

## A simple way to reconstruct the Wigner function

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**Abstract:** We show a simple mechanism to measure the Wigner function of a harmonic oscillator. For this system we also show that autocorrelation and Wigner functions are equivalent.

**Keywords:** Harmonic Oscillator, Quasiprobability Distribution Functions

Non classical states of ions [1,2] and cavity fields [3] have been produced recently in experiments around the world [4–8]. This is one of the reasons why the Nobel prize was awarded last year to Serge Haroche and David Wineland. Once a given nonclassical state has been produced, it is important to count with mechanisms that allow us to measure them, making the gathering of such information a key problem in quantum mechanics. For instance, information about the position or momentum allows us to look for non classicality of the system. However, it is possible to obtain full information from a system by measuring, not some of its observables, but directly the density matrix [9,10], i.e. obtaining information about all possible observables. One of the possible ways of obtaining such information is via a quasiprobability distribution function, that may be related to the density matrix by using the equation [11]

$$F(\alpha, s) = \frac{2}{\pi(1-s)} \sum_{k=0}^{\infty} \left( \frac{s+1}{s-1} \right)^k \langle \alpha, k | \rho | \alpha, k \rangle \quad (1)$$

with  $s$  the quasiprobability function's parameter that indicates which is the relevant distribution ( $s = -1$  Husimi [12],  $s = 0$  Wigner [13] and  $s = 1$  Glauber-Sudarshan [14,15] distribution functions),  $\rho$  the density matrix and the states  $|\alpha, k\rangle$  are the so-called displaced number states [16].

It is well known that the Glauber-Sudarshan  $P$ -function is highly singular (note the term  $s - 1$  in the denominator) and may be used to measure non-classicality of states [17]. From it one can write the density matrix in a (diagonal) coherent state basis

$$\rho = \frac{1}{\pi} \int d^2\alpha P(\alpha) |\alpha\rangle \langle \alpha| = \frac{1}{\pi} \int d^2\alpha F(\alpha, 1) |\alpha\rangle \langle \alpha|, \quad (2)$$

that may be used to derive Fokker-Planck equations from master equations [18].

The (Husimi)  $Q$ -function may be obtained from (1) by taking  $s = -1$ . In such a case, the only term that survives in the sum is  $k = 0$ , that allow us to write

$$Q(\alpha) = F(\alpha, -1) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle. \quad (3)$$

Moreover, besides applications in classical optics [19], it has been shown that these phase space distributions can be expressed, in thermofield dynamics, as overlaps between the state of the system and *thermal* coherent states [20], that is probably the reason by which, systems subject to decay may still be "measured" [21,22].

Wineland's [9] and Haroche's [10] groups used the above expression to measure the Wigner function ( $s = 0$  case) of the quantized motion of an ion and the quantized cavity field, respectively. It is somehow simple to obtain a quasiprobability distribution function from experimental data from the above equation as there is already there a direct recipe. Let us write equation (1) as

$$F(\alpha, s) = \frac{2}{\pi(1-s)} \sum_{k=0}^{\infty} \left( \frac{s+1}{s-1} \right)^k \langle k | D^\dagger(\alpha) \rho D(\alpha) | k \rangle \quad (4)$$

where  $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ , with  $a$  and  $a^\dagger$  the annihilation and creation operators respectively, is the

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Glauber displacement operator. Note that, in order to obtain a quasiprobability distribution function we need to do the following: displace the system by an amplitude  $\alpha$  and then measure the diagonal elements of the displaced density matrix.

Equation (4) may be rewritten also as a trace

$$F(\alpha, s) = \frac{2}{\pi(1-s)} \text{Tr} \left\{ \left( \frac{s+1}{s-1} \right)^{a^\dagger a} D^\dagger(\alpha) \rho D(\alpha) \right\}. \quad (5)$$

By using the commutation properties under the symbol of trace, and for simplicity, considering the system in a *pure* state  $|\psi\rangle$ , the above equation may be casted into

$$\begin{aligned} F(\alpha, s) &= \frac{2}{\pi(1-s)} \text{Tr} \left\{ D(\alpha) \left( \frac{s+1}{s-1} \right)^{a^\dagger a} D^\dagger(\alpha) \rho \right\} \\ &= \frac{2}{\pi(1-s)} \langle \psi | D(\alpha) \left( \frac{s+1}{s-1} \right)^{a^\dagger a} D^\dagger(\alpha) | \psi \rangle. \end{aligned} \quad (6)$$

