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# A Fitted Operator Method for a System of Delay Model of Tumor Cells Dynamics within their Micro-Environment

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**Abstract:** This paper deals with the extension of the dynamics modeling the interaction among transformed epithelial cells (TECs), fibroblasts, myofibroblasts, transformed growth factor (TGF$-\beta$), and epithelial growth factor (EGF), in silico, in a setup mimicking experiments in a tumor chamber invasion assay. In the sequel of establishing solution continuously depends on the data and existence of unique solution, we were able to extend the Gronwall's inequality for linear ordinary differential equations, to the Gronwall's inequality for linear, delay ordinary differential equations. The method of upper and lower solutions is utilized to present that the equilibrium points are globally stable, whereas, equilibrium points are analyzed and the conditions for the existence of Hopf bifurcation are also established. Since it is not possible to solve the extended dynamics, nor the original dynamics, we derive, analyze, implement a fitted operator method and present our results. Analysis of the basic properties of the fitted operator method presents that it is consistent, stable and convergent. Since our numerical results are in agreement with our findings, we thus believe that our findings in this study, can indeed contribute more toward the design of the drug which can slow and/or confine tumor invasion.

**Keywords:** Tumor cells' micro-environment, proliferation, migration, delay partial differential equations, Hopf bifurcation; fitted operator, stability analysis.

## 1 Introduction

Since cell types such as epithelial cells, fibroblasts, myofibroblasts, endothelial cells, and inflammatory cells are well known to form an integral part of a tumor micro-environment [1] then, the composition of the surrounding extra-cellular matrix (ECM) may play an important role in confining cancer. This can be achieved by either modulating cell adhesion or blocking Matrix Metalloproteinase (MMP) [1]. In human Ductal Carcinoma In Situ (DCIS), it is understood that Matrix Metalloproteinase (MMP) material have shown that several classes of MMPs are expressed in periductal fibroblasts and myofibroblasts, indicating an intense stromal involvement during early invasion [1,2]. Thus, a situation which corresponds to the case of a more aggressive carcinoma, where tumor cells are degrading the basal membrane and invade into the stroma is considered. The invasion of transformed epithelial cells (TECs) into stroma is an important and complex step toward metastasis [3]. Thus, let $D_n, D_f, D_m, D_E, D_G, D_P$ denote constant diffusion coefficients for the density of transformed epithelial cells, density of fibroblasts (f), density of myofibroblasts (m), concentration of epidermal growth factor (EGF), concentration of transformed growth factor (TGF-$\beta$), and concentration of matrix metalloproteinase (MMP). Therefore, in an effort to understand the complex step toward metastasis, Kim and Friedman in [1] derived the following dynamics

$$\left.\begin{array}{l}\frac{\partial n}{\partial t} = \nabla\cdot(D_n\nabla n) - \nabla\cdot\underbrace{\chi_n n\frac{\nabla E}{\sqrt{1+(|\nabla E|/\lambda_E)^2}}}_{\text{chemotaxis}}\\ -\underbrace{\nabla\chi_n^1 I_s n\frac{\nabla\rho}{\sqrt{1+(|\nabla\rho|/\lambda_\rho)^2}}}_{\text{haptotaxis}}\end{array}\right\}$$

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$$
\left.\begin{array}{l}
+\underbrace{a_{11}\frac{E^4}{k_E^4+E^4}n\left(1-\frac{n}{n_*-a_{12}\rho I_s}\right)}_{\text{proliferation}} \in \Omega_+, t>0,\\
\frac{\partial f}{\partial t}=\nabla\cdot(D_f\nabla f)-\underbrace{a_{12}Gf}_{f\to m}+\underbrace{a_{22}f}_{\text{proliferation}} \in \Omega_-, t>0,
\end{array}\right\}
$$

