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The Performance of Quantile Regression and Linear Regression with Heteroskedasticity was Compared in a Simulated Study

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Abstract: The least-square estimator has several drawbacks when dealing with heteroscedasticity; this estimate will not be a Best Linear Unbiased Estimator (BLUE). Quantile Regression is a dependable option; however, it has some substantial computational problems. We compare five resampling approaches to estimate the standard error of the coefficients, in the situation of heterogeneity, for inference. According to simulation study, quantile regression beats linear regression and is also better when predicting errors in the presence of heterogeneity.

Keywords: Bootstrap Approach, Heteroskedasticity Problem, Linear Programming, Linear Regression, Quantile Regression.

1 Introduction

An estimator approach must be employed to estimate the parameter model when modelling the relationship between covariates and responses. When estimating parameter values, the Ordinary Least Squares (OLS) method is used to reduce the sum of squares of the error to the smallest possible value. In this case, the OLS is known as BLUE (Best Linear Unbiased Estimator), when all model assumptions are fulfilled. However, if one or more of the assumptions are not satisfied, it may be deceptive. While the great majority of regression models focus on assessing the conditional mean of the dependent variable, there is growing interest in approaches for modelling other features of the conditional distribution. Quantile Regression (QR) is a common method for modelling the quantiles of a dependent variable given a collection of covariate variables (Huang et al., 2017)[1].

While it is well known that Simple Linear Regression (SLR) may be used to predict the anticipated value of a continuous outcome given the variables in the model, Quantile Regression (QR) is a statistical approach used to estimate and infer conditional quantile functions. Quantile regression can be used to compare across groups the full distribution of a continuous response or a single quantile of the answer (Cade & Noon, 2003)[2]. The quantile regression approach has the benefit of allowing for the analysis of connections between factors other than the conditional mean of the response. The most appealing aspect of QR is that it does not implicitly impose limiting assumptions of changing location on the way variables impact the response by allowing covariate effects to be investigated at different quantiles. This similar property makes QR estimates more robust to outliers than LSR estimates. Furthermore, in the presence of a patterned dataset, a skewed distributed data set, or outliers, it would be conceivable to assess the influence of a set of independent factors on a particular conditional quantile of a certain outcome using variant estimates (John & Nduka, 2009)[3].

Quantile regression differs from Ordinary Least Squares (OLS) in various ways:

I. Given a collection of explanatory variables, quantile regression may be used to define the full conditional distribution of a dependent variable.

II. QR provides a more complete view of the independent factors' influence on the dependent variable.

III. QR has a linear programming model that allows for straightforward estimate; it provides a robust measure of location.

IV. When the error term is non-normal, the quantile regression estimator outperforms the least squares estimator.

V. Modeling flexibility for data with diverse conditional distributions.

VI. The median regression is less sensitive to outliers than the OLS regression.

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(Ibrahim & Yahaya, 2015)[4] and (Rodriguez & Yao, 2017)[5]

Koenker and Basset (1978)[6] created quantile regression to model the quantile of a response variable, such as the median, based on the values of a set of predictors. The conditional quantile gets its name from the fact that it is calculated depending on a set of predictors. Because researchers are often more interested in the more extreme values of the conditional distribution than the mean. Therefore, quantile regression is an acceptable approach of study in the field of public health.

In this study, we compare modelling generated data with non-constant error variance using the quantile regression technique with straightforward linear regression. The theoretical underpinning of quantile regression is described in Section 2 of the study. Section 3 discusses two approaches for estimating the simplex and the interior point quantile models. We provide five different techniques for conducting standard error using the "bootstrap approach" for the inference quantile model. In section 4, we demonstrated how to compare quantile regression and simple regression using a simulated case study with and without Leave-One-Out-cross validation (LOOCV) and some error measures such as, Mean Absolute Error (MAE), Mean Absolute Percent Error (MAPE), and Root Mean Square Error (RMSE).

The simulation study was conducted for various conditional quantiles of the response variable (10th, 25th, 50th, 75th, and 90th) and with different sample sizes (15, 25, 30, 50, 200,500, and 700), and two methods for estimating coefficients of the quantile model; simplex, and interior point methods. In Section 4, we wrap off with a brief discussion.

2 The Essentials of the Quantile Regression Model (QRM)

Quantile Regression introduced by "Koenker and Basset (1978)" [6] is a good alternative to ordinary least squares regression. While simple least squares regression minimizes the sum of squared errors, the median regression estimator minimized the total of absolute errors. By minimizing an asymmetrically weighted sum of absolute errors, the remaining conditional quantile functions are estimated.

Quantile Regression model specifies changes in the conditional quantiles. Quantiles defined to be a particular location for a given random variable (y); therefore, $\tau - th$ quantile denotes the value of (y) such that $\mathrm{prob}\ (\mathrm{Y} \leq \mathrm{y}) = \tau$ by using the definition of cdf, $\mathrm{F(y)} = \mathrm{prob}\ (\mathrm{Y} \leq \mathrm{y})$ the quantile function can be denoted as its inverse as following:

$$Q_{\mathrm{y}}\ (\tau) = \mathrm{F}_{\mathrm{y}}^{-1}\ (\tau) = \inf\{\mathrm{y}: \mathrm{F(y)} \geq\ \tau\} \qquad (1)$$

A linear regression model for the τ -the conditional quantile of $\mathrm{y_i}$ can be expressed as

$$\mathrm{Q}_{\tau}(\mathrm{y}|x) = x'\beta_{\tau} \qquad (2)$$

Where y is a scalar dependent variable, x' is the k × 1 vector of explanatory variables, β is the coefficient vector, τ is the conditional quantile of interest for $\tau \in (0,1)$

An estimate for the $\tau - th$ quantile of (Y) can be obtained by minimizing the next objective function

$$\hat{\beta}_{\tau} = \min_{\beta} \sum_{\mathrm{i}=1}^{n} \rho_{\tau}(\mathrm{y}_i - x_i{}'\beta) \qquad i = 1 \dots. n \qquad (3)$$

Where the loss function:

$$\rho_{\tau}(u) = u(\tau - I(u < 0))$$

Or

$$\hat{\beta}_{\tau} = \min_{\beta} \sum\nolimits^{n}_{\substack{i=1 \\ \mathrm{y}_i < x_i{}'\beta}} (1 - \tau)(|\mathrm{y}_i - x_i{}'\beta|) + \sum\nolimits^{n}_{\substack{i=1 \\ \mathrm{y}_i \geq x_i{}'\beta}} (\tau)(|\mathrm{y}_i - x_i{}'\beta|) \qquad (4)$$

Since, minimizing a sum of asymmetrically weighted absolute residuals (giving different weights to positive and negative residuals) produces the quantiles, the symmetry of absolute value yields the median (by setting τ =0.5).

Since, the symmetry of absolute value yields the median (by setting τ = 0.5) (l_1 problem), then minimizing a sum of asymmetrically weighted absolute residuals (giving different weights to positive and negative residuals) yields the quantiles. So, we can use (loss function ρ_{τ}) that is an asymmetric weight to the error depending on the quantile and the overall sign of the error (Koenker & Hallock, 2001)[7] and (John & Nduka, 2009)[3].

3 Aspects of Quantile Regression Computation

Since, the objective function (3) is not differentiable; the traditional method of differentiating the objective function is no longer applicable. A feasible method for estimating the parameters β of quantile regression is the linear programming method (Zhao & Yu, 2020)[8].

$$\text{Let} \begin{cases} \mu = [y - X\beta]_+ \\ v = [X\beta - Y]_+ \\ \emptyset = [\beta]_+ \\ \varphi = [-\beta]_+ \\ y = [y_1, \dots . y_n] \\ X = [x_1, \dots . x_n] \\ [z]_+ \text{ is the non} - \text{negative part of z} \end{cases} \quad (5)$$

Let $D_{LAR}(\beta) = \sum_{i=1}^{n} |y_i - x_i'\beta|$ and $D_{\rho_\tau}(\beta) = \sum_{i=1}^{n} \rho_\tau(y_i - x_i'\beta)$. The l_1 problem, $\min_\theta D_{LAR}(\theta)$, can be reformulated as:

$$\min_\beta \{e'\mu + e'v | y = X\beta + \mu - v, (\mu, v) \in \mathbb{R}^{n+}\} \quad (6)$$

Where e denotes a n vector of ones.

$$\text{Let} \begin{cases} B = [X, -X, I, -I] \\ \theta = [\emptyset', \varphi', \mu', v']' \\ d = [0'\ 0'\ e'\ e']' \end{cases}$$

Where $0' = [0\ 0 \dots .]_p$

The reformulation presents a standard LP problem

$$\min_\theta \{d'\theta\} \quad \text{Subjcted to} \quad B\theta = y, \quad \theta \geq 0 \qquad \text{(Primal Problem)(P)} \quad (7)$$

The primal problem has the dual formulation

$$\max_d \{y'z\} \quad \text{Subjcted to} \quad B'z \leq d \qquad \text{(Dual Problem)(D)} \quad (8)$$

This can be simplified as:

$$\max_z \{y'z | X'z = 0, z \in [-1, 1]^n\}$$

By setting $\gamma = (\frac{1}{2}z + \frac{1}{2}e)$, $b = (\frac{1}{2}X'e)$

The problem becomes

$$\max_\gamma \{y'\gamma | X'\gamma = b, \gamma \in [0, 1]^n\} \quad (9)$$

For quantile regression, the minimization problem is $\min_\beta \sum \rho_\tau(y_i - x_i'\beta)$, and a similar set of steps leads to the dual formulation:

$$\max_z \{y'z | X'z = (1-\tau)X'e,\ z \in [0,1]^n\}, \quad (10)$$

(Davino et al., 2013)[9] and (Chen &Wei, 2005)[10].

