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On Nondifferentiable Multiobjective Programming Involving Type-I α-Invex Functions

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The aim of this paper is to study a nondifferentiable multiobjective programming problem with inequality constraints. In this paper we introduce the concept of type-I α-invex, weak strictly pseudo-quasi type-I α-invex, strong pseudo-quasi type-I α-invex, weak quasi-strictly-pseudo type-I α-invex and weak strictly-pseudo type-I α-invex functions. By utilizing these new notions we derive a Fritz John type sufficient optimality condition and establish Mond-Weir type and general Mond-Weir type duality results for the nondifferentiable multiobjective programming problem.

Keywords: Type-I α-invexity, nondifferentiable multiobjective programming, convexity, duality.

1 Introduction

Convexity plays a vital role in many aspects of mathematical programming (see, for example, Bazaraa *et al.* [3] and Mangasarian [12]). In order to study the optimization problems in a wider context various useful generalizations of the notion of convexity have been introduced. Hanson [8] introduced the class of invex functions. Later, Hanson and Mond [9] defined two new classes of functions called type-I and type-II functions. This concept was extended by Rueda and Hanson [29] to pseudo-type-I and quasi-type-I functions. Univex functions were introduced and studied by Bector *et al.* [4]. Rueda *et al.* [30] studied optimality and duality results for several mathematical programs by combining

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the concept of type-I and univex functions. Kaul *et al.* [11] considered a multiple objective problem with type-I functions and obtained some results on optimality and duality. Mishra [15] studied a multiple objective nonlinear programming problem by combining the cocept of type-I, pseudo-type-I, quasi-type-I, quasi-pseudo-type-I, pseudo-quasi-type-I and univex functions. More details on type-I functions can be found in Ye [33], Suneja and Srivastava [31], Mishra *et al.* [19, 21, 22] and Mishra *et al.* [23, 24]. Aghezzaf and Hachimi [1] introduced new class of generalized type-I vector valued functions and derived various duality results for a nonlinear multiobjective programming problem.

Theoretical problems of differentiable programming can be solved by substituting invexity for convexity e.g. Hanson [8], Craven [5], Egudo and Hanson [7], and Jayakumar and Mond [10]. But corresponding conclusion can not be obtained in nondifferentiable programming with the aid of invexity introduced by Hanson [8] because the existence of a derivative is required in the definition of invexity.

Generalization of invexity to locally Lipschitz functions, with derivative replaced by Clarke generalized gradient has been considered by Craven [6], Reiland [28], Mishra and Mukherjee [17], Mishra [13, 14], and Mishra and Giorgi [16]. However, Antczak [2] used directional derivative, in association with a hypothesis of an invex kind, following Ye [33].

Noor [26] and Mishra and Noor [18] have studied some properties of the α-preinvex functions and their differentials. Recently Mishra, Pant and Rautela [20] and Pant and Rautela [27] introduced the concepts of strict pseudo α-invex, quasi α-invex, weak strictly pseudo quasi α-invex, strong pseudo quasi α-invex, weak quasi strictly pseudo α-invex and weak strictly pseudo α-invex functions.

In the present paper, as an application of the new classes of type-I α-invex functions we consider a nondifferentiable multiobjective programming problem and derive Fritz John type sufficient optimality conditions for a (weakly) Pareto efficient solution to the problem. Further the Mond-Weir type and general Mond-Weir type of duality results are also obtained.

2 Preliminaries

Throughout this paper, we will use the following conventions for vectors in R^n:

$$
\begin{aligned}
x = y &\Leftrightarrow x_i = y_i,\ i = 1, \ldots, n;\\
x > y &\Leftrightarrow x_i > y_i,\ i = 1, \ldots, n;\\
x \geqq y &\Leftrightarrow x_i \geqq y_i,\ i = 1, \ldots, n;\\
x \geq y &\Leftrightarrow x_i \geqq y_i,\ i = 1, \ldots, n \text{ but } x \neq y.
\end{aligned}
$$

Let X be a nonempty subset of R^n, $\eta : X \times X \to R^n$ is an n-dimensional vector valued function and $\alpha(x, y) : X \times X \to R_+ \backslash 0$ be a bifunction. First, we recall some known results and concepts.

Definition 2.1. A subset $X \subseteq R^n$ is said to be α-*invex set*, if there exist $\eta : X \times X \to R^n$ and $\alpha(x, u) : X \times X \to R_+$ such that for all $x \in X$

$$u + \lambda\alpha(x, u)\eta(x, u) \in X,\ \forall\, x, u \in X,\ \lambda \in [0, 1].$$

Note that α-invex set need not to be convex set.

The following example from Noor (2004) shows that α-invex set need not to be convex set.

Example 2.1. The set $X = R\backslash(-1/2, 1/2)$ is an invex set with respect to $\alpha(x, u) = 1$ and η, where

$$\eta = \begin{cases} x - u, & \text{for } x > 0,\ u > 0 \\ u - x, & \text{for } x < 0,\ u > 0. \end{cases}$$

It is clear that X is not a convex set.

From now onward we assume that the set X is a nonempty α-invex set with respect to $\alpha(\cdot, \cdot)$ and $\eta(\cdot, \cdot)$ unless otherwise specified.

