Weighted Balance loss function and Minimum Expected loss function, the Bayes estimates and Posterior variance of the posterior distribution are as follows where posterior variances are in parentheses.

6 Conclusion

We compared mean squared error under various loss functions using non-informative and informative priors in Table 1, and observed that the informative prior (Gamma prior) gives lower mean squared error, making it a suitable for the inverse Power function.In comparison to all the loss functions studied here, the Minimum Expected loss function is better because the mean square error is lower. As the sample size grows, the mean square error decreases.

Also, from Real data analysis (Table 2), When the Bayes posterior variances of several loss functions are evaluated, the Minimum Expected loss function has the lowest Bayes posterior variance. As a result, we find that the Minimum Expected Loss Function is the preferable alternative.

Table 2: Bayes estimates and Posterior variances

Prior	LINEX	ELF	WBLF	MELF
Gamma	1.6014	1.5881	1.6325	1.5659
	(0.00014)	(0.00010)	(0.00024)	(0.000006)
Power density	1.63763	1.6241	1.6696	1.6014
	(0.00026)	(0.00021)	(0.00040)	(0.00014)
Uniform	1.6489	1.6355	1.6809	1.6128
	(0.00031)	(0.00025)	(0.00046)	(0.00017)

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References

[1] C. Harris, The pareto distribution as a queue service discipline, *Oper Res.*, **16**, 307–313 (1968).
[2] E. A. Rady, W. A. Hassanein, and T. A. Elhaddad, The power lomax distribution with an application to bladder cancer data, *SpringerPlus*, **5(1)**, 1838 (2016).
[3] A. S. Hassan and M. Abd-Allah, On the inverse power lomax distribution, *Ann. Data Sci.*, **6**, 259–278 (2019).
[4] A. S. Hassan, A. H. S. Mohamed, and A. Elsehetry, Truncated power lomax distribution with application to flood data, *Journal of Statistics Applications and Probability*, **9(2)**, 347–359 (2020).
[5] H. Schabe, Bayes estimates under asymmetric loss, *IEEE Trans. Reliab.*, **40(1)**, 63–67 (1991).
[6] S. K. Singh, U. Singh, and A. S. Yadav, Reliability estimation for inverse lomax distribution under type ii censored data using markov chain monte carlo method, *International Journal of Mathematics and Statistics*, **17(1)**, 128–146 (2016).
[7] H. M. Reyad and S. A. Othman, E-bayesian estimation of two-component mixture of inverse lomax distribution based on type-i censoring scheme, *Journal of Advances in Mathematics and Computer Science*, **26(2)**, 1–22 (2018).
[8] S. Dey and S. S. Maiti, Bayesian estimation of the parameter of maxwell distribution under different loss functions, *J. Stat. Theory Pract.*, **4**, 279–287 (2010).
[9] A. M. S. Kazmi, M. Aslam, and S. Ali, Preference of prior for the class of life-times distributions under different loss functions, *Pakistan Journal of Statistics*, **28(4)**, 467–487 (2012).
[10] A. Afaq. , S. P. Ahmad, and A. Ahmed, Preference of priors for the expotentiated expotential distribution under different loss functions, *International Journal of Modern Mathematical Sciences*, **13(3)**, 307–321 (2015).
[11] H. R. Varian, A bayesian approach to real estate assessment, *Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage*, 195–208 (1975).
[12] D. Dey, M. Gosh, and C. Srinivasan, Simultaneous estimation of parameter under entropy loss, *Journal of Statistical Planning and Inference*, **15**, 347–363 (1987).
[13] W. Nasir and M. Aslam, Bayes approach to study shape parameter of frechet distribution, *Journal of Basic Applied Sciences*, **4(3)**, 246–254 (2015).
[14] V. M. R. Tummala and P. T. Sathe, Minimum exprcted loss estimators of reliability and parameters of certain lifetime distributions, *Reliability, IEEE Transactions*, **27(4)**, 283–285 (1978).
[15] T. Bjerkedal, Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacill, *Am. J. Epidemiol.*, **72**, 130–148 (1960).