3.1 Simplex Algorithm

Median regression (l_1 regression) has been defined as linear programming problems that can be addressed fast using a simplex approach since the 1950s. Barrodale and Roberts (1974)[11] devised an efficient variant of the simplex method that uses the special structure of the coefficient matrix (B) to solve the fundamental LP problem (P) in two steps. In the first step, only the columns ending in (X) or (-X) are chosen as crucial columns. In the second step, only the columns in (I) or (-I) as a basis or non-basic columns are exchanged. The algorithm obtains an optimal solution by executing these two stages interactively (Chen, 2004)[12]. The approach of Barrodale and Roberts was the first to use the bounded variables dual form of the median regression issue. Koenker & d'Orey (1993)[13] utilized a specific version of the simplex method for median regression to conduct quantile regression with any given quantile, encompassing the complete quantile process. The worst case for this simplex technique is computationally demanding for large datasets. The algorithm's careful and exact coding, on the other hand, makes it suitable for datasets with fewer than 5,000 observations and 50 variables (Chen & Wei, 2005)[10] and (Chen, 2004)[12].

3.2 Interior Point Algorithm

Alternative methods have been developed for addressing huge LP problems. Karmarkar's (1984)[14] inner point technique solves a series of quadratic issues by using an ellipsoid to approximate the relative interior of the constraint set, rather than proceeding from vertex to vertex around the constraint set's outer surface as the simplex requires (Chen &Wei, 2005)[10]. Karmarkar's method was a new polynomial-time algorithm with polynomial computational complexity in linear programming. Despite, the fact that a single iteration of the Karmarkar algorithm is expensive, optimality is achieved after just a few iterations, making the technique computationally appealing (Zhao & Yu, 2020)[8]. The interior point method is shown to perform better than the Simplex algorithm in the worst-case scenario (Chen, 2004)[12]. Interior point algorithms come in a variety of shapes and sizes. The Primal-Dual with Predictor-Corrector method is the most often used l_1 regression or quantile regression technique (Chen & Wei, 2005)[10].

Let $c = \mathrm{y}, b = (1-\tau)X'e$ and $A = X'$. The dual problem, eq. (10), with a general upper bound (u) is

$$\max\ \{c'z\}$$

Subjcted to

$$Az = b$$
$$0 \le z \le u$$

To solve this LP problem, $0 \le z \le u$ is split in to $z \ge 0$ and $z \le u.$ Let (v) be the primal slack that $z + v = u,$ and associated dual variable (w) with these constraints. The interior point solves the system of equations to satisfy the "Karush-Kuhn-Tuker (KKT) conditions for optimality:

$$\text{(KKT)}\quad \begin{cases} Az = b \\ z + v = u \\ A't + s - w = c \\ ZSe = 0 \\ VWe = 0 \\ z, s, v, w \ge 0, \end{cases} \qquad (11)$$

where $Z = \text{diag}\ (\text{z})$, that is $Z_{i,j} = z_{i,j}$ if $i = j$, $Z_{i,j} = 0$ otherwise, $S = \text{diag}\ (\text{s})$, $W = \text{dig}(w), V = \text{diag}\ (v)$ these are the conditions for feasibility with the complementarity conditions $ZSe = 0$, $VWe = 0$ added $c'z = b't - u'w$ must occur at the optimum. Complementarity conditions force the optimal objectives of the primal and dual to be equal, $c'z_{opt} = b't_{opt} - u'w_{opt}$ (Chen & Wei,2005)[10].

The interior point algorithm works by iteratively using Newton's method to locate a decent direction $(\Delta z^k, \Delta t^k, \Delta s^k, \Delta v^k, \Delta w^k)$ to move from the current solution $(z^k, t^k, s^k, v^k, w^k)$ towards a better solution (Chen, 2004)[12].

This is accomplished in two steps. The first step is known as *an affine step*, and it involves solving a linear problem using Newton's method in order to determine a direction $(\Delta z^k_{aff}, \Delta t^k_{aff}, \Delta s^k_{aff}, \Delta v^k_{aff}, \Delta w^k_{aff})$ for reducing complementarity toward zero. The second step is known as the *centering step*, and it involves solving another linear system to find a centering vector $(\Delta z^k_c, \Delta t^k_c, \Delta s^k_c, \Delta v^k_c, \Delta w^k_c)$ in order to reduce complementarity even further. The centering phase may not significantly reduce complementarity, but it does strengthen the central path and makes significant progress toward the optimal in the next iteration. After you've completed these two steps, then

$$(\Delta z^k, \Delta t^k, \Delta s^k, \Delta v^k, \Delta w^k) = (\Delta z^k_{aff}, \Delta t^k_{aff}, \Delta s^k_{aff}, \Delta v^k_{aff}, \Delta w^k_{aff}) + (\Delta z^k_c, \Delta t^k_c, \Delta s^k_c, \Delta v^k_c, \Delta w^k_c) \quad (12)$$

$$(z^{k+1}, t^{k+1}, s^{k+1}, v^{k+1}, w^{k+1}) = (z^k, t^k, s^k, v^k, w^k) + \alpha\,(\Delta z^k, \Delta t^k, \Delta s^k, \Delta v^k, \Delta w^k), \quad (13)$$

where, α is the step length assigned a value as large as possible but not so large that a $z_i^{k+1}, s_i^{k+1}, v_i^{k+1}$ or w_i^{k+1} is "too close" to zero. The Predictor-Corrector variant usually takes less iteration to reach the optimum, although requires solving two linear equations instead of one. In both the affine step and the centering step, factorization of the $(X'[\emptyset^k]^{-1}X)$ matrix, where $\emptyset^k$ is a diagonal matrix computed in k^{th} iteration, takes the majority computing time when solving the linear systems. However, the additional overhead of calculating the second linear system is small, as the factorization of the $(X'[\emptyset^k]^{-1}X)$ matrix has already been performed to solve the first linear system (Chen & Wei, 2005)[10].

3.3 Resampling Methods

Resampling procedures can be utilized as a viable alternative to asymptotic inference since they allow parameter standard errors to be calculated without any assumptions about the error distribution (Yanuar et al., 2019)[15].

In QR analysis, several contributions in the literature imply that bootstrap is the best resampling approach. This simulation compares empirically the bootstrap approaches in two quantile regression models for predicting standard error of coefficients at various conditional quantiles for each sample size:

- "XY" pair method or design matrix bootstrap (Kocherginsky et al.,2005)[16].
- "WXY" uses the generalized bootstrap with unit exponential weights (Bose & Chatterjee, 2003)[17].
- "WILD" uses the wild bootstrap method (Feng et al, 2011)[18].
- "PWY" method based on pivotal estimating functions (Parzen et al., 1994)[19].
- "MCMB" Markov chain marginal bootstrap (He & Hu, 2002)[20].

4 Simulation Study

In practice, we want to compare many statistical prediction scenarios and select the most effective one. To assess a model's performance on a dataset, we must assess how well the model's predictions match the observed data. Leave-One-Out Cross-Validation (LOOCV) is a popular approach for accomplishing this. Cross-validation is a superior approach for evaluating models than residuals. The difficulty with residual assessments is that they do not show how well the learner will do when asked to generate new predictions for data it has not seen before (Wong, 2015)[21].

One way to overcome this problem is to not use the entire data set when training a learner. Some of the data is removed before training begins. After training, the data that was deleted can be used to assess the learned model's performance with "fresh" data. This is the fundamental concept behind a broad family of model assessment techniques known as cross-validation. The purpose of our simulation is to know the ability of both quantile regression and simple linear regression models can generate new predictions with cases it has not already seen (using situations they haven't encountered before). To measure the difference between the predictions made by the model and the actual observations, some measures are used in the comparison such as Mean Absolute Error "MAE", Mean Absolute Percentage Error "MAPE", and Root Mean Square Error "RMSE" with and without the use of Leave One - Out of Cross-Validation Method (LOOCV) in the heterogeneous case. Simulation study compares simple linear regression to quantile regression with "the simplex algorithm and the interior point algorithm" at each sample sizes for n=15, 25, 30, 50, 200,500 and 700 in order to observe the performance of SLR and QR as sample size increased.

The generated response variable was defined as $y_i = 14 + 0.4x_i + \varepsilon_i$. With x_i is generated from a Uniform distribution ranging from 1 to 6, ε_i is generated from a Normal distribution with mean (0) and variance is $0.5 + 0.03 * x_i^2$. Generating the random error term with a function depending on x_i yields a response variable that is heteroscedastic for the range of x_i. Therefore, the problem of heterogeneity of variance is summarized in the error distribution, which is $\varepsilon_i \sim N(0, \sqrt{0.5 + 0.03 * x_i^2})$. The number of replication was set to be S=1000 times, which is the average number of iterations that were used in previous researches; the results of this part presented in Table 1(a) and Table 1(b).

The second purpose of the simulation estimate parameter and its standard errors for the quantile regression analyses with simplex method (br) and interior point algorithm frich newton or (fn) using different methods of the bootstrap approach such as (XY- WXY- PWY- Wild- MCMB). With $y_i = 14 + 0.4x_i + \varepsilon_i$ for the sample sizes, n=15, 25, 30, 50, 200, 500 and 700 at different conditional quantiles (10th, 25th, 50th, 75th, and 90th) were conducted, these results are presented in Table 2(a), Table 2(b), Table 2(c) and Table 2(d).