Definition 2.2. The function $f : X \subseteq R^n \to R^k$ on the α-invex set is said to be α-*preinvex function* if there exist $\eta : X \times X \to R^n$ and $\alpha(x, u) : X \times X \to R_+$ such that for all $x \in X$

$$f(x + \lambda\alpha(x, u)\eta(x, u)) \leq (1 - \lambda)f(u) + \lambda f(x),\ \forall\, x, u \in X,\ \lambda \in [0, 1].$$

We consider the following mathematical programming problem:

(P) Minimize $f(x)$, subject to $g(x) \leq 0,\ x \in X$, where $f : X \subseteq R^n \to R^k$ and $g : X \subseteq R^n \to R^m$ are functions on a set $X \subseteq R^n$ (a nonempty α-invex set).

Throughout this paper we use the notation

$$\alpha(x, u)f'(u, \eta(x, u)) = \lim_{\lambda \to 0^+} \frac{f(u + \lambda\alpha(x, u)\eta(x, u)) - f(u)}{\lambda},$$

and a similar notation for $\alpha(x, u)g'(u, \eta(x, u))$.

Let D be a nonempty α-invex set such that $D = \{x \in X : g(x) \leqq 0\}$ is the set of all the feasible solutions for (P) and denote $I = \{1, \ldots, k\},\ M = \{1, \ldots, m\},\ J(x) = \{j \in M : g_j(x) = 0\}$ and $\overline{J}(x) = \{j \in M : g_j(x) < 0\}$. This implies $J(x) \cup \overline{J}(x) = M$.

Now, we introduce the concept of type-I α-invex, weak strictly pseudo-quasi type-I α-invex, strong pseudo-quasi type-I α-invex, weak quasi-strictly-pseudo type-I α-invex and weak strictly-pseudo type-I α-invex functions.

Definition 2.3. The pair (f, g) is said to be *type-I α-invex with respect to α and η at $u \in X$*, if there exist functions $\alpha(x,u) : X \times X \to R_+$ and $\eta : X \times X \to R^n$ such that

$$f(x) - f(u) \geqq \alpha(x,u) f'(u, \eta(x,u)),\ \forall\, x, u \in X;$$
$$-g(u) \geqq \alpha(x,u) g'(u, \eta(x,u)),\ \forall\, x, u \in X.$$

Definition 2.4. The pair (f, g) is said to be *weak strictly pseudo-quasi type-I α-invex with respect to α and η at $u \in X$*, if there exist functions $\alpha(x,u) : X \times X \to R_+$ and $\eta : X \times X \to R^n$ such that

$$f(x) - f(u) \leq 0 \Rightarrow \alpha(x,u) f'(u, \eta(x,u)) < 0,\ \forall\, x, u \in X;$$
$$-g(u) \leqq 0 \Rightarrow \alpha(x,u) g'(u, \eta(x,u)) \leqq 0,\ \forall\, x, u \in X.$$

Definition 2.5. The pair (f, g) is said to be *strong pseudo-quasi type-I α-invex with respect to α and η at $u \in X$*, if there exist functions $\alpha(x,u) : X \times X \to R_+$ and $\eta : X \times X \to R^n$ such that

$$f(x) - f(u) \leq 0 \Rightarrow \alpha(x,u) f'(u, \eta(x,u)) \leq 0,\ \forall\, x, u \in X;$$
$$-g(u) \leqq 0 \Rightarrow \alpha(x,u) g'(u, \eta(x,u)) \leqq 0,\ \forall\, x, u \in X.$$

Example 2.2. Consider the function $f = (f_1, f_2) : [-1, 4) \to R$ defined by

$$f_1 = \begin{cases} x^3, & -1 \leq x \leq 2 \\ 8, & 2 \leq x \leq 4, \end{cases}$$

$$f_2 = \begin{cases} 0, & -1 \leq x \leq 2 \\ 2x^2 - 8, & 2 \leq x \leq 4 \end{cases}$$

and the function $g = (g_1, g_2) : [-1, 4) \to R$ defined by

$$g_1 = \begin{cases} -x^2, & -1 \leq x \leq 2 \\ -4, & 2 \leq x \leq 4, \end{cases}$$

$$g_2 = \begin{cases} 5x, & -1 \leq x \leq 2 \\ x^4 - 6, & 2 \leq x \leq 4. \end{cases}$$

Clearly, $f_1,\ f_2,\ g_1$ and g_2 are not differentiable functions at $x = 2$. The feasible region is nonempty. Let $\alpha(x, \overline{x}) = 1$, $\eta(x, \overline{x}) = x^2(x - \overline{x})/2$ and $\overline{x} = 2$.

(i) If $x \in [-1, 2)$ and $f_1(x) + f_2(x) \leq f_1(2) + f_2(2)$, then it implies that $x \leq 2$, which further implies that $\alpha(x, \overline{x}) f_1'(\overline{x}, \eta(x, \overline{x})) + \alpha(x, \overline{x}) f_2'(\overline{x}, \eta(x, \overline{x})) = 6x^2(x - 2) \leq 0$ and $-g_1(\overline{x}) - g_2(\overline{x}) \leq 0$, which implies that $\alpha(x, \overline{x}) g_1'(\overline{x}, \eta(x, \overline{x})) + \alpha(x, \overline{x}) g_2'(\overline{x}, \eta(x, \overline{x})) \leq 0$.

(ii) The case $x \in [2, 4)$ can be verified similarly.