$$
\left.\begin{array}{l}
\frac{\partial m}{\partial t}=\nabla\cdot(D_m\nabla m)-\nabla\cdot\underbrace{\left(\chi_m m\frac{\nabla G}{\sqrt{1+(|\nabla G|/\lambda_G)^2}}\right)}_{\text{chemotaxis}},\\
+\underbrace{a_{21}Gf}_{f\to m}+\underbrace{a_{31}m}_{\text{proliferation}} \in \Omega_-, t>0,\\
\frac{\partial \rho}{\partial t}=-\underbrace{a_{41}Pn}_{\text{degradation}}\underbrace{+(a_{42}f+a_{43}m)\left(1-\frac{\rho}{\rho_*}\right)}_{\text{release/reconstruction}} \in S, t>0,\\
\frac{\partial E}{\partial t}=\nabla\cdot(D_E\nabla E)+\underbrace{I_{\Omega_-}(a_{51}f+a_{52}m)}_{\text{production}}-\underbrace{a_{53}E}_{\text{decay}} \in \Omega_*, t>0,\\
\frac{\partial G}{\partial t}=\nabla\cdot(D_G\nabla G)+\underbrace{a_{61}I_{\Omega_+}n}_{\text{production}}-\underbrace{a_{62}G}_{\text{decay}} \in \Omega_*, t>0,\\
\frac{\partial P}{\partial t}=\nabla\cdot(D_p\nabla P)+\underbrace{a_{71}I_{\Omega_-}m}_{\text{production}}-\underbrace{a_{72}P}_{\text{decay}} \in \Omega_*, t>0,
\end{array}\right\}
$$

where, $\Omega$ denotes the 3-dimensional domain

$\Omega=\{x=(x_1,x_2,x_3); -L_i<x_i<L_i \text{ for } 1\le i\le 3\}$,

and set

$$
\begin{array}{ll}
\Omega_+=\Omega\cap\{x_1>0\}, & \Omega_-=\Omega\cap\{x_1<0\}\\
\Omega_*=\Omega_+\cup\Omega_-, & \\
\Gamma_+=\partial\Omega_+, & \Gamma_-=\partial\Omega_-,
\end{array}
$$

where, the semi-permeable membrane occupies the planar region

$M=\{-L_i<x_i<L_i,\quad x_1=0, \text{ for } i=2,3\}$,

the extracellular matrix (ECM) occupies a 3-dimensional region

$$
\begin{array}{l}
S=\{-L_0<x_1<L_0, x_1\neq 0,\\
-L_i<x_i<L_i, i=2,3\}
\end{array}
$$

where, $0<L_0<L_1$ and the characteristic function of a set $A$ is denoted by $I_A$, so that

$I_A(x)=1$ if $x\in A$, $\quad A(x)=0$ if $x\notin A$.

The dependent variables are defined in Table 1

and transmission conditions at the semi-permeable membrane are

$$
\left.\begin{array}{ll}
\frac{\partial E^+}{\partial x_1}=\frac{\partial E^-}{\partial x_1}, & -\frac{\partial E^+}{\partial x_1}+\gamma(E^+-E^-)=0, M, t>0,\\
\frac{\partial G^+}{\partial x_1}=\frac{\partial G^-}{\partial x_1}, & -\frac{\partial G^+}{\partial x_1}+\gamma(G^+-G^-)=0, M, t>0,\\
\frac{\partial P^+}{\partial x_1}=\frac{\partial P^-}{\partial x_1}, & -\frac{\partial P^+}{\partial x_1}+\gamma(P^+-P^-)=0, M, t>0,
\end{array}\right\}
$$

**Table 1:** Dependent variables for the model in equation (3-1)

| | |
|---|---|
| $n(x,t)$ | density of transformed epithelial cells (TECs) |
| $f(x,t)$ | density of fibroblasts |
| $m(x,t)$ | density of myofibroblasts |
| $\rho(x,t)$ | concentration of extracellular matrix (ECM) |
| $E(x,t)$ | concentration of epidermal growth factor (EGF) |
| $G(x,t)$ | concentration of transformed growth factor (TGF-$\beta$) |
| $P(x,t)$ | concentration of matrix metalloproteinase (MMP) |

where,

$$
E(x,t)=\begin{cases} E^+(x,t) \text{ if } x_1>0,\\ E^-(x,t) \text{ if } x_1<0,\end{cases}
$$

$$
G(x,t)=\begin{cases} G^+(x,t) \text{ if } x_1>0,\\ G^-(x,t) \text{ if } x_1<0,\end{cases}
$$

$$
P(x,t)=\begin{cases} P^+(x,t) \text{ if } x_1>0,\\ P^-(x,t) \text{ if } x_1<0.\end{cases}
$$