Table 1(a): Comparison of simple linear regression and quantile regression with both the simplex and the interior-point algorithms across "MAE, MAPE and RMSE" with and without "leave one out cross-validation" in case of heteroscedasticity with sample sizes 15, 25, 30, and 50.

Sample size	Criteria		Simple Regression	Quantile Regression									
				$\tau = 0.1$		$\tau = 0.25$		$\tau = 0.5$		$\tau = 0.75$		$\tau = 0.9$	
				br	fn	br	fn	br	fn	br	fn	br	fn
n=15	LOOCV	MAE	0.61767722	1.1150383	1.115039	0.98911847	0.98911847	0.5424120	0.5424120	0.7960522	0.7960522	0.9231220	0.9231221
		MAPE	0.27307395	0.49295663	0.49295675	0.43728743	0.43728743	0.2397994	0.2397994	0.3519332	0.3519332	0.40811060	0.40811062
		RMSE	0.67486695	1.27101019	1.27101020	0.91295520	0.91295520	0.6654364	0.6654364	0.8782305	0.8782305	1.08904746	1.08904759
	Without LOOCV	MAE	0.52951206	0.97422120	0.97422121	0.76571426	0.76571426	0.5289507	0.5289507	0.6916220	0.6916220	0.85120311	0.85120347
		MAPE	0.03524703	0.06272881	0.06272882	0.04970719	0.04970719	0.0352291	0.0352291	0.0472457	0.0472457	0.05792767	0.05792769
		RMSE	0.66535925	1.16595468	1.16595469	0.91722817	0.91722817	0.6654364	0.6654364	0.8782305	0.8782305	1.08904746	1.08904788
n=25	LOOCV	MAE	0.81626190	1.5244690	1.5244692	1.21927664	1.21927664	0.7896364	0.78963646	0.87690042	0.87690042	1.2946591	1.2946592
		MAPE	0.21278923	0.3974096	0.3974099	0.31785011	0.31785011	0.20584831	0.20584831	0.22859693	0.22859693	0.33750139	0.33750140
		RMSE	0.93223218	1.7905153	1.7905154	1.27101554	1.27101554	0.94268831	0.942688313	1.11644855	1.11644855	1.56485051	1.56485055
	Without LOOCV	MAE	0.75144802	1.4887560	1.4887561	1.0631811	1.0631811	0.73875697	0.73875697	0.85739611	0.85739611	1.25224111	1.25224119
		MAPE	0.04966508	0.0935124	0.0935125	0.06721518	0.06721518	0.04950694	0.04950694	0.05847572	0.05847572	0.08528960	0.08528962
		RMSE	0.93054370	1.7905151	1.7905154	1.2710155	1.2710155	0.95306472	0.953064672	1.12416136	1.12416136	1.56485053	1.56485055
n=30	LOOCV	MAE	0.72552387	1.19094505	1.19094506	1.06980964	1.06980964	0.69241344	0.69241344	0.93645192	0.93645192	1.05968525	1.05968528
		MAP	0.15892681	0.07671633	0.07671636	0.06954358	0.06954358	0.15167393	0.15167393	0.06390930	0.06390930	0.07246	0.07246

												300	302
		RMSE	0.8553 3632	1.365 18750	1.365 18752	1.152 40069	1.152 40069	0.861 09917	0.8610 9917	1.128 02262	1.128 02262	1.27 585 570	1.27 585 571
	Without LOOCV	MAE	0.6725 7706	1.158 17740	1.158 17743	0.938 26636	0.938 26636	0.662 00104	0.6620 0104	0.886 10632	0.886 10632	1.03 659 946	1.03 659 953
		MAPE	0.0447 2044	0.074 26160	0.074 26163	0.060 20736	0.060 20736	0.044 01631	0.0440 1631	0.060 76575	0.060 76575	0.07 105 363	0.07 105 366
		RMSE	0.8550 6335	1.365 18753	1.365 18756	1.152 40069	1.152 40069	0.864 40330	0.8644 0330	1.128 02262	1.128 02262	1.27 585 570	1.27 585 577
n=50	LOOCV	MAE	0.7066 6761	1.188 51227	1.188 51227	0.814 23564	0.814 23564	0.701 10940	0.7011 0940	0.902 15870	0.902 15870	1.26 357 166	1.26 357 198
		MAPE	0.0907 5675	0.152 63967	0.152 63967	0.104 57162	0.104 57162	0.090 04291	0.0900 4291	0.115 86351	0.115 86351	0.16 227 948	0.16 227 953
		RMSE	0.8355 9279	1.469 85672	1.469 85672	0.982 54600	0.982 54600	0.835 17793	0.8351 7793	1.043 28798	1.043 28798	1.46 874 730	1.46 874 732
	Without LOOCV	MAE	0.6747 3311	1.174 23460	1.174 23460	0.778 31943	0.778 31943	0.674 31496	0.6743 1496	0.836 37168	0.836 37168	1.23 569 990	1.23 569 991
		MAPE	0.0433 8722	0.072 58479	0.072 58479	0.048 52791	0.048 52791	0.043 32856	0.0433 28556	0.055 17642	0.055 17642	0.08 193 149	0.08 193 150
		RMSE	0.8351 0629	1.412 01004	1.412 01004	0.982 42085	0.982 42085	0.835 17793	0.8351 7793	1.043 28798	1.043 28798	1.46 874 730	1.46 874 732

Table 1(b): Comparison of simple linear regression and quantile regression with both the simplex and the interior-point algorithms across "MAE, MAPE and RMSE" with and without "leave one out cross-validation" in case of heteroscedasticity with sample sizes 200, 500, and 700.

Sample size	Criteria		Simple Regression	Quantile Regression									
				$\tau = 0.1$		$\tau = 0.25$		$\tau = 0.5$		$\tau = 0.75$		$\tau = 0.9$	
				br	fn	br	fn	br	fn	br	fn	br	fn
n=200	LOOCV	MAE	0.77 9129 93	1.300 25520	1.300 25520	0.925 93823	0.925 93823	0.7732 5244	0.7732 5244	0.928 56418	0.928 56418	1.270 81959	1.270 81960
		MAPE	0.02 5358 71	0.042 32001	0.042 32001	0.030 13694	0.030 13694	0.0251 6741	0.0251 6741	0.030 22241	0.030 22241	0.041 36192	0.041 36196
		RMSE	0.98 1356 34	1.531 83824	1.531 83824	1.157 67227	1.157 67227	0.9825 6352	0.9825 6352	1.175 94741	1.175 94741	1.528 29715	1.528 29718
	Without LOOCV	MAE	0.77 1200 87	1.294 85455	1.294 85455	0.924 39021	0.924 39021	0.7699 1157	0.7699 1157	0.926 71232	0.926 71232	1.267 45071	1.267 45074
		MAPE	0.05 0543 91	0.081 72777	0.081 72777	0.058 61890	0.058 61890	0.0503 5119	0.0503 5119	0.062 53366	0.062 53366	0.085 70370	0.085 70371

		RMSE	0.98132821	1.53183823	1.53183823	1.16138025	1.16138025	0.98300454	0.98300454	1.17594721	1.17594721	1.52829715	1.52829718
n=500	LOOCV	MAE	0.734403893	1.27580294	1.27580294	0.91964965	0.91964965	0.73234535	0.73234535	0.86335570	0.86335571	1.22814110	1.22814111
		MAPE	0.09550891	0.01659176	0.01659176	0.01196000	0.01196000	0.0952412	0.0952412	0.01122790	0.01122791	0.01597191	0.01597192
		RMSE	0.90742611	1.49210468	1.49210468	1.11377176	1.11377176	0.908589356	0.908589356	1.10071040	1.10071041	1.47719477	1.47719479
	Without LOOCV	MAE	0.72943091	1.27371382	1.27371382	0.90849912	0.90849912	0.727974576	0.727974576	0.86138944	0.86138948	1.22004030	1.22004032
		MAPE	0.04787024	0.08030161	0.08030161	0.05764926	0.05764926	0.047703962	0.047903962	0.05822842	0.05822848	0.08249321	0.08249328
		RMSE	0.90742610	1.49210468	1.49210468	1.10748103	1.10748103	0.908427428	0.908427428	1.10071040	1.10071041	1.47719472	1.47719479
n=700	LOOCV	MAE	0.74104027	1.28670576	1.28670576	0.91937315	0.91937315	0.74099904	0.74099904	0.89574630	0.89574632	1.22106976	1.22106978
		MAPE	0.04853962	0.08089447	0.08089447	0.05810915	0.05810915	0.04830981	0.04830981	0.06045874	0.06045875	0.08264519	0.08264520
		RMSE	0.92105826	1.52198347	1.52198347	1.13465600	1.13465600	0.92171752	0.92171752	1.11748230	1.11748230	1.47232428	1.47232428
	Without LOOCV	MAE	0.73918608	1.28297122	1.28297122	0.91868856	0.91868856	0.73862270	0.73862270	0.89214722	0.89214727	1.22072931	1.22072933
		MAPE	0.04841852	0.08062301	0.08062301	0.05806065	0.05806065	0.04843504	0.04835504	0.06023630	0.06023636	0.08262582	0.08262589
		RMSE	0.92105823	1.52198347	1.52198347	1.13465600	1.13465600	0.92162495	0.92162495	1.12164396	1.12164396	1.47232428	1.47232428

Table 2(a): Bootstrap Simulation Study Results.