Thus (f, g) is strong pseudo-quasi type-I α-invex with respect to α and η at $x = 2$. However, (f, g) is not type-I α-invex with respect to same α and η at $x = 2$.

Definition 2.6. The pair (f, g) is said to be *weak quasi-strictly-pseudo type-I α-invex with respect to α and η at $u \in X$*, if there exist functions $\alpha(x, u) : X \times X \to R_+$ and $\eta : X \times X \to R^n$ such that

$$f(x) - f(u) \leq 0 \Rightarrow \alpha(x, u) f'(u, \eta(x, u)) \leqq 0,\ \forall\, x, u \in X;$$
$$-g(u) \leqq 0 \Rightarrow \alpha(x, u) g'(u, \eta(x, u)) \leq 0,\ \forall\, x, u \in X.$$

Definition 2.7. The pair (f, g) is said to be *weak strictly-pseudo type-I α-invex with respect to α and η at $u \in X$*, if there exist functions $\alpha(x, u) : X \times X \to R_+$ and $\eta : X \times X \to R^n$ such that

$$f(x) - f(u) \leq 0 \Rightarrow \alpha(x, u) f'(u, \eta(x, u)) < 0,\ \forall\, x, u \in X;$$
$$-g(u) \leqq 0 \Rightarrow \alpha(x, u) g'(u, \eta(x, u)) < 0,\ \forall\, x, u \in X.$$

Definition 2.8. A point $\overline{x} \in D$ is said to be a *weak Pareto efficient solution for (P)* if the relation $f(\overline{x}) < f(x)$ holds for all $x \in D$.

Definition 2.9. A point $\overline{x} \in D$ is said to be a *locally weak Pareto efficient solution for (P)* if there is a neighborhood $N(\overline{x})$ around $\overline{x}$ such that $f(\overline{x}) < f(x)$, holds for all $x \in N(\overline{x}) \cap D$.

The following results from Antczak (2002) and Weir and Mond (1988) type will be needed in the next section.

Lemma 2.1. *If $\overline{x}$ is a locally weak Pareto or a weak Pareto efficient solution of (P) and if g_j is continuous at $\overline{x}$ for $j \in \overline{J}(\overline{x})$, then the following system of inequalities*

$$f'(\overline{x}, \eta(x, \overline{x})) < 0,$$
$$g'_{J(\overline{x})}(\overline{x}, \eta(x, \overline{x})) < 0,$$

has no solution for $x \in X$.

Definition 2.10. Function g is said to satisfy the generalized Slaters constraint qualification at $\overline{x} \in D$ if g is α-invex at $\overline{x}$, and there exist $\overline{x} \in D$ such that $g_j(\overline{x}) < 0,\ j \in J(\overline{x})$.

Lemma 2.2 (Fritz John type necessary optimality condition)**.** *Let x be a weak Parato efficient solution for (P). Moreover we assume that g_j is continuous for $j \in \overline{J}(\overline{x})$, f and g are directionally differentiable at $\overline{x}$ with $f'(\overline{x}, \eta(x, \overline{x}))$ and $g'_{J(\overline{x})}(\overline{x}, \eta(x, \overline{x}))$ α-preinvex functions of x on X. Moreover, we assume that g satisfies the generalized Slaters constraint* *qualification at $\overline{x}$. Then there exist $\overline{\xi} \in R_+^k$, $\overline{\mu} \in R_+^m$, such that $(\overline{x}, \overline{\xi}, \overline{\mu})$ satisfies the following conditions:*

$$\overline{\xi}^T f'(\overline{x}, \eta(x, \overline{x})) + \overline{\mu}^T g'_{J(\overline{x})}(\overline{x}, \eta(x, \overline{x})) \geq 0, \forall x \in X, \tag{2.1}$$

$$\overline{\mu}^T g(\overline{x}) = 0, \tag{2.2}$$

$$g(\overline{x}) \leqq 0. \tag{2.3}$$

3 Sufficient Optimality Conditions

In this section, we establish a Fritz John type sufficient optimality condition.

Theorem 3.1. *Let $\overline{x}$ be a feasible solution for (P) at which conditions (1)-(3) are satisfied. Moreover, if any one of the following conditions is satisfied:*

(a) *$(\overline{\xi}^T f, \overline{\mu}^T g)$ is strong pseudo-quasi type-I α-invex at $\overline{x}$ with respect to some α_0, α_1 and η;*

(b) *$(\overline{\xi}^T f, \overline{\mu}^T g)$ is weak strictly pseudo-quasi type-I α-invex at $\overline{x}$ with respect to some α_0, α_1 and η;*

(c) *$(\overline{\xi}^T f, \overline{\mu}^T g)$ is weak strictly pseudo type-I α-invex at $\overline{x}$ with respect to some α_0, α_1 and η;*

then $\overline{x}$ is a weak Pareto efficient solution for (P).