Here, $\gamma$ is a positive parameter which depends on the size and density of the holes in the membrane [4], where, Kim and Friedman in [1] have assumed that the membrane is not permeable to cells, thus, no flux boundary conditions

$$
\left.\begin{array}{l}
\left(D_n\nabla n-\chi_n n\frac{\nabla E}{\sqrt{1+(|\nabla E|/\lambda_E)^2}}-\chi_n^1 I_S n\frac{\nabla\rho}{\sqrt{1+(|\nabla\rho|/\lambda_\rho)^2}}\right)\cdot\nu\\
=0, \Gamma_+, t>0,\\
D_f\nabla f\cdot\nu=0,\ \Gamma_-, t>0,\\
\left(D_m\nabla m-\chi_m m\frac{\nabla G}{\sqrt{1+(|\nabla G|/\lambda_G)^2}}\right)\cdot\nu=0,\\
\Gamma_-, t>0, D_\rho\nabla\rho\cdot\nu=0,\\
D_E\nabla E\cdot\nu=0,\quad D_G\nabla G\cdot\nu=0,\\
D_P\nabla P\cdot\nu=0, \partial\Omega-\{M\}, t>0,
\end{array}\right\}
$$

are imposed, where, $\nu$ denotes the outward normal vector. The prescribed one-dimensional space initial conditions are given as

$$
\left.\begin{array}{l}
n_0(x)=\frac{1}{2}\left(1+\tanh\left(-\frac{1}{\varepsilon}(0.8-x)\right)\right)\in\Omega_+,\\
f_0(x)=0,143\frac{1}{2}\left(1+\tanh\left(-\frac{1}{\varepsilon}(x-0.2)\right)\right)\in\Omega_-,\\
m_0(x)=0.0\in\Omega_-, \rho_0(x)=1.0\in S, E_0(x)=1.0\in\Omega_*,\\
G_0(x)=1.0\in\Omega_*, P_0(x)=0.0\in\Omega_*.
\end{array}\right\} \quad (1)
$$

Thus, one is of the view that the incorporation of time $\tau>0$ required for some transformations and/or productions to take place should be incorporated into the dynamics in equation 3-1. Such transformation are fibroblasts into myfibroblasts, degradation of the extra-cellular matric (ECM), production of fibroblasts and myfibroblasts by the transformed epithelial cells (TEC), production of transformed epithelial cells by the transformed growth factor (TGF$-\beta$) and the production of fibroblasts by the Matrix Metalloproteinase (MMP). Therefore, the spatial distribution for the dynamics are due to the role played by the ECM [1,2]. Hence, its spatial domain should be increased from $S$ to the entire domain $\Omega_*$. Thus, incorporating the time required, the ECM spatial distributions into the dynamics for equation (3-1) and ignoring the vertical variables, then the model in equation (3-1) becomes

$$\left.\begin{array}{l}
\frac{\partial n}{\partial t} - D_n \Delta n = -\nabla \cdot \chi_n n \frac{\nabla E}{\sqrt{1+(|\nabla E|/\lambda_E)^2}} - \nabla \chi_n^1 I_s n \frac{\nabla \rho}{\sqrt{1+(|\nabla \rho|/\lambda_\rho)^2}} \\
\quad + \frac{a_{11} n E^4}{k_E^4 + E^4}\left(1 - \frac{n}{n_* - a_{12}\rho I_s}\right) \in [0, L_1], \\
\frac{\partial f}{\partial t} - D_f \Delta f = (-a_{12} G(x, t-\tau) + a_{22}) f(x, t-\tau) \in [-L_1, 0), \\
\frac{\partial m}{\partial t} - D_m \Delta m = -\nabla \cdot \left(\chi_m m \frac{\nabla G}{\sqrt{1+(|\nabla G|/\lambda_G)^2}}\right) \\
\quad + a_{21} G(x, t-\tau) f(x, t-\tau) + a_{31} m \in [-L_1, 0), \\
\frac{\partial \rho}{\partial t} - D_\rho \Delta \rho = -a_{41} P(x, t-\tau) n(x, t-\tau) \\
\quad + (a_{42} f + a_{43} m)\left(1 - \frac{\rho}{\rho_*}\right) \in [-L_1, L_1], \\
\frac{\partial E}{\partial t} - D_E \Delta E = I_{\Omega_-}(a_{51} f(x, t-\tau) + a_{52} m(x, t-\tau)) \\
\quad - a_{53} E \in [-L_1, L_1], \\
\frac{\partial G}{\partial t} - D_G \Delta G = a_{61} I_{\Omega_+} n(x, t-\tau) - a_{62} G \in [-L_1, L_1], \\
\frac{\partial P}{\partial t} - D_p \Delta P = a_{71} I_{\Omega_-} m(x, t-\tau) - a_{72} P \in [-L_1, L_1],
\end{array}\right\} \tag{1.2}$$