Sample Size	Quantiles		Coefficients and standard errors of different estimation method at various conditional quantiles									
			XY		WXY		Wild		PWY		MCMB	
			br	fn	br	fn	br	fn	br	fn	br	fn
n=15	$\tau = 0.1$	b_0 (S.E)	13.2000109 (0.5707098)	13.2000109 (0.7170460)	13.2000109 (0.5045398)	13.2000109 (0.4347610)	13.2000109 (0.4253688)	13.2000109 (0.4551295)	13.2000109 (34.7804600)	13.2000109 (36.3458315)	---------	-------
		b_1 (S.E)	0.3092765 0.1902672	0.3092765 (0.2106324)	0.3092765 0.1599374)	0.3092765 (0.1427698)	0.3092765 (0.1251882)	0.3092765 0.1357287)	0.3092765 7.5043117)(	0.3092765 (7.8096834	-------	-------

			)(		(			(		)		
	$\tau = 0.2$	b_0 (S.E)	13.5805549 (0.7747437)	13.5805550 (0.9214046)	13.5805549 (0.7550544)	13.5805550 (0.6477737)	13.5805549 (0.7192004)	13.5805550 (0.7721260)	13.5805549 (4.0561319)	13.580555 (5.069988)	---------	-------
		b_1 (S.E)	0.3702841 0.2177864)(	0.3702841 (0.2535473)	0.3702841 0.2022040) (	0.3702841 0.1834732)(	0.3702841 0.1924623)(	0.3702841 0.2127359) (	0.370284 0.9038805)(	0.370284 1.123042)(	-------	-------
	$\tau = 0.5$	b_0 (S.E)	14.5644123 (1.0947280)	14.5644123 (0.9992879)	14.5644123 (1.0741791)	14.5644123 (1.0617473)	14.5644123 (1.0278951)	14.5644123 (1.0323510)	14.5644123 (1.1165838)	14.5644123 (1.8406719)	---------	-------
		b_1 (S.E)	0.2126863 0.3460653)(	0.2126863 0.3031749)(	0.2126863 (0.3388922)	0.2126863 (0.3333361)	0.2126863 (0.2784370)	0.2126863 (0.3185852)	0.2126863 0.3445505)(	0.2126863 (0.4803286)	-------	-------
	$\tau = 0.7$	b_0 (S.E)	15.3223247 (0.9643505)	15.3223248 (0.9353268)	15.3223247 (0.8898476)	15.3223248 (0.9631866)	15.3223247 (0.8822022)	15.3223248 (0.8859767)	15.3223247 (1.6013828)	15.3223248 (1.9015461)	---------	-------
		b_1 (S.E)	0.1575487 0.3550746)(	0.1575487 0.3414888)(	0.1575487 (0.3271975)	0.1575487 (0.3117396)	0.1575487 0.3045120)(	0.1575487 (0.3073315)	0.1575487 0.7926737)(	0.1575487 (0.9314562)	-------	-------
	$\tau = 0.9$	b_0 (S.E)	15.1634252 (0.6509846)	15.1634249 (0.6551189)	15.1634252 (0.6022785)	15.1634249 (0.6010100)	15.1634252 (1.1240403)	15.1634249 (1.2930095)	15.1634252 (3.1597951)	15.1634249 (2.2329777)	---------	-------
		b_1 (S.E)	0.4371506 0.2532182)(	0.4371506 (0.2534684)	0.4371506 0.2244259) (	0.4371506 (0.2412366)	0.4371506 (0.4148632)	0.4371506 0.4616529) (	0.4371506 1.5358851)(	0.4371506 (1.3003938)	-------	-------
n=25	$\tau = 0.1$	b_0 (S.E)	11.9713655 (1.0983279)	11.9713655 (1.0799559)	11.9713655 (1.0668898)	11.9713655 (1.0764742)	11.9713655 (1.3651568)	11.9713655 (1.43953140)	11.9713655 (95.4233071)	11.9713655 (4.1181646)	---------	-------
		b_1 (S.E)	0.6535483 (0.2488961)	0.6535483 0.2401527)(	0.6535483 0.2399803) (	0.6535483 0.2433904)(	0.6535483 0.3161381)(	0.6535483 (0.3393415)	0.6535483 1.2296740)(	0.6535483 0.9922930) (	---------	-------
	τ	b_0	13.06	13.063	13.06	13.063	13.063	13.06	13.063	13.06	-------	-------

	= 0.2	(S. E)	31294 (0.9285010)	1294 (1.0110607)	31294 (0.7212653)	1294 (0.8718788)	1294 (0.7104839)	31294 (0.7143362)	1294 (0.9524745)	31294 (1.0723410)		
		b_1 (S. E)	0.4663367 0.2113682)(	0.4663367 (0.2339501)	0.4663367 (0.1676842)	0.4663367 (0.2013055)	0.4663367 0.1743010)(	0.4663367 (0.1791747)	0.4663367 0.2205762)(	0.4663367 0.2776870)(	---------	-------
	τ = 0.5	b_0 (S. E)	13.6018247 (0.8766436)	13.6018248 (0.8970655)	13.6018247 (0.8427379)	13.6018248 (0.8540242)	13.6018247 (0.6706197)	13.6018248 (0.6899513)	13.6018247 (0.9421437)	13.6018248 (0.8434673)	-------	-------
		b_1 (S. E)	0.4429737 0.1826778)(	0.4429737 0.1823267)(	0.4429737 (0.1058435)	0.4429737 0.1946474)(	0.4429737 (0.1623970)	0.4429737 0.1678885)(	0.4429737 (0.2214462)	0.4429737 (0.1977849)	---------	-------
	τ = 0.7	b_0 (S. E)	14.3053836 (1.0345696)	14.3053836 (1.1045071)	14.3053836 (1.0678386)	14.3053836 (1.0828197)	14.3053836 (1.01326945)	14.3053836 (1.0140564)	14.3053836 (1.0901656)	14.3053836 (1.0466680)	-------	-------
		b_1 (S. E)	0.4145441 (0.2386106)	0.4145441 0.2710297)(	0.4145441 0.2442832)(	0.4145441 0.2596042)(	0.4145441 (0.2245814)	0.4145441 (0.2354877)	0.4145441 0.3087461)(	0.4145441 0.2414805)(	---------	-------
	τ = 0.9	b_0 (S. E)	15.3080701 (0.7184563)	15.3080697 (0.6703760)	15.3080701 (0.5057041)	15.3080697 (0.5100743)	15.3080701 (0.6304444)	15.3080697 (0.6457548)	15.3080701 (2.5018992)	15.3080697 (1.7935776)	---------	-------
		b_1 (S. E)	0.3136087 0.1632574)(	0.3136087 0.1526860)(	0.3136089 0.1185471)(	0.3136089 (0.1194674)	0.3136089 0.1380578)(	0.3136089 (0.1419836)	0.3136087 (1.4222036)	0.3136087 0.9946198)(	-------	-------

Table 2(b): Bootstrap Simulation Study Results.

Sample Size	Quantiles		Coefficients and standard errors of different estimation method at various conditional quantiles									
			XY		WXY		Wild		PWY		MCMB	
			br	fn	br	fn	br	fn	br	fn	br	fn
n=30	τ = 0.1	b_0 (S. E)	12.7300193 (0.6710330)	12.7300193 (0.6933380)	12.7300193 (0.5040184)	12.7300193 (0.5210013)	12.7300193 (0.3283073)	12.7300193 (0.3537431)	12.730019 (14.373496)	12.730019 (9.956973)	---------	-------
		b_1	0.404	0.4044	0.404	0.4044	0.4044	0.105	0.4044	0.404	-------	-------

		(S. E)	4950 0.193 8495) (	950 0.1889 (584)	4950 (0.141 1655)	950 0.1495 (226)	950 0.1111 (093)	4547 0.404 4950) (	95 2.5796 (38)	495 (1.789 425)		
	τ = 0.2	b_0 (S. E)	12.74 6211 2 (0.82 1183 2)	12.746 2123 (0.829 0897)	12.74 62112 (0.839 9858)	12.746 2123 (0.857 6918)	12. 746211 2 (0.7151 080)	12.74 62123 (0.737 5095)	12.746 2112 (1.5163 1900)	12.74 62123 (0.841 5016)	------- --	-------
		b_1 (S. E)	0.584 6809 (0.23 4092 2	0.5846 804 (0.237 1166)	0.584 6809 (0.231 0246)	0.5846 804 0.2408 (090)	0.5846 804 0.2408 (090)	0.584 6804 (0.169 2123)	0.5846 809 (0.3519 569)	0.584 6804 0.238 8603) (	-------	-------
	τ = 0.5	b_0 (S. E)	14.23 3633 4 (0.64 6939 2)	14.233 6334 (0.668 0720)	14.23 36334 (0.690 0778)	14.233 6334 (0.697 5387)	14.233 6334 (0.6975 387)	14.23 36334 (0.580 7569)	14.233 6334 (0.6985 993)	14.23 36334 (0.658 3637)	------- --	-------
		b_1 (S. E)	0.312 9341 0.176 1038) (	0.3129 341 (0.169 9850)	0.312 9341 0.173 2867) (	0.3129 341 (0.173 9151)	0.3129 341 (0.1519 304)	0.312 9341 (0.152 5903)	0.3129 341 0.1718 (602)	0.312 9341 (0.169 4058)	-------	-------
	τ = 0.7	b_0 (S. E)	14.79 1038 6 (0.54 3703 9)	14.791 0386 (0.561 8364)	14.79 10386 (0.501 5292)	14.791 0386 (0.519 8659)	14.791 0386 (0.5523 533)	14.79 10386 (0.599 3602)	14.791 0386 (0.7683 011)	14.79 10386 (0.671 9141)	------- --	-------
		b_1 (S. E)	0.378 4980 0.160 6587) (	0.3784 980 0.1730 (406)	0.378 4980 (0.127 1904)	0.3784 980 (0.151 7791)	0.3784 980 (0.1525 111)	0.378 4980 (0.164 2814)	0.3784 980 0.2881 (093)	0.378 4980 0.190 9988) (	-------	-------
	τ = 0.9	b_0 (S. E)	15.04 3812 2 (0.87 8438 4)	15.043 8122 (0.891 2758)	15.04 38122 (0.770 9910)	15.043 8122 (0.771 8941)	15.043 8122 (0.7219 908)	15.04 38122 (0.741 2098)	15.043 8122 (4.2602 714)	15.04 38122 (2.635 3400)	------- --	-------
		b_1 (S. E)	0.422 1348 (0.22 7008 4)	0.4221 348 (0.234 2686)	0.422 1348 (0.199 6093)	0.4221 348 (0.200 5876)	0.4221 348 (0.2280 205)	0.422 1348 0.228 3029) (	0.4221 348 2.8433 (730)	0.422 1348 (1.402 6491)	-------	-------
n=5 0	τ = 0.1	b_0 (S. E)	12.81 2642 0 (0.41 0130 1)	12.812 6420 (0.453 9058)	12.81 26420 (0.410 7345)	12.812 6420 (0.388 5764)	12.812 6420 (0.4157 610)	12.81 26420 (0.485 3485)	12.812 6420 (0.8463 566)	12.81 26420 (0.885 8923)	------- --	-------