Proof. We prove the theorem by contradiction. Let us assume that $\overline{x}$ is not a weak Pareto efficient solution of (P). Then there is a feasible solution x of (P) such that

$$f_i(x) < f_i(\overline{x}) \text{ for any } i = 1, 2, \ldots, k$$
$$\Rightarrow f_i(x) - f_i(\overline{x}) < 0$$
$$\Rightarrow \overline{\xi}^T f_i(x) - \overline{\xi}^T f_i(\overline{x}) < 0, \text{ (since } \overline{\xi}^T > 0). \tag{3.1}$$

Now from the feasibility of x and (2.2), we get

$$\overline{\mu}^T g(x) - \overline{\mu}^T g(\overline{x}) \leq 0.$$

If the condition (a) is satisfied, then from the above two inequalities, we get

$$\overline{\xi}^T \alpha_0(x, \overline{x}) f'(\overline{x}, \eta(x, \overline{x})) < 0 \text{ and } \overline{\mu}^T \alpha_1(x, \overline{x}) g'(\overline{x}, \eta(x, \overline{x})) \leqq 0.$$

By the positivity of α_0 and α_1 the above two inequalities reduces to

$$\overline{\xi}^T f'(\overline{x}, \eta(x, \overline{x})) < 0 \text{ and } \overline{\mu}^T g'(\overline{x}, \eta(x, \overline{x})) \leqq 0.$$

From above two inequalities, we get

$$\overline{\xi}^T f'(\overline{x}, \eta(x, \overline{x})) + \overline{\mu}^T g'(\overline{x}, \eta(x, \overline{x})) < 0.$$

This contradicts (2.1).

If condition (b) is satisfied, we assume that $\overline{x}$ is not a weak Pareto efficient solution of (P). Then there is a feasible solution x of (P) such that

$$\begin{aligned} &f_i(x) - f_i(\overline{x}) < 0 \\ \Rightarrow\ &\overline{\xi}^T f_i(x) - \overline{\xi}^T f_i(\overline{x}) < 0,\ (\text{since } \overline{\xi}^T > 0). \end{aligned}$$

Now by condition (b) and (2.2) we get,

$$\overline{\xi}^T \alpha_0(x, \overline{x}) f'(\overline{x}, \eta(x, \overline{x})) < 0 \text{ and } \overline{\mu}^T \alpha_1(x, \overline{x}) g'(\overline{x}, \eta(x, \overline{x})) < 0.$$

By the positivity of α_0 and α_1 the above two inequalities reduces to

$$\overline{\xi}^T f'(\overline{x}, \eta(x, \overline{x})) < 0 \text{ and } \overline{\mu}^T g'(\overline{x}, \eta(x, \overline{x})) < 0.$$

From the above two inequalities, we get

$$\overline{\xi}^T f'(\overline{x}, \eta(x, \overline{x})) + \overline{\mu}^T g'(\overline{x}, \eta(x, \overline{x})) < 0.$$

This is again a contradiction to (2.1).

Now for the part (c), following the similar process, we get

$$\overline{\xi}^T f'(\overline{x}, \eta(x, \overline{x})) + \overline{\mu}^T g'(\overline{x}, \eta(x, \overline{x})) < 0.$$

This contradicts (2.1) and complete the proof. □

Example 3.1. Consider function $f = (f_1, f_2)$ defined on $X = R$, by $f_1(x) = x^2,\ f_2(x) = x^3$ and function g defined on $X = R$, by

$$g = \begin{cases} -2x^2, & -1 \leq x \leq 2 \\ -x^3, & 2 \leq x \leq 2.5. \end{cases}$$

Clearly, g is not differentiable at $x = 2$, but only directionally differentiable at $x = 2$. The feasible region is nonempty. Let $\alpha(x, \overline{x}) = 1$, $\eta(x, \overline{x}) = (x - \overline{x})/2$ and $\overline{x} = 0$.

(i) If $x \in [-1, 2)$, $-g(\overline{x}) = 0$, implies that $\alpha(x, \overline{x}) g'(\overline{x}, \eta(x, \overline{x})) = 0$.

(ii) The case $x \in [2, 2.5)$ can be verified similarly.

$$f(x) \leqq f(\overline{x}) \Rightarrow \alpha(x, \overline{x}) f'(\overline{x}, \eta(x, \overline{x})) = 0, \text{ for all } x.$$

Thus (f, g) is strong pseudo-quasi type-I α-invex at $x = 0$. But (f, g) is not type-I α-invex at $x = 0$ with respect to $\alpha(x, \overline{x}) = 1$ and $\eta(x, \overline{x}) = (x - \overline{x})/2$. Then, by Theorem 3.1(a), $\overline{x}$ is a weak Pareto efficient solution for the given multiobjective programming problem.

4 Mond-Weir Duality

Now in relation to (P) we consider the following dual problem in the format of Mond-Weir (1981):

(MWD) Maximize $f(y) = (f_1(y), f_2(y), \ldots, f_k(y))$, subject to

$$\begin{aligned} &(\xi^T f' + \mu^T g')(y, \eta(x, y)) \geqq 0, \text{ for all } x \in D, \\ &\qquad\qquad \mu_j g_j(y) \geq 0,\ j = \{1, 2, \ldots, m\}, \end{aligned} \tag{4.1}$$

$$\xi^T e = 1, \tag{4.2}$$

$$\xi \in R^k_+,\ \mu \in R^m_+, \tag{4.3}$$

where $e = (1, 1, \ldots, 1) \in R^k$.

Let

$$W = \left\{ \begin{aligned} &(y, \xi, \mu) \in X \times R^k \times R^m : \xi^T f'(y, \eta(x, y)) + \mu^T g'(y, \eta(x, y)) \geqq 0, \\ &\mu_j g_j(y) \geq 0,\ j = 1, 2, \ldots, m,\ \xi \in R^k_+,\ \xi^T e = 1,\ \mu \in R^m_+ \end{aligned} \right\}$$

denote the set of all feasible solutions of (MWD). We also denote by $pr_x W$ the projection of set W on X.