Consider now a displaced harmonic oscillator with frequency  $\omega$

$$H = \omega a^\dagger a + \beta a^\dagger + \beta^* a \quad (7)$$

with  $\beta$  the amplitude of the displacement. One can directly write the evolved wave function as (we set  $\hbar = 1$ )

$$\begin{aligned} |\psi(t)\rangle &= e^{-iHt} |\psi(0)\rangle \\ &= D^\dagger(\beta/\omega) e^{-i\omega t a^\dagger a} D(\beta/\omega) |\psi(0)\rangle \end{aligned} \quad (8)$$

From equation (8) we may obtain the *autocorrelation function* [23]

$$\begin{aligned} A(t) &= \langle \psi(0) | \psi(t) \rangle \\ &= \langle \psi(0) | D^\dagger(\beta/\omega) e^{-i\omega t a^\dagger a} D(\beta/\omega) | \psi(0) \rangle \end{aligned} \quad (9)$$

that is very similar to equation (6). In fact, if we choose  $t = \pi/\omega$  in the above equation, it produces a term

$$e^{-i\pi a^\dagger a} = (-1)^{a^\dagger a}, \quad (10)$$

that is essential in the production of the Wigner function (the alternating term), so that by setting  $s = 0$  in equation (6), the Wigner and autocorrelation functions become proportional:

$$F(\beta/\omega, 0) = W(\beta/\omega) = \frac{2}{\pi} A(\pi/\omega) \quad (11)$$

which is not surprising as the Wigner function is the generating function for all spatial autocorrelation functions of the wave function [24].

Thus, an eigenstate of the harmonic oscillator, namely, a number state  $|n\rangle$ , may be easily measured, simply by choosing as initial state  $|\psi(0)\rangle = |n\rangle$ , and projecting it with the same number state. This can be done for instance in cavity QED, by obtaining the density matrix from the evolved wavefunction and then

measuring its diagonal elements by passing atoms through the cavity [21, 22]. Note however, that, for every displacement of the harmonic oscillator, a single value of the Wigner function is obtained. Therefore for the reconstruction of the Wigner function it is necessary a big number of experiments in order to fill the phase space up.

Note that such systems may be emulated in classical light propagation through waveguide arrays [25] due to the analogy between linear lattices and the atom-field interaction [26]. Therefore, experiments leading to measurements of quasiprobability distribution functions may be easier to implement in classical optical systems.

In conclusion, we have shown a simple method to reconstruct the Wigner function for the harmonic oscillator and have shown that for this system, the autocorrelation function is proportional to the Wigner function.

## References

- [1] H. Moya-Cessa and P. Tombesi, Filtering number states of the vibrational motion of an ion, *Phys. Rev. A* 61, 025401 (2000).
- [2] H. Moya-Cessa, D. Jonathan and P.L. Knight, A family of exact eigenstates for a single trapped ion interacting with a laser field, *J. of Mod. Optics* 50, 265 (2003).
- [3] H. Moya-Cessa, P.L. Knight and A. Rosenhouse-Dantsker, Photon amplification in a two-photon lossless micromaser, *Phys. Rev. A* 50, 1814 (1994).
- [4] D. Leibfried, E. Knill, S. Seidelin, J. Britton, R. B. Blakestad, J. Chiaverini, D. B. Hume, W. M. Itano, J. D. Jost, C. Langer, R. Ozeri, R. Reichle and D. J. Wineland, Creation of a six-atom 'Schrodinger cat' state, *Nature* 438, 639 (2005).
- [5] A. Ben-Kish, B. De Marco, V. Meyer, M. Rowe, J. Britton, W. M. Itano, B. M. Jelenkovic, C. Langer, D. Leibfried, T. Rosenband, and D. J. Wineland, Experimental Demonstration of a Technique to Generate Arbitrary Quantum Superposition States of a Harmonically Bound Spin-1/2 Particle, *Phys. Rev. Lett.* 90, 037902 (2003).
- [6] D.J. Wineland, C.R. Monroe, C. Sackett, D. Kielpinski, M. Rowe, V. Meyer, W. Itano, Superposition and quantum measurement of trapped atoms, *Ann. der Physik* 9, 11 (2000).
- [7] S. Haroche, M. Brune, and J.M. Raimond, Schrodinger cat states and decoherence studies in cavity QED, *Eur. Phys. J. Special Topics* 159, 19 (2008).
- [8] P. Bertet, S. Osnaghi, P. Milman, A. Auffeves, P. Maioli, M. Brune, J. M. Raimond, and S. Haroche, Generating and Probing a Two-Photon Fock State with a Single Atom in a Cavity, *Phys. Rev. Lett.* 88, 143601 (2002).
- [9] D. Leibfried, D. M. Meekhof, B. E. King, C. Monroe, W. M. Itano, and D. J. Wineland, Experimental Determination of the Motional Quantum State of a Trapped Atom, *Phys. Rev. Lett.* 77, 4281 (1996).