		b_1 (S. E)	0.3552351 0.1149459) (	0.3552351 (0.1270548)	0.3552351 (0.1110114)	0.3552351 (0.1054487)	0.3552351 0.1181 (989)	0.3552351 (0.1276330)	0.3552351 0.2018 (122)	0.3552351 0.1940885) (	---------	-------
	τ = 0.2	b_0 (S. E)	13.73694330 (0.38636303)	13.7369433 (0.4041277)	13.73694330 (0.39061659)	13.7369433 (0.4249780)	13.73694330 (0.35987854)	13.7369433 (0.4038099)	13.7369433 (0.4144503)	13.7369433 (0.4933480)	-------	-------
		b_1 (S. E)	0.23194699 0.0867200 (6)	0.2319470 0.1058 (229)	0.23194699 0.09476137 ()	0.2319470 0.1031 (175)	0.23194699 (0.09424508)	0.2319470 0.1070911) (	0.2319470 0.1052 (772)	0.2319470 (0.1527233)	---------	-------
	τ = 0.5	b_0 (S. E)	13.9650195 (0.5165271)	13.9650195 (0.4756555)	13.9650195 (0.4253507)	13.9650195 (0.4682694)	13.9650195 (0.5250806)	13.9650195 (0.5368228)	13.9650195 (0.4993309)	13.9650195 (0.5062857)	-------	-------
		b_1 (S. E)	0.3868798 0.1760076) (	0.3868798 0.1671 (737)	0.3868798 0.1752756) (	0.1701708 0.3868 (798)	0.3868798 0.1525 (493)	0.3868798 (0.1654687)	0.3868798 0.1679 (039)	0.3868798 0.1783012) (	---------	-------
	τ = 0.7	b_0 (S. E)	14.2164774 (0.3982058)	14.2164774 (0.3540801)	14.2164774 (0.3704471)	14.2164774 (0.4134776)	14.2164774 (0.3593911)	14.2164774 (0.4103692)	14.2164774 (0.5413691)	14.2164774 (0.7530600)	-------	-------
		b_1 (S. E)	0.5142862 (0.1622304)	0.5142862 0.1437 (561)	0.5142862 0.1493434) (	0.5142862 0.1601 (326)	0.5142862 0.1399 (584)	0.5142862 0.1556043) (	0.5142862 (0.1804779)	0.5142862 0.2238824) (	---------	-------
	τ = 0.9	b_0 (S. E)	14.8688462 (0.6748245)	14.8688461 (0.6845897)	14.8688462 (0.5557487)	14.8688461 (0.6059213)	14.8688462 (0.7281399)	14.8688461 (0.7535891)	14.868846 (1.501650)	14.8688461 (1.1827868)	---------	-------
		b_1 (S. E)	0.5448430 0.1672983) (	0.5448430 0.1648 (302)	0.5448430 0.1363276) (	0.5448430 0.1372 (621)	0.5448430 (0.2155226)	0.5448430 (0.21929430	0.544843 0.3168 (80)	0.5448430 0.3229455) (	-------	-------

Table 2(c): Bootstrap Simulation Study Results.

Sample Size	Quantiles	Coefficients and standard errors of different estimation method at various conditional quantiles									
		XY		WXY		Wild		PWY		MCMB	
		br	fn	br	fn	br	fn	br	fn	br	fn

n=200	$\tau = 0.1$	b_0 (S. E)	13.14283379 (0.18458392)	13.14283379 (0.17802201)	13.14283379 (0.18235164)	13.1428338 (0.1645578)	13.14283379 (0.16617240)	13.14283379 (0.16234749)	13.14283379 (0.22685298)	13.14283379 (0.20475323)	13.14283379 (0.17860148)	13.14283379 (0.19055186)
		b_1 (S. E)	0.27318815 0.0677681(1)	0.27318815 0.0604(8671)	0.27318815 (0.06347104)	0.2731882 0.0594(808)	0.27318815 0.0567(1353)	0.27318815 0.05442034()	0.08033402 0.2731(8815)	0.27318815 0.06967245()	0.27318815 0.05970739()	0.27318815 0.0557(0625
	$\tau = 0.2$	b_0 (S. E)	13.34333492 (0.22983606)	13.34333492 (0.23775469)	13.34333492 (0.23749582)	13.34333492 (0.23570058)	13.34333492 (0.22109014)	13.34333492 (0.20561813)	13.34333492 (0.23527573)	13.34333492 (0.23770188)	13.34333492 (0.22378647)	13.34333492 (0.23225180)
		b_1 (S. E)	0.35588146 0.0744563(5)	0.35588146 0.0753(8099)	0.35588146 (0.07321328)	0.35588146 0.0739(5501)	0.35588146 (0.07249207)	0.35588146 0.06610478()	0.35588146 0.0737(0259)	0.35588146 (0.073872680	0.35588146 0.06889019()	0.35588146 0.0697(2237)
	$\tau = 0.5$	b_0 (S. E)	13.83897552 (0.20375838)	13.83897552 (0.20857639)	13.83897552 (0.18843754)	13.83897552 (0.19430959)	13.83897552 (0.20073800)	13.83897552 (0.19995587)	13.83897552 (0.20457488)	13.83897552 (0.20028358)	13.83897552 (0.20053376)	13.83897552 (0.20685797)
		b_1 (S. E)	0.44390877 (0.06435703)	0.44390877 0.0665(3589)	0.44390877 0.05898371()	0.44390877 0.0598(9468)	0.44390877 (0.06088287)	0.44390877 0.06005614()	0.44390877 0.0631(3212)	0.44390877 0.06361385()	0.44390877 0.06425358()	0.44390877 0.0658(5652)
	$\tau = 0.7$	b_0 (S. E)	14.16482714 (0.26853014)	14.16482714 (0.32129018)	14.16482714 (0.26485261)	14.16482714 (0.26645980)	14.16482714 (0.22114173)	14.16482714 (0.22084545)	14.16482714 (0.28978113)	14.16482714 (0.29844708)	14.16482714 (0.26934473)	14.1648271 (0.2965669)
		b_1 (S. E)	0.49745375 0.0816074(4)	0.49745375 0.0942(1194)	0.49745375 0.07965122()	0.49745375 0.0816(6465)	0.49745375 0.0681(6395)	0.49745375 (0.06805794)	0.49745375 0.0852(8743)	0.49745375 0.08601236()	0.49745375 0.07677711()	0.4974538 0.0840(198)
	$\tau = 0.9$	b_0 (S. E)	14.67588571 (0.26658921)	14.67588571 (0.23205763)	14.67588571 (0.22424049)	14.67588571 (0.24193307)	14.67588571 (0.21522371)	14.67588571 (0.20603881)	14.67588571 (0.24766289)	14.67588571 (0.25339701)	14.67588571 (0.26068564)	14.67588571 (0.23991592)
		b_1 (S.	0.5433258	0.54332589	0.54332589	0.54332589	0.54332589	0.54332589	0.54332589	0.54332589	0.54332589	0.54332589