Theorem 4.1 (Weak Duality)**.** *Let x and (y, ξ, μ) be feasible solutions for (P) and (MWD) respectively. Moreover, we assume that any one of the following conditions holds:*

(a) $(f, \overline{\mu}^T g)$ *is strong pseudo-quasi type-I α-invex at y on $D \cup pr_x W$ with respect to some α_0, α_1 and η;*
(b) $(f, \overline{\mu}^T g)$ *is weak strictly pseudo-quasi type-I α-invex at y on $D \cup pr_x W$ with respect to some α_0, α_1 and η;*
(c) $(f, \overline{\mu}^T g)$ *is weak strictly pseudo type-I α-invex at y on $D \cup pr_x W$ with respect to some α_0, α_1 and η.*

Then the following can not hold:

$$f(x) \leq f(y).$$

Proof. Suppose that

$$f(x) \leq f(y), \quad \text{i.e.} \quad f(x) - f(y) \leq 0. \tag{4.4}$$

Since x is feasible for (P) and (y, ξ, μ) is feasible for (MWD). It follows that

$$-\sum_{j=1}^{m} \mu_j g_j(y) \leq 0. \tag{4.5}$$

If condition (a) is satisfied, (4.4) and (4.5) imply

$$\alpha_0(x,y)f'(y,\eta(x,y)) \leq 0 \text{ and } \sum_{j=1}^{m}\mu_j\alpha_1(x,y)g'(y,\eta(x,y))\leqq 0.$$

By the positivity of α_0 and α_1 the above two inequalities reduce to

$$f'(y,\eta(x,y)) \leq 0 \tag{4.6}$$

and

$$\sum_{j=1}^{m}\mu_j g'(y,\eta(x,y))\leqq 0. \tag{4.7}$$

Since $\xi \geq 0$, from (4.6) and (4.7), we get

$$\sum_{i=1}^{k}\xi_i f_i'(y,\eta(x,y)) + \sum_{j=1}^{m}\mu_j g_j'(y,\eta(x,y)) < 0. \tag{4.8}$$

This contradicts (4.1). Hence the assertion.

If the condition (b) is satisfied, from (4.4) and (4.5), we get

$$\alpha_0(x,y)f'(y,\eta(x,y)) < 0 \text{ and } \sum_{j=1}^{m}\mu_j\alpha_1(x,y)g'(y,\eta(x,y))\leqq 0.$$

By the positivity of α_0 and α_1 the above inequalities reduce to

$$f'(y,\eta(x,y)) < 0 \tag{4.9}$$

$$\sum_{j=1}^{m}\mu_j g'(y,\eta(x,y))\leqq 0. \tag{4.10}$$

Since $\xi \geq 0$, (4.9) and (4.10) imply (4.8), again a contradiction to (4.1).

If the condition (c) is satisfied, from (4.4) and (4.5), we get

$$\alpha_0(x,y)f'(y,\eta(x,y)) < 0 \text{ and } \sum_{j=1}^{m}\mu_j\alpha_1(x,y)g'(y,\eta(x,y)) < 0.$$

By the positivity of α_0 and α_1 the above inequalities reduce to

$$f'(y,\eta(x,y)) < 0 \tag{4.11}$$

$$\sum_{j=1}^{m}\mu_j g'(y,\eta(x,y)) < 0. \tag{4.12}$$

But $\xi\geqq 0$, (4.11) and (4.12) imply (4.8), which contradicts (4.1). This completes the proof.

□

Theorem 4.2 (Strong duality). *Let $\overline{x}$ be a locally weak Pareto efficient solution for (P) at which the generalized Slaters constraint qualification is satisfied. Let $f,\ g$ be directionally differentiable at $\overline{x}$ with $f'(\overline{x},\eta(x,\overline{x}))$ and $g'(\overline{x},\eta(x,\overline{x}))$ are α-preinvex functions on X. Let g_j be continuous for $j \in \overline{J}(\overline{x})$, then there exist $\overline{\mu} \in R^m_+$ such that $(\overline{x},1,\overline{\mu})$ is feasible for (MWD). If the weak duality between (P) and (MWD) in Theorem 4.1 holds, then $(\overline{x},1,\overline{\mu})$ is a locally weak Pareto efficient solution for (MWD).*

Proof. Since $\overline{x}$ satisfies all the conditions of Lemma 2.2, there exist $\overline{\mu} \in R^m_+$ such that conditions (1)-(3) hold. By (1)-(3), we have $(\overline{x},1,\overline{\mu})$ is feasible for (MWD). By the weak duality, it follows that $(\overline{x},1,\overline{\mu})$ is a locally weak Pareto efficient solution for (MWD). □

Theorem 4.3 (Converse duality). *Let $(\overline{y},\overline{\xi},\overline{\mu})$ be a weak Pareto efficient solution for (MWD). Moreover we assume that the hypothesis of Theorem 3.1 hold for $\overline{y}$ in $D \cup pr_x W$, then $\overline{y}$ is a weak Pareto efficient solution for (P).*