for $(x,t) \in [-\bar{L_1}, L_1] \times [-\tau, 0]$. The dynamics in equation (1.2) is a system of discrete delay reaction-diffusion equations. Delay differential equations (DDEs) are widely used for analysis and predictions in various areas of life sciences [5], epidemiology [6], immunology [7], physiology [8], and neural networks [9,10]. Since time-delays and/or time-lags, can be related to the duration of certain hidden processes like the stages of life cycle, time between infection of a cell and production of new viruses, duration of the infectious period, immune period, then introduction of such time-delays in a differential dynamics significantly increases the complexity of a dynamic. Therefore, the first aim in this paper, is to carry out mathematical analysis, which leads to the investigation of how time delay $\tau$ affects the dynamics in equation (1.2). By applying Poincar*é* normal form and the center manifold theorem as in [11], one finds conditions for the functions and derives formulas which determine the properties of Hopf bifurcation. More specifically, the paper presents that equilibrium point losses its stability and the dynamics exhibit Hopf bifurcation under certain conditions.The second aim is to develop a reliable numerical method based on the qualitative features of the dynamics in equation (1.2). Thus, below, we highlight some of the recent developments.

In [12], Hafez and Youssri developed a numerical scheme to solve the variable-order fractional linear sub-diffusion and nonlinear reaction-sub-diffusion equations using the shifted Jacobi collocation method, whereas, in [13], an overview of numerical problems encountered when determining the coefficients and rich variety of techniques proposed to solve these problems with regard to a series of explicit formulae expressing the derivatives, integrals and moments of a class of orthogonal polynomials of any degree and for any order in terms of the same polynomials are addressed.

Abd-Elhameed and Youssri in [14] proposed a new numerical solutions for certain coupled system of fractional differential equations through the employment of the so-called generalized Fibonacci polynomials. The polynomials include two parameters in which they generalize some important well-known polynomials such as Fibonacci, Pell, Fermat, second kind Chebyshev, and Dickson polynomials. The proposed numerical algorithm is essentially built on applying the spectral tau method together with utilizing a Fejer quadrature formula.

Delay differential equations are one of the most powerful mathematical modeling tools and they appear in various applications from life sciences to engineering and physics. They model dynamical systems, when their evolution depends on prior times. A large class of epidemiological models can be formulated as a system of differential equations, frequently involving spatial structure and time delays see for example [15,16].

The rest of the paper is structured as follow. Mathematical analysis of the extended model is presented in Section 2. A robust numerical scheme based on the fitted finite difference technique is formulated, implemented and analysed for convergence in Section 3. To justify the effectiveness of the proposed scheme, some numerical results are presented in Section 4 and Section 5 concludes the paper.

## 2 Mathematical analysis

In this section, well-posedness of the existence of unique solution, local stability, Hopf Bifurcation and global stability analysis of the equilibrium points are established. Let

$$\mathbf{u} = [n, f, m, \rho, E, G, P]^T$$
$$\mathbf{D} = [D_n, D_f, D_m, D_\rho, D_E, D_G, D_P]^T$$

where $T$ denote a transpose. Then the extended dynamics in equation (1.2) can be rewritten as