		E)	9 0.0915978(3)	0.0869(7541)	(0.07738097)	0.0916(2944)	0.0856(3741)	0.08066467()	0.0904(7083)	0.09868119()	(0.09164325)	(0.08138216)
n=500	$\tau = 0.1$	b_0 (S.E)	13.36113396 (0.21825425)	13.36113395 (0.21711872)	13.36113396 (0.20739731)	13.36113395 (0.22062680)	13.36113396 (0.16527503)	13.36113395 (0.15470078)	13.36113396 (0.22116159)	13.36113395 (0.21492036)	13.36113396 (0.20451120)	13.36113395 (0.19400448)
		b_1 (S.E)	0.19649741 0.0696876(3)	0.19649741 0.0676(9167)	0.19649741 0.06508688()	0.06754907 0.1964(9741)	0.19649741 0.0940(973)	0.05147236 0.06649741()	0.19649741 (0.06895719)	0.19649741 0.06591897()	0.19649741 0.06021259()	0.19649741 (0.05926069)
	$\tau = 0.2$	b_0 (S.E)	13.45796841 (0.15411368)	13.45796841 (0.14507985)	13.45796841 (0.14480593)	13.45796841 (0.14131638)	13.4579684 (0.1435010)	13.45796841 (0.1313536)	13.45796841 (0.14077720)	13.45796841 (0.14566235)	13.45796841 (0.13343102)	13.45796841 (0.15068411)
		b_1 (S.E)	0.35717246 (0.04691785)	0.35717246 0.0446(7465)	0.35717246 0.04420146()	0.35717246 0.0458(1778)	0.3571725 0.0419(297)	0.3571725 0.0404007)(	0.35717246 0.0447(7365)	0.35717246 (0.04612409)	0.35717246 0.04280595()	0.04648076 (0.35717246)
	$\tau = 0.5$	b_0 (S.E)	14.02845033 (0.15250179)	14.02845033 (0.14727767)	14.02845033 (0.15231958)	14.02845033 (0.15891293)	14.02845033 (0.13231353)	14.02845033 (0.13111836)	14.0284503 (0.1477933)	14.02845033 (0.1437833)	14.02845033 (0.13396469)	14.02845033 (0.14231323)
		b_1 (S.E)	0.39824639 0.0370523(1)	0.39824639 (0.03979369)	0.39824639 0.03887015()	0.39824639 0.0403(0645)	0.39824639 (0.03041311)	0.39824639 0.03015527()	0.3982464 0.0395(280)	0.39824639 0.03657738()	0.39824639 0.03385846()	0.39824639 0.0345(036)
	$\tau = 0.7$	b_0 (S.E)	14.1686228 (0.1361631)	14.16862277 (0.12428989)	14.16862277 (0.12426196)	14.1686228 (0.1266623)	14.16862277 (0.11332027)	14.16862277 (0.10797200)	14.1686228 (0.1287110)	14.16862277 (0.12630567)	14.16862277 (0.13108771)	14.16862277 (0.15829412)
		b_1 (S.E)	0.5445988 0.0385513)(	0.54459882 (0.03442417)	0.54459882 (0.03404164)	0.5445988 0.0344(185)	0.54459882 (0.03033834)	0.03883981 0.01459882()	0.5445988 (0.0365621)	0.03731259 0.54459882()	0.54459882 0.03446117()	0.54459882 0.0360(2435)
	$\tau = 0.9$	b_0 (S.E)	14.42086259 (0.19959371)	14.42086259 (0.21394292)	14.42086259 (0.19141528)	14.42086259 (0.18834413)	14.42086259 (0.16037539)	14.42086259 (0.15070290)	14.42086259 (0.17337869)	14.42086259 (0.16350420)	14.42086259 (0.18497644)	14.4208626 (0.1791312)

		b_1 (S.E)	0.62931157 (0.06522583)	0.62931157 0.0642(0037)	0.62931157 0.05957375()	0.62931157 (0.05789011)	0.62931157 0.0518(7316)	0.62931157 (0.04942184)	0.62931157 (0.05441719)	0.62931157 0.05541073()	0.62931157 (0.05925703)	0.6293116 0.0585(164)

Table 2(d): Bootstrap Simulation Study Results.

Sample Size	Quantiles		Coefficients and standard errors of different estimation method at various conditional quantiles									
			XY		WXY		Wild		PWY		MCMB	
			Br	Fn	br	Fn	Br	fn	br	Fn	br	fn
n=700	$\tau = 0.1$	b_0 (S.E)	13.26520591 (0.25345903	13.26520591 (0.23559228)	13.26520591 (0.25174687)	13.26520591 (0.22765399)	13.2652059 (0.2012906)	13.26520591 (0.20040118)	13.26520591 (0.23491763)	13.2652059 (0.2576933)	13.26520591 (0.25345903	13.26520591 (0.23559228)
		b_1 (S.E)	0.26781097 (0.06744843)	0.26781097 (0.06037501)	0.26781097 0.06694987()	0.26781097 (0.05911688)	0.2678110 (0.0519626)	0.26781097 0.05067935()	0.26781097 (0.06170839)	0.2678110 (0.0673241)	0.26781097 (0.06744843)	0.26781097 (0.06037501)
	$\tau = 0.2$	b_0 (S.E)	13.56989248 (0.09305391)	13.56989248 (0.09528379)	13.56989248 (0.09390353)	13.56989248 (0.09037328)	13.56989248 (0.10285546)	13.56989248 (0.09707435)	13.56989248 (0.10375223)	13.56989248 (0.09871386)	13.56989248 (0.09305391)	13.56989248 (0.09528379)
		b_1 (S.E)	0.33726639 (0.0259040)	0.33726639 0.0303(9945)	0.33726639 (0.02850255)	0.33726639 0.0276(8527)	0.33726639 0.0306(7456)	0.33726639 0.02913177()	0.33726639 0.0318(5442)	0.33726639 (0.03087455)	0.33726639 (0.0259040)	0.33726639 0.0303(9945)
	$\tau = 0.5$	b_0 (S.E)	13.94006997 (0.09595077)	13.94006997 (0.09739919)	13.9400700 (0.1059499)	13.94006997 (0.09162193)	13.94006997 (0.07921619)	13.94006997 (0.07563667)	13.94006997 (0.08640105)	13.94006997 (0.09105309)	13.94006997 (0.09595077)	13.94006997 (0.09739919)
		b_1 (S.E)	0.42511664 (0.03238733)	0.42511664 0.0320(6496)	0.4251166 0.0346486)(	0.42511664 0.0310(5137)	0.42511664 0.0269(9180)	0.42511664 (0.02538714)	0.42511664 (0.02749462)	0.42511664 0.03322176()	0.42511664 (0.03238733)	0.42511664 0.0320(6496)
	$\tau = 0.7$	b_0 (S.E)	14.16439095 (0.09828792)	14.16439095 (0.10321454)	14.16439095 (0.09580734)	14.16439095 (0.09145564)	14.16439095 (0.09799735)	14.16439095 (0.09179467)	14.1643909 (0.1006565)	14.16439095 (0.10119435)	14.16439095 (0.09828792)	14.16439095 (0.10321454)
		b_1 (S.	0.5359934	0.53599346	0.53599346	0.53599346	0.53599346	0.53599346	0.5359935	0.53599346	0.5359934	0.53599346

		E)	6 0.0320135(3)	(0.03240598)	(0.03017646)	0.0304(8352)	0.0293(2217)	0.02218227)	0.0325(470)	(0.03345967)	6 0.0320135(3)	(0.0324059
	$\tau = 0.9$	b_0 (S.E)	14.71885037 (0.17963723)	14.71885037 (0.16409051)	14.71885037 (0.16894576)	14.71885037 (0.17417147)	14.71885037 (0.16148812)	14.71885037 (0.14121731)	14.7188504 (0.1879247)	14.71885037 (0.19088438)	14.71885037 (0.17963723)	14.71885037 (0.16409051)
		b_1 (S.E)	0.54211762 0.0415461(5)	0.54211762 0.0378(9154)	0.54211762 0.03775214()	0.54211762 0.0391(6738)	0.54211762 0.0396(3913)	0.54211762 0.03218176()	0.5421176 0.0425(027)	0.54211762 (0.04161641)	0.54211762 0.0415461(5)	0.54211762 0.0378(9154)

Table 1(a) and Table 1(b) show, in the instance of LOOVE, that the values of MAE, MAPE, and RMSE, for the quantile regression at the conditional quantile (τ =0.5), "median regression" are less than their values for the simple linear regression with a small sample size (n=15). When the sample size is increased to n=25, 30, 50, 200, 500, and 700, the results show that the values of MAE and MAPE for quantile regression (τ =0.5) is smaller than their values for simple linear regression, but the value of RMSE is lower for simple linear regression than the median regression (τ =0.5).

In the absence of LOOVE, with all sample sizes, the RMSE for quantile regression at the conditional quantile (τ =0.5) "median regression” is bigger than the RMSE for simple linear regression, although the values of MAE and MAPE for quantile regression (τ =0.5) are fewer than their values for simple linear regression.

When we compare the performance of the Simplex method to the Interior-Point algorithm, we see that with and without (LOOVE), for all used measures MAE, MAPE, and RMSE, for extreme values of conditional quantiles (τ =0.1 and 0.9) with a small sample size n ≤ 30, the simplex algorithm gives results that are relatively smaller than the interior point, which is implying that the simplex algorithm is more accurate than the interior-point algorithm in the lower and upper quantiles. However, in the middle conditional quantile (τ = 0.25, 0.5, and 0.75), the two approaches perform nearly identically.

The performance of the two algorithms is equal in large data, with increasing sample size (n=50,200,500, and 700), except at the upper conditional quantile, where the simplex algorithm gives results that are lower than the interior-point algorithm. This ensured that, while the interior-point algorithm's performance has improved in the lower quantile (τ =0.1), it still has a lower performance than the simplex algorithm at the upper quantile (τ =0.9).

The XY, WXY, Wild bootstrap, PWY, and MCMB approaches are compared in terms of their capacity to estimate standard errors of coefficients. The differences between the five bootstrap approaches may be better highlighted by comparing the results of a heterogeneous quantile model based on calculating coefficients using the simplex and interior-point algorithms. Table 2(a), Table 2(b), Table 2(c), and Table 2(d) display the results.

With a small sample of n=15 and 25, both methods WXY and Wild bootstrap are the best ways to estimate the standard error of coefficients. We notice that with the Wild bootstrap, the simplex algorithm outperforms the interior-point algorithm, whereas with the WXY method, the interior point algorithm outperforms the simplex algorithm.

In reality, its performance improves as (n) increases, such as (n=30 and 50). The XY technique performs somewhat better, while the PWY method performs the poorest at each conditional quantile. One restriction of MCMB's asymptotic validity is that both the number of observations and the duration of the Markov chain must be infinite. As a result, it may not be appropriate for situations with small sample sizes, as MCMB is only valid for high sample sizes.