Proof. We prove the theorem by contradiction. Suppose that ($\overline{y}$ is not a weak Pareto efficient solution for (P), that is, there exist $\overline{x} \in D$ such that $f(\overline{x}) < f(\overline{y})$. Since condition (a) of Theorem 4.1 holds, we get

$$\sum_{i=1}^{k} \overline{\xi}_i \alpha_0(\overline{x},\overline{y}) f_i'(\overline{y},\eta(\overline{x},\overline{y})) < 0.$$

By the positivity of α_0 the above inequality reduce to

$$\sum_{i=1}^{k} \overline{\xi}_i f_i'(\overline{y},\eta(\overline{x},\overline{y})) < 0. \tag{4.13}$$

From the feasibility of $\overline{x}$ and $(\overline{y},\overline{\xi},\overline{\mu})$ for (P) and (MWD) respectively, we have

$$\sum_{j=1}^{m} \overline{\mu}_j g_j(\overline{y}) \leq 0.$$

The above inequality in the light of condition (a) of Theorem 4.1, yields

$$\sum_{j=1}^{m} \overline{\mu}_j \alpha_1(\overline{x},\overline{y}) g_j'(\overline{y},\eta(\overline{x},\overline{y})) \leqq 0.$$

Since $\alpha_1 > 0$, we get

$$\sum_{j=1}^{m} \overline{\mu}_j g_j'(\overline{y},\eta(\overline{x},\overline{y})) \leqq 0. \tag{4.14}$$

By (4.13) and (4.14), we get

$$\sum_{i=1}^{k} \overline{\xi}_i f_i'(\overline{y},\eta(\overline{x},\overline{y})) + \sum_{j=1}^{m} \overline{\mu}_j g_j'(\overline{y},\eta(\overline{x},\overline{y})) < 0. \tag{4.15}$$

This contradicts the dual constraint (4.1).

Similarly by condition (b) in Theorem 4.1, we get

$$\sum_{i=1}^{k} \overline{\xi}_i \alpha_0(\overline{x},\overline{y}) f_i'(\overline{y},\eta(\overline{x},\overline{y})) < 0 \text{ and } \sum_{j=1}^{m} \overline{\mu}_j \alpha_1(\overline{x},\overline{y}) g_j'(\overline{y},\eta(\overline{x},\overline{y})) \leq 0.$$

By the positivity of α_0 and α_1 the above two inequalities reduce to

$$\sum_{i=1}^{k} \overline{\xi}_i f_i'(\overline{y},\eta(\overline{x},\overline{y})) < 0 \text{ and } \sum_{j=1}^{m} \overline{\mu}_j g_j'(\overline{y},\eta(\overline{x},\overline{y})) \leq 0.$$

Since $\xi \geq 0$, the above two inequalities imply (4.15), which yields contradiction to (4.1).

By condition (c), we have

$$\sum_{i=1}^{k} \overline{\xi}_i \alpha_0(\overline{x},\overline{y}) f_i'(\overline{y},\eta(\overline{x},\overline{y})) < 0 \text{ and } \sum_{j=1}^{m} \overline{\mu}_j \alpha_1(\overline{x},\overline{y}) g_j'(\overline{y},\eta(\overline{x},\overline{y})) < 0.$$

By the positivity of α_0 and α_1 the above two inequalities reduce to

$$\sum_{i=1}^{k} \overline{\xi}_i f_i'(\overline{y},\eta(\overline{x},\overline{y})) < 0 \text{ and } \sum_{j=1}^{m} \overline{\mu}_j g_j'(\overline{y},\eta(\overline{x},\overline{y})) < 0.$$

Since $\xi \geq 0$, the above two inequalities imply (4.15), which yields again a contradiction to (4.1). Hence, the proof is completed. □

5 General Mond-Weir Duality

We shall continue our discussion on duality for (P) in the present section by considering a general Mond-Weir type dual problem and proving weak and strong duality theorem under the assumption of type-I α-invexity introduced in section 2.

We consider the following general Mond-Weir type dual to (P)

(GMWD) Maximize $\phi(y,\xi,\mu) = f(y) + \mu_{J_0}^T g_{J_0}(y)e$, subject to

$$(\xi^T f' + \mu^T g')(y,\eta(x,y)) \geqq 0, \text{ for all } x \in D, \tag{5.1}$$

$$\mu_{J_t} g_{J_t}(y) \geq 0, \ 1 \leqq t \leqq r, \tag{5.2}$$

$$\xi^T e = 1, \tag{5.3}$$

$$\xi \in R_+^k, \ \mu \in R_+^m,$$

where $J_t, \ 1 \leqq t \leqq r$ are partitions of set M and $e = (1,1,\ldots,1) \in R^k$.

Let

$$W = \left\{ \begin{array}{l} (y,\xi,\mu) \in X \times R^k \times R^m : \xi^T f'(y,\eta(x,y)) + \mu^T g'(y,\eta(x,y)) \geqq 0, \\ \mu_j g_j(y) \geq 0, \ j = 1,2,\ldots,m, \ \xi \in R_+^k, \ \xi^T e = 1, \ \mu \in R_+^m \end{array} \right\}$$

denote the set of all feasible solutions of (GMWD).