- [10] P. Bertet, A. Auffeves, P. Maioli, S. Osnaghi, T. Meunier, M. Brune, J. M. Raimond, and S. Haroche, Direct Measurement of the Wigner Function of a One-Photon Fock State in a Cavity, *Phys. Rev. Lett.* 89, 200402 (2002).
- [11] H. Moya-Cessa and P.L. Knight, Series representation of quantum-field quasiprobabilities, *Phys. Rev. A* 48, 2479 (1993).
- [12] K. Husimi, Some Formal Properties of the Density Matrix, *Proc. Phys. Math. Soc. Jpn.* 22, 264 (1940).
- [13] E.P. Wigner, On the Quantum Correction For Thermodynamic Equilibrium, *Phys. Rev.* 40, 749 (1932).
- [14] R.J. Glauber, Coherent and Incoherent States of the Radiation Field, *Phys. Rev.* 131, 2766 (1963).
- [15] E.C.G. Sudarshan, Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams, *Phys. Rev. Lett.* 10, 277 (1963).
- [16] F.A.M. de Oliveira, M.S. Kim, P.L. Knight and V. Buzek, Properties of displaced number states, *Phys. Rev. A* 41, 2645 (1990).
- [17] J. Janszky, Min Gyu Kim, and M. S. Kim, Quasiprobabilities and the nonclassicality of fields, *Phys. Rev. A* 53, 502 (1996).
- [18] H. Risken, *The Fokker-Planck Equation: Methods of Solutions and Applications*, Springer, Berlin, 1984.
- [19] J.R. Moya-Cessa, L.R. Berriel-Valdos, and H. Moya-Cessa, Optical Production of the Husimi Function of Two Gaussian Functions, *Appl. Math. Inf. Sc.* 2, 309 (2008).
- [20] S Chaturvedi, V Srinivasan and G S Agarwal, Quantum phase space distributions in thermofield dynamics, *J. Phys. A* 32 1909 (1999).
- [21] H. Moya-Cessa, J.A. Roversi, S.M. Dutra and A. Vidiella-Barranco, Recovering coherence from decoherence: A method of quantum-state reconstruction, *Phys. Rev. A.* 60, 4029 (1999).
- [22] H. Moya-Cessa, S.M. Dutra, J.A. Roversi, and A. Vidiella-Barranco, Quantum state reconstruction in the presence of dissipation, *J. of Mod. Optics* 46, 555 (1999).
- [23] H. Eleuch, Quantum Trajectories and Autocorrelation Function in Semiconductor Microcavity, *Appl. Math. Inf. Sc.* 3, 185 (2009).
- [24] C.K. Zachos, D.B. Fairlie and T.L. Curtright, in *Quantum Mechanics in Phase Space, An Overview with Selected Papers* (World Scientific Series in 20th Century Physics, Vol. 34, 2005).
- [25] R. Keil, A. Perez-Leija, F. Dreisow, M. Heinrich, H. Moya-Cessa, S. Nolte, D. N. Christodoulides, and A. Szameit, Classical Analogue of Displaced Fock States and Quantum Correlations in Glauber-Fock Photonic Lattices, *Phys. Rev. Lett.* 107, 103601 (2011).
- [26] Crisp M D 1993 Ed Jaynes' steak dinner problem II ( *Physics and Probability, Essays in honor of Edwin T. Jaynes*) ed. W.T. Grandy, Jr, and P.W. Milonni (Cambridge: Cambridge University Press).



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