According to Table 2(c) and Table 2(d), with a large sample size (n=200, 500, 700), the estimated coefficients of the quantile model were derived using the two methods mentioned above (the simplex and interior-point), and the standard errors of the estimated coefficients were determined individually in each technique. As a result, we will first provide discussion on both strategies for estimating our quantile model.

```
Warning message:
In rq.fit.br(x, y, tau = tau, ...) : Solution may be nonunique
```

The warning message in the above code cautions about the solution's non–uniqueness while using the simplex technique. According to Barrodale and Roberts (1974), the inclusion of a null variable in the basis is a clue but not a necessary condition for non–uniqueness. This suggests that the ideal solution is an edge rather than a single vertex of the simplex. The interior-point approach may produce the same warning message about the solution's non-uniqueness, but for a different cause. Because the optimal solution is already near a central path, the non–uniqueness from the interior point implies that a group of points may be the best optimum solution.

The results will be compared for five variants of the bootstrap approach for the standard error according to the two prior algorithms to provide an educated comparison for standard errors. The comparison is carried out in two scenarios: one with a small sample size and the other with a large sample size.

With small sample sizes, n=15, 25, and 30, each WXY and Wild bootstrap methods are the best ways to estimate the standard error of coefficients. We notice that the simplex algorithm outperforms the interior-point algorithm for the Wild bootstrap, but not for the WXY method. In reality, the performance improves as n increases, such as n=30 and 50. Although the XY technique performs somewhat better, the PWY method performs the poorest at each conditional quantile. One disadvantage of MCMB's asymptotic validity is that it needs infinity for both the number of observations and the length of the Markov chain. As a result, it may not be appropriate for situations with small sample sizes, as MCMB is only valid for high sample sizes.

The performance of WXY continues to appear impressive with high sample sizes, n=200, 500, 700, especially with the interior point approach. Both XY and WXY are clearly inferior performance when estimating the model using the interior-point technique to the Wild bootstrap when estimating the model using the simplex technique. The efficiency of the PWY method steadily improves with large samples, especially in the upper quantiles, until it becomes competitive with the XY, but it still falls short of the Wild bootstrap.

The MCMB method is too time-consuming for large problems, and the most important characteristic of this method is that it has higher performance in the middle conditional quantile (0.25, 0.5, and 0.75) when estimating the model with the simplex algorithm, and the best performance in the extreme quantile when estimating the model using the interior point algorithm (0.1 and 0.9).

The violation of the homoskedastic assumption is perhaps the most common reason for using QR, although it is far from the only one. In the following portion of this section, the same prior simulation is utilized to demonstrate many QR properties with regard to the error component. The goal of this application is to demonstrate the SLR "mean regression" and QR behavior for various typologies of homogeneous and heterogeneous error terms using graphs with regression lines at each quantile (10th, 25th, 50th, 75th, and 90th) for large sample sizes of 200,500, and 700. Two models are shown in Table 3: A homogeneous error model with a normal error term is represented by model (1). A heterogeneous error model is represented by Model (2).

Table 3: A summary of the several illustrative models utilized in this article

Error model	Error term	$\boldsymbol{model}_i$
Homogeneous	$\varepsilon \sim N(0,1)$	model (1) $\rightarrow$ $y_i = 14 + 0.4x_i + \varepsilon_i$
Heterogeneous	$\varepsilon \sim N(0, \sqrt{0.5 + 0.03 * x_i^2}\ )$	model (2) $\rightarrow$ $y_i = 14 + 0.4x_i + \varepsilon_i$

The first column indicates the kind of error model, the second column indicates the error term, and the third column indicates the resultant model.

This part focuses on the interpretation of the QR estimated coefficients by drawing a parallel between homogeneous and heterogeneous regression models.

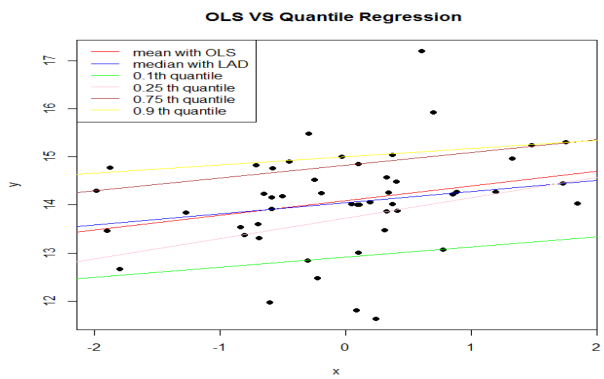

Fig. 1(a). Model (1): homogeneous error model (normal errors)

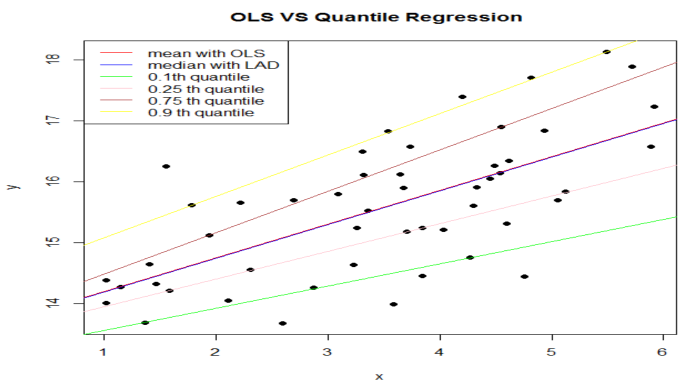

Fig. 1(b). Model (2) : heterogeneous error model

Fig. 1: Scatterplots, OLS line (red line), and conditional quantile lines (other lines), τ = {0.1, 0.25, 0.5, 0.75, 0.9}, for a homogeneous error model (1) and for a heterogeneous error model (2) with sample size n=50.

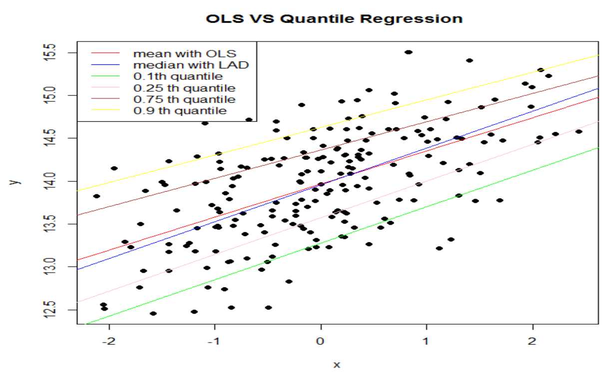

Fig. 2(a). Model (1): homogeneous error model (normal errors)

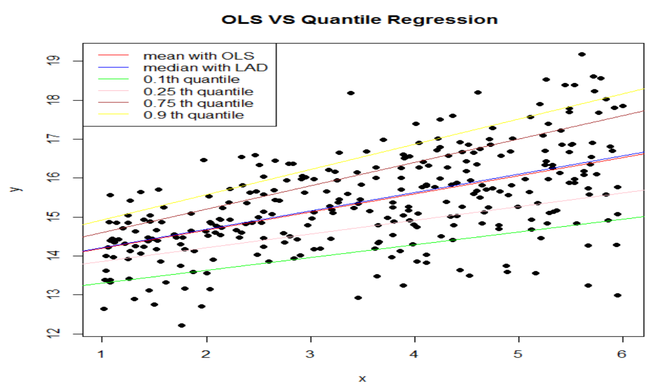

Fig. 2(b). Model (2): heterogeneous error model

Fig. 2: Scatterplots, OLS line (red line), and conditional quantile lines (other lines), τ = {0.1, 0.25, 0.5, 0.75, 0.9}, for a homogeneous error model (1) and for a heterogeneous error model (2) with sample size n=200.

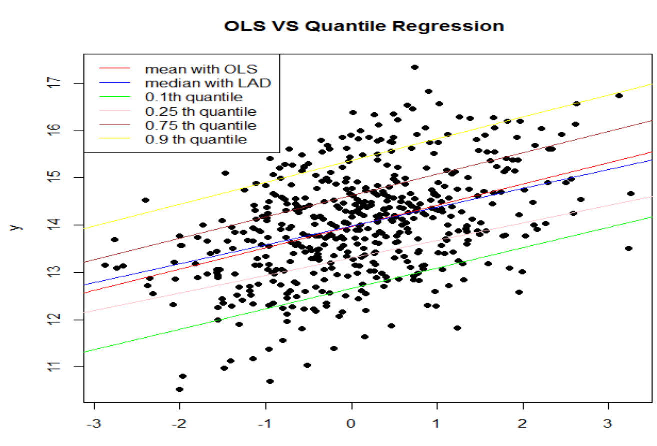

Fig. 3(a). Model (1): homogeneous error model (normal errors)

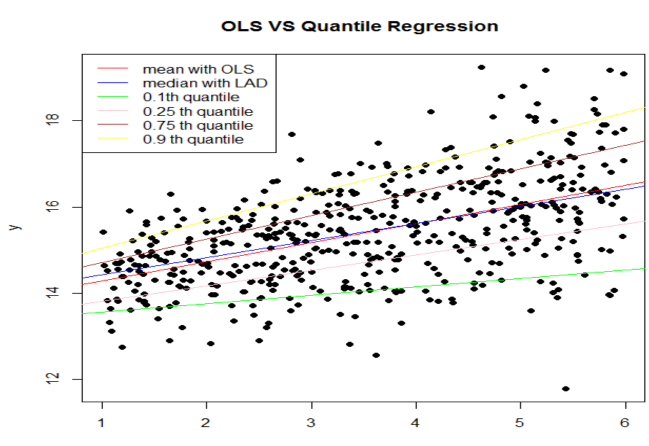

Fig. 3(b). Model (2) : heterogeneous error model

Fig 3: Scatterplots, OLS line (red line), and conditional quantile lines (other lines), τ = {0.1, 0.25, 0.5, 0.75, 0.9}, for a homogeneous error model (1) and for a heterogeneous error model (2) with sample size n=500.