Theorem 5.1 (Weak Duality). *Let x and (y, ξ, μ) be a feasible solution for (P) and (GMWD) respectively. Assume that one of the following condition holds:*

(a) $\xi > 0$ *and* $(f + \mu_{J_0} g_{J_0}, \mu_{J_t} g_{J_t})$ *is strong pseudo-quasi type-I α-invex at y on $D \cup pr_x W$ with respect to some α_0, α_1 and η for any t, $1 \leqq t \leqq r$;*
(b) $(f + \mu_{J_0} g_{J_0}, \mu_{J_t} g_{J_t})$ *is weak strictly pseudo-quasi type-I α-invex at y on $D \cup pr_x W$ with respect to some α_0, α_1 and η for any t, $1 \leqq t \leqq r$;*
(c) $(f + \mu_{J_0} g_{J_0}, \mu_{J_t} g_{J_t})$ *is weak strictly pseudo type-I α-invex at y on $D \cup pr_x W$ with respect to some α_0, α_1 and η for any t, $1 \leqq t \leqq r$.*

Then the following condition can not hold:

$$f(x) \leq \phi(y, \xi, \mu).$$

Proof. We prove the theorem by contradiction. Suppose

$$f(x) \leq \phi(y, \xi, \mu). \tag{5.4}$$

Since x is feasible for (P) and $\mu \geq 0$, (5.4) implies that

$$\begin{aligned} & f(x) + \mu_{J_0}^T g_{J_0}(x) e \leq f(y) + \mu_{J_0}^T g_{J_0}(y) e \\ \Rightarrow\ & f(x) + \mu_{J_0}^T g_{J_0}(x) e - f(y) + \mu_{J_0}^T g_{J_0}(y) e \leq 0. \end{aligned} \tag{5.5}$$

From the feasibility of x for (P) and (5.2), we have

$$-\mu_{J_t}^T g_{J_t}(y) \leqq 0, \text{ for any } 1 \leqq t \leqq r. \tag{5.6}$$

By condition (a), from (5.5) and (5.6), we have

$$\alpha_0(x, y) f'(y, \eta(x, y)) + \mu_{J_0} \alpha_0(x, y) g'_{J_0}(y, \eta(x, y)) \leq 0$$

and

$$\mu_{J_t} \alpha_1(x, y) g'_{J_t}(y, \eta(x, y)) \leq 0, \text{for any } 1 \leqq t \leqq r.$$

By the positivity of α_0 and α_1 the above two inequalities reduce to

$$f'(y, \eta(x, y)) + \mu_{J_0} g'_{J_0}(y, \eta(x, y)) \leq 0$$

and

$$\mu_{J_t} g'_{J_t}(y, \eta(x, y)) \leq 0, \text{for any } 1 \leqq t \leqq r.$$

Since $\xi > 0$, the above two inequalities yield

$$f'(y, \eta(x, y)) + \sum_{t=0}^{r} \mu_{J_t} g'_{J_t}(y, \eta(x, y)) < 0. \tag{5.7}$$

Since $J_0, \ldots, J_r$ are partition of M, (5.7) is equivalent to

$$f'(y, \eta(x, y)) + \mu^T g'(y, \eta(x, y)) < 0. \tag{5.8}$$

which contradicts the dual constraint (5.2).

Similarly by condition (b) we have

$$\alpha_0(x, y) f'(y, \eta(x, y)) + \mu_{J_0} \alpha_0(x, y) g'_{J_0}(y, \eta(x, y)) < 0$$

and

$$\mu_{J_t} \alpha_1(x, y) g'_{J_t}(y, \eta(x, y)) \leq 0, \text{for any } 1 \leqq t \leqq r.$$

By the positivity of α_0 and α_1 the above two inequalities reduce to

$$f'(y, \eta(x, y)) + \mu_{J_0} g'_{J_0}(y, \eta(x, y)) < 0$$

and

$$\mu_{J_t} g'_{J_t}(y, \eta(x, y)) \leq 0, \text{for any } 1 \leqq t \leqq r.$$

Since $\xi \geq 0$, the above two inequalities yield

$$f'(y, \eta(x, y)) + \sum_{t=0}^{r} \mu_{J_t} g'_{J_t}(y, \eta(x, y)) < 0.$$

The above inequality leads to (5.8), which contradicts (5.1).

Now for the part (c) following the similar process we get (5.8), which contradicts (5.1). Hence, the proof is completed. □

Theorem 5.2 (Strong duality). *Let $\overline{x}$ be a locally weak Pareto efficient solution for (P) at which the generalized Slaters constraint qualification is satisfied. Let $f,\ g$ be directionally differentiable at $\overline{x}$ with $f'(\overline{x}, \eta(x, \overline{x}))$ and $g'(\overline{x}, \eta(x, \overline{x}))$ are α-preinvex functions on X. Let g_j be continuous for $j \in \overline{J}(\overline{x})$, then there exist $\overline{\mu} \in R^m_+$ such that $(\overline{x}, 1, \overline{\mu})$ is feasible for (GMWD). If the weak duality between (P) and (MWD) in Theorem 5.1 holds, then $(\overline{x}, 1, \overline{\mu})$ is a locally weak Pareto efficient solution for (GMWD).*

Proof. The proof of this theorem is similar to the proof of Theorem 4.2. □

References

[1] B. Aghezzaf and M. Hachimi, Generalized invexity and duality in multiobjective programming problems, *Journal of Global Optimization* **18** (2000), 91–101.

[2] T. Antczak, Multiobjective programming under d-invexity, *European Journal of Operational Research* **137** (2002), 28–36.