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{D}\Delta\mathbf{u} = \mathbf{F}(\mathbf{u}(x,t), \nabla\mathbf{u}(x,t), \mathbf{u}(x,t-\tau)),$$

$$t \in [t_0 - \tau, t_0],$$

where,

$$\left.\begin{array}{l}F_1(\mathbf{u}(x,t),\nabla\mathbf{u}(x,t),\mathbf{u}(x,t-\tau))\\ \quad=-\nabla\cdot\chi_n n\frac{\nabla E}{\sqrt{1+(|\nabla E|/\lambda_E)^2}}\\ \quad-\nabla\chi_n^1 I_s n\frac{\nabla\rho}{\sqrt{1+(|\nabla\rho|/\lambda_\rho)^2}}\\ \quad+\frac{a_{11}nE^4}{k_E^4+E^4}\left(1-\frac{n}{n_*-a_{12}\rho I_s}\right)\in[0,L_1],\\ F_2(\mathbf{u}(x,t),\nabla\mathbf{u}(x,t),\mathbf{u}(x,t-\tau))\\ \quad=(-a_{12}G(x,t-\tau)+a_{22})\\ \quad\times f(x,t-\tau)\in[-L_1,0),\\ \\ F_3(\mathbf{u}(x,t),\nabla\mathbf{u}(x,t),\mathbf{u}(x,t-\tau))\\ =-\nabla\cdot\left(\chi_m m\frac{\nabla G}{\sqrt{1+(|\nabla G|/\lambda_G)^2}}\right)\\ \quad+a_{21}G(x,t-\tau)f(x,t-\tau)\\ \quad+a_{31}m\in[L_1,0),\\ F_4(\mathbf{u}(x,t),\nabla\mathbf{u}(x,t),\mathbf{u}(x,t-\tau))\\ \quad=-a_{41}P(x,t-\tau)n(x,t-\tau)\\ \quad+(a_{42}f+a_{43}m)\left(1-\frac{\rho}{\rho_*}\right)\in[-L_1,L_1],\\ F_5(\mathbf{u}(x,t),\nabla\mathbf{u}(x,t),\mathbf{u}(x,t-\tau))\\ \quad=I_{\Omega_-}(a_{51}f(x,t-\tau)\\ \quad+a_{52}m(x,t-\tau))-a_{53}E\in[-L_1,L_1],\\ F_6(\mathbf{u}(x,t),\nabla\mathbf{u}(x,t),\mathbf{u}(x,t-\tau))\\ \quad=a_{61}I_{\Omega_+}n(x,t-\tau)\\ \quad-a_{62}G\in[-L_1,L_1],\\ F_7(\mathbf{u}(x,t),\nabla\mathbf{u}(x,t),\mathbf{u}(x,t-\tau))\\ \quad=a_{71}I_{\Omega_-}m(x,t-\tau)\\ \quad-a_{72}P\in[-L_1,L_1],\end{array}\right\}$$

with all other terminal conditions remained unchanged, as given in (3)-(1) through to (1.1).

### 2.1 Solution continuously depending on the data

Let $\mathbf{v}(x,t,t-\tau),\mathbf{z}(x,t,t-\tau)\in\mathbb{C}_1^2[-L_1,L_1]$ denote two solutions for the dynamics in equation (1.2), such that $\mathbf{v}(x,t,t-\tau)-\mathbf{z}(x,t,t-\tau)=:\mathbf{u}(x,t,t-\tau)$, (where $-\tau<t<0$), yields the following results.

**Theorem 21***Let*

$$\left.\begin{array}{l}\frac{\partial\mathbf{u}(\mathbf{x},\mathbf{t},\mathbf{t}-\tau)}{\partial t}-\mathbf{D}\Delta\mathbf{u}(x,t,t-\tau)=0;\\ \quad x\in[-L_1,L_1],\ t>0,\\ u_1(x,0)=\frac{1}{2}\left(1+\tanh\left(-\frac{1}{\varepsilon}(0.8-x)\right)\right)\\ \quad\in[0,L_1]\\ u_2(x,0)=0,143\frac{1}{2}\left(1+\tanh\left(-\frac{1}{\varepsilon}(x-0.2)\right)\right)\\ \quad\in[-L_1,0),\\ u_3(x,0)=0.0\in[L_1,0),\\ \rho_0(x)=1.0\in[-L_1,L_1],u_4(x,0)\\ \quad=1.0\in[-L_1,L_1],\\ u_5(x,0)=1.0\in[-L_1,0),\\ u_6(x,0)(x)=0.0\in[-L_1,L_1].\end{array}\right\}$$

*Then $\mathbf{u}(x,t,t-\tau)$ is identically zero.*

**Proof:**
Proceeding component-wise for elements of the vector $\mathbf{u}(x,t)$, one obtains,

$$\left.\begin{array}{l}\mathbf{u}(\cdot,t,t-\tau)\mathbf{u}_t(\cdot,t,t-\tau)=\frac{1}{2}\partial_t\mathbf{u}^2(\cdot,t,t-\tau),\\ \mathbf{u}(\cdot,t,t-\tau)\mathbf{u}_{t-\tau}(\cdot,t,t-\tau)=\frac{1}{2}\partial_{t-\tau}\mathbf{u}^2(\cdot,t,t-\tau).\end{array}\right\}$$

Similarly, one also finds

$$\begin{array}{l}\mathbf{u}(\cdot,t,t-\tau)\mathbf{u}_{xx}(\cdot,t,t-\tau)=\partial_x(\mathbf{u}(\cdot,t,t-\tau)\mathbf{u}_x(\cdot,t,t-\tau))\\ \quad-\mathbf{u}_x^2(\cdot,t,t-\tau),\end{array}$$

then in view of **Theorem 21**

$$\begin{array}{l}\frac{1}{2}\frac{d}{dt}\int_{-L_1}^{L_1}\mathbf{u}^2(\cdot,t,t-\tau)dx\\ \quad=\int_{-L_1}^{L_1}\frac{1}{2}\partial_t\mathbf{u}^2(\cdot,t,t-\tau)dx\\ \quad=\int_{-L_1}^{L_1}\mathbf{u}(\cdot,t,t-\tau)\mathbf{u}_t(\cdot,t,t-\tau)dx\\ \quad=\mathbf{D}\int_{-L_1}^{L_1}\mathbf{u}(\cdot,t)\mathbf{u}_{xx}(\cdot,t,t-\tau)dx,\\ \quad=\mathbf{D}\int_{-L_1}^{L_1}\partial_x(\mathbf{u}(\cdot,t,t-\tau)\mathbf{u}_x(\cdot,t,t-\tau))dx\\ \quad\ -\int_{-L_1}^{L_1}\mathbf{u}_x^2(\cdot,t,t-\tau)dx,\\ \quad=\mathbf{D}\mathbf{u}(\cdot,t,t-\tau)\mathbf{u}_x(\cdot,t,t-\tau)|_{-L_1}^{L_1}\\ \quad-\int_{-L_1}^{L_1}\mathbf{u}_x^2(\cdot,t,t-\tau)dx\\ \quad=-\mathbf{D}\int_{-L_1}^{L_1}\mathbf{u}_x^2(\cdot,t,t-\tau)dx\le 0.\end{array}$$

These implies that the function

$$t\to\int_{-L_1}^{L_1}\mathbf{u}^2(\cdot,t,t-\tau)dx$$

is a non-increasing function. Hence

$$\begin{array}{l}0\le\int_{-L_1}^{L_1}\mathbf{u}^2(\cdot,t,t-\tau)dx.\\ \quad\le\int_{-L_1}^{L_1}\mathbf{u}^2(\cdot,0,0)dx=0,\forall x,t\end{array}$$

A similar results can be established that,

$$\begin{array}{l}\frac{1}{2}\frac{d}{d(t-\tau)}\int_{-L_1}^{L_1}\mathbf{u}^2(\cdot,t,t-\tau).\\ \quad=-\mathbf{D}\int_{-L_1}^{L_1}\mathbf{u}_x^2(\cdot,t,t-\tau)dx\le 0\end{array}$$

This proves uniqueness of solution to the system of delay time-depended non-linear quasi-parabolic partial differential equations (PPDEs) in equation (1.2).

**Corollary 22***Let $\mathbf{v},\mathbf{z}\in\mathbb{C}_1^2[-L_1,L_1]$ denote solutions to the dynamics in equation (1.2) with initial states $\mathbf{v}_0,\mathbf{z}_0$, such that $\mathbf{v}_0-\mathbf{z}_0=:\mathbf{u}_0$ and $\mathscr{F}_u,\mathscr{F}_v$ denote the real-valued functions bounding $\mathbf{v}$ and $\mathbf{z}$, respectively. Then*

$$\begin{array}{l}\|\mathbf{v}-\mathbf{z}\|^2\le\exp(-\eta t)\|\mathbf{v}_0-\mathbf{z}_0\|^2\\ \qquad+\varpi\int_0^t\exp(-\eta(t-s))\|\mathscr{F}_u-\mathscr{F}_v\|^2ds,\end{array}$$

*and*

$$\begin{array}{l}\|\mathbf{v}-\mathbf{z}\|^2\le\exp(-\eta(t-\tau))\|\mathbf{v}_0-\mathbf{z}_0\|^2\\ \qquad+\varpi_\tau\int_{t-\tau}^t\exp(-\eta(t-\tau-s))\|\mathscr{F}_u-\mathscr{F}_v\|^2ds,\end{array}$$

*for some $-\tau<t<0$, and $\varpi,\varpi_\tau,\eta\in\mathbb{R}^+$.*

In order to prove the **Corollary 22**, the following are preliminaries to the proof.

**Definition 23** *Let* $\mathbf{v},\mathbf{z} \in \mathbb{C}_1^2[-L_1,L_1]$ *then*

$$(\mathbf{v},\mathbf{z}) := \int_{-L_1}^{L_1} \mathbf{v}(\cdot,t),\mathbf{z}(\cdot,t)dx,$$

*defines an in inner product [17].*

**Lemma 24** *If* $\mathbf{v},\mathbf{z} \in \mathbb{C}_1^2[-L_1,L_1]$ *then*

$$(\mathbf{v}_{xx}+\sigma\mathbf{v},\mathbf{z}) = \sigma(\mathbf{v},\mathbf{z})-(\mathbf{v}_x,\mathbf{z}_x) = (\mathbf{v},\mathbf{z}_{xx}+\sigma\mathbf{z}),$$

*where,* $\sigma \in \mathbb{R}$.

**Proof:**
Since

$$\partial_x[\mathbf{v}_x\mathbf{z}] = \mathbf{v}_{xx}\mathbf{z}+\mathbf{v}_x\mathbf{z}_x,$$

then

$$(\partial_x[\mathbf{v}_x,\mathbf{z}]) = (\mathbf{v}_{xx},\mathbf{z})+(\mathbf{v}_x,\mathbf{z}_x),$$

$$\Rightarrow (\mathbf{v}_x,\mathbf{z})|_{-L_1}^{L_1} = (\mathbf{v}_{xx},\mathbf{z})+(\mathbf{v}_x,\mathbf{z}_x),$$

$$\Rightarrow (\mathbf{v}_{xx},\mathbf{z}) = -(\mathbf{v}_x,\mathbf{z}_x).$$

Similarly,

$$\partial_x[\mathbf{v}\mathbf{z}_x] = \mathbf{v}\mathbf{z}_{xx}+\mathbf{v}_x\mathbf{z}_x, \tag{2.3}$$

then it follows from equation in (2.3) that

$$(\mathbf{v},\mathbf{z}_{xx}) = -(\mathbf{v}_x,\mathbf{z}_x). \tag{2.4}$$

Combining equation in (2.3) with equation in (2.4) yields

$$(\mathbf{v}_{xx},\mathbf{z}) = (\mathbf{v},\mathbf{z}_{xx}).$$

Hence, the results follows. □

**Lemma 25** *Let* $\mathbf{v} \in \mathbb{C}_1^2[-L_1,L_1]$, *then*

$$\|\mathbf{v}\| := (\mathbf{v},\mathbf{v})^{\frac{1}{2}} = \left(\int_{-L_1}^{L_1} \mathbf{v}^2(\cdot,t)dx\right)^{\frac{1}{2}},$$

*defines a norm.*

**Proof:** See [17].

**Corollary 26** *Let* $\mathbf{v},\mathbf{z} \in \mathbb{C}_1^2[-L_1,L_1]$ *then*