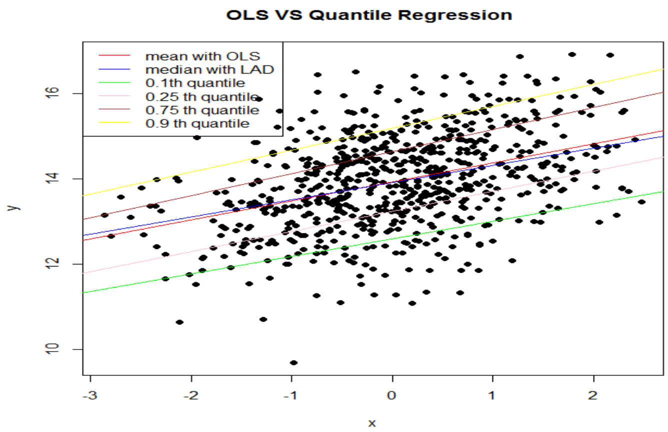

Fig. 4(a). Model (1): homogeneous error model (normal errors)

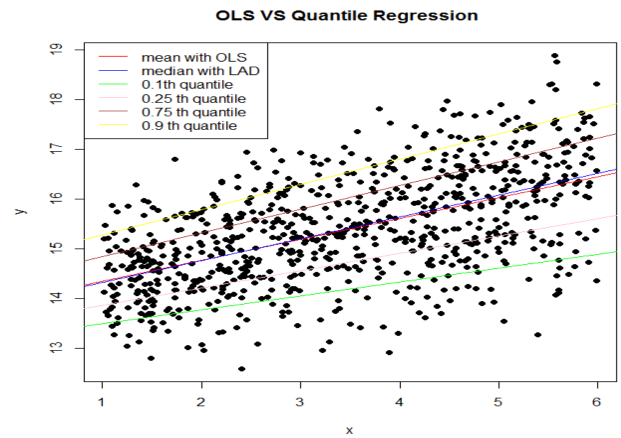

Fig. 4(b). Model (2) : heterogeneous error model

Fig. 4: Scatterplots, OLS line (red line), and conditional quantile lines (other lines), τ = {0.1, 0.25, 0.5, 0.75, 0.9}, for a homogeneous error model (1) and for a heterogeneous error model (2) with sample size n=700.

The previous section focuses on the interpretation of the QR estimated coefficients results by drawing a parallel between homogeneous and heterogeneous regression models. Starting from a comparison of QR behavior on such different models. In fact, it is easy to understand how QR is important in dealing with heteroskedasticity, also, the utilize of different distributions for the error term will offer the opportunity to observe QR capability to estimate the whole conditional distribution in the presence of homoskedasticity.

For both models (1) and (2), scatter plots of conditional mean, conditional median, and conditional quantiles were presented in each figure for τ= (0.1, 0.25, 0. 5, 0.75, and 0.9). Model (1) in each image (1, 2, 3, and 4) shows estimated impact in the homogeneous variance regression: a change in the mean of the (y) distribution conditional on the value of (x). Model (2) in each image (1, 2, 3, and 4) shows impact in the heterogeneous variance regression: a change in each part "quantiles" of the (y) distribution conditional on the value of (x).

We notes that scatter plot of model (2) in case of heterogeneous variance is more scattered than model (1). For the homogeneous variance regression model (1) the only estimated effect is a change in the mean of the distribution of (y) conditional on the value of (x), where SLR slope estimates are then the same at all QR's. In contrast, in the (model 2), QR shows that slope estimates differ across quantiles because the variance in (y) changes as a function of (x). Thus, in such a case, OLS regression analysis provides an incomplete picture of the relationship between variables, as it only focuses on changes at the conditional mean. Therefore, the importance of QR is clear to describe the entire conditional distribution of a dependent variable.

From the previous discussion we conclude that QR offers a more complete view of relationships among variables for heterogeneous regression models, providing a method for modeling the rates of changes in the response variable at multiple points of the distribution when such rates of change are different. However, QR is also a useful tool in the case of homogeneous regression models outside of the classical normal regression model. When the error term satisfies the classical normal assumption.

5: Conclusion

Both MAE and MAPE, in the case of LOOCV, show that quantile regression offers more accuracy than simple linear regression when predicting error in the presence of a heteroscedasticity problem of variance. With all sample sizes, the simplex approach provides greater accuracy of prediction, especially at extreme conditional quantiles (t=0.1 and 0.9). In the intermediate conditional quantiles (t=0.25, 0.5, and 0.75), both the simplex and the interior point algorithms do equally well in predicting errors with and without LOOCV. For simple linear regression, the RMSE with LOOCV is the best metric. Based on the preceding simulation results, we may deduce and suggest the following inferential statistics surrounding standard errors of the coefficients of the quantile regression model:

1- The estimates of the "Wild bootstrap" approach are closer to the beginning point in our simulation in the median regression at (t=0.5) values than the other methods in terms of the capacity to estimate parameter values.

2- With small sample sizes, the "PWY" approach delivers the poorest results, especially in lower quantiles (0.1 and 0.25).

3- While the "XY" approach produces adequate results, it is inefficient in comparison to "Wild" and "WXY".

4- When using the simplex approach to estimate the coefficients, both the "WXY" and "Wild" methods are competitive with all sample sizes.

5- When using the interior point approach to estimating coefficients, there is no objection to using "WXY" with all sample sizes.

6- The "MCMB" method is considered invalid with small samples; however, with large samples, it is recommended to use the "MCMB" with the interior-point algorithm if the values of interest are in the extreme conditional quantiles (0.1 and 0.9), and the "MCMB" with the simplex algorithm if the values of interest are in the middle quantiles (0.25, 0.5, and 0.75).

References:

[1] Huang, Q., Zhang, H., Chen, J., & He, M. J. J. B. B., Quantile regression models and their applications: A review, *Journal of Biometrics & Biostatistics*, **8(3)**, 1-6 (2017).

[2] Cade, B. S., & Noon, B. R., A gentle introduction to quantile regression for ecologists, *Frontiers in Ecology and the Environment*, ***1*(8)**, 412-420 (2003).

[3] John, O. O., & Nduka, E. C., Quantile regression analysis as a robust alternative to ordinary least squares, *Scientia Africana*, ***8*(2)**, 61-65 (2009).

[4] Ibrahim, A., & Yahaya, A,. Analysis of quantile regression as alternative to ordinary least squares, *International Journal of Advanced Statistics and Probability*, **3(2)**, 138-145 (2015).

[5] Rodriguez, R. N., & Yao, Y., Five things you should know about quantile regression. In Proc. The SAS global forum 2017 conference, 2-5, (2017).

[6] Koenker, R., & Basset, G., Asymptotic theory of least absolute error regression, *Journal of the American Statistical Association*, **73(363)**, 618-622 (1978).

[7] Koenker, R., & Hallock, K. F., Quantile regression, *Journal of economic perspectives*, ***15*(4)**, 143-156 (2001).

[8] Zhao, P., & Yu, S., An improved interior point algorithm for quantile regression, *IEEE Access*, **8**, 139647-139657 (2020).

[9] Davino, C., Furno, M., & Vistocco, D., *Quantile regression: theory and applications,* John Wiley & Sons, **988**, (2013).

[10] Chen, C., & Wei, Y., Computational issues for quantile regression, *Sankhyā: The Indian Journal of Statistics*, 399-417 (2005).

[11] Barrodale, I., & Roberts, F. D. K., Solution of an overdetermined system of equations in the l 1 norm [F4]. *Communications of the ACM*, **17(6)**, 319-320 (1974).

[12] Chen, C., An adaptive algorithm for quantile regression, In Proc. Theory and applications of recent robust methods*,* 39-48, (2004).

[13] Koenker, K. R. & D'Orey, V., A remark on computing regression quantiles, *Applied Statistics*, **43**, 410-414 (1993).

[14] Karmarkar, N., A new polynomial-time algorithm for linear programming, In Proc. The sixteenth annual ACM symposium on theory of computing, 302-311 (1984).

[15] Yanuar, F., Yozza, H., Firdawati, F., Rahmi, I., & Zetra, A., Applying bootstrap quantile regression for the construction of a low birth weight model, *Makara Journal of Health Research*, **23(2)**, 5 (2019).

[16] Kocherginsky, M., He, X., & Mu, Y., Practical confidence intervals for regression quantiles, *Journal of Computational and Graphical Statistics*, **14(1)**, 41-55 (2005).

[17] Bose, A., & Chatterjee, S., Generalized bootstrap for estimators of minimizers of convex functions, *Journal of Statistical Planning and Inference*, **117(2)**, 225-239 (2003).

[18] Feng, X., He, X., & Hu, J., Wild bootstrap for quantile regression, *Biometrika*, **98(4)**, 995-999 (2011).

[19] Parzen, M. I., Wei, L. J., & Ying, Z., A resampling method based on pivotal estimating functions, *Biometrika*, **81(2)**, 341-350 (1994).

[20] He, X., & Hu, F., Markov chain marginal bootstrap, *Journal of the American Statistical Association*, **97(459)**, 783-795 (2002).

[21] Wong, T. T., Performance evaluation of classification algorithms by k-fold and leave-one-out cross validation, *Pattern Recognition*, **48(9)**, 2839-2846 (2015).