[3] M. S. Bazaraa, H. D. Sherali and C. M. Shetty, *Nonlinear Programming: Theory and Algorithms*, Wiley, New York, 1991.

[4] C. R. Bector, S. K. Suneja and S. Gupta, Univex functions and univex nonlinear programming, In: *Proceedings of the Administrative Sciences Association of Canada* (1992), pp.115-124.

[5] B. D. Craven, Invex functions and constrained local minima, *Bulletin of Australian Mathematical Society* **24** (1981), 357–366.

[6] B. D. Craven, Nondifferentiable optimization by smooth approximation, *Optimization* **17** (1986), 3–17.

[7] R. R. Egudo and M. A. Hanson, Multi-objective duality with invexity, *Journal of Mathematical Analysis and Applications* **80** (1987), 469–477.

[8] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, *Journal of Mathematical Analysis and Applications* **80** (1981), 545–550.

[9] M. A. Hanson and B. Mond, Necessary and sufficient conditions in constrained optimization, *Mathematical programming* **37** (1987), 51–58.

[10] V. Jeyakumar and B. Mond, On generalized convex mathematical programming, *Journal of Australian Mathematical Society Series B* **34** (1992), 43–53.

[11] R. N. Kaul, S. K. Suneja and M. K. Srivastava, Optimality criteria and duality in multiple objective optimization involving generalized invexity, *Journal of Optimization Theory and Applications* **80** (1994), 465–482.

[12] O. L. Mangasarian, *Nonlinear Programming*, McGraw Hill, New York, 1969.

[13] S. K. Mishra, Lagrange multipliers saddle point and scalarizations in composite multiobjective nonsmooth programming, *Optimization* **38** (1996), 93–105.

[14] S. K. Mishra, On sufficiency and duality in nonsmooth multiobjective programming, *Opsearch* **34** (1997), 221–231.

[15] S. K. Mishra, On multiple objective optimization with generalized univexity, *Journal of Mathematical Analysis and Applications* **224** (1998), 131–148.

[16] S. K. Mishra and G. Giorgi, Optimality and duality with generalized semi-univexity, *Opsearch* **37** (2000), 340–350.

[17] S. K. Mishra and R. N. Mukherjee, On generalized convex multiobjective nonsmooth programming, *Journal of Australian Mathematical Society Series B* **38** (1996), 140–148.

[18] S. K. Mishra and M. A. Noor, On vector variational-like inequality problems, *Journal of Mathematical Analysis and Applications* **311** (2005), 69–75.

[19] S. K. Mishra, G. Giorgi and S. Y. Wang, Duality in vector optimization in Banach spaces with generalized convexity, *Journal of Global Optimization* **29** (2004), 415–424.
[20] S. K. Mishra, R. P. Pant and J. S. Rautela, Generalized α-invexity and nondifferentiable minimax fractional programming, *Journal of Computational and Applied Mathematics* **206** (2007), 122–135.
[21] S. K. Mishra, S. Y. Wang and K. K. Lai, Complex minimax programming under generalized convexity, *Journal of Computational and Applied Mathematics* **167** (2004), 59–71.
[22] S. K. Mishra, S. Y. Wang and K. K. Lai, Optimality and duality in nondifferentiable and multiobjective programming under generalized d-invexity, *Journal of Global Optimization* **29** (2004), 425–438.
[23] S. K. Mishra, S. Y. Wang and K. K. Lai, Optimality and duality for multiple objective optimization under generalized type-I univexity, *Journal of Mathematical Analysis and Applications* **303** (2005), 315–326.
[24] S. K. Mishra, S. Y. Wang and K. K. Lai, Multiple objective fractional programming involving semilocally type-I preinvex and related functions, *Journal of Mathematical Analysis and Applications* **310** (2005), 626–640.
[25] B. Mond and T. Weir, Generalized concavity and duality. In: Schaible, S. and Ziemba, W.T. (eds.), *Generalized concavity Optimization and Economics*, Academic Press, New York (1981), pp. 263-280.
[26] M. A. Noor, On generalized preinvex functions and monotonicities, *Journal of Inequalities in Pure and Applied Mathematics* **5** (2004), 1–9.
[27] R. P. Pant and J. S. Rautela, α-invexity and duality in mathematical programming, *Journal of Mathematical Analysis and Approximation Theory* **1** (2006), 14–114.
[28] T. W. Reiland, Nonsmooth invexity, *Bulletin of the Australian Mathematical Society* **42** (1990), 437–446.
[29] N. G. Rueda and M. A. Hanson, Optimality criteria in mathematical programming involving generalized invexity, *Journal of Mathematical Analysis and Applications* **130** (1988), 375–385.
[30] N. G. Rueda, M. A. Hanson and C. Singh, Optimality and duality with generalized convexity, *Journal of Optimization Theory and Applications* **86** (1995) 491–500.
[31] S. K. Suneja and M. K. Srivastava, Optimality and duality in nondifferentiable multiobjective optimization involving d-type-I and related functions, *Journal of Mathematical Analysis and Applications* **206** (1997), 465–479.
[32] T. Weir and B. Mond, Preinvex functions in multiple objective optimization, *Journal of Mathematical Analysis and Applications* **136** (1988), 29–38.
[33] Y. L. Ye, d-invexity and optimality conditions, *Journal of Mathematical Analysis and Applications* **162** (1991), 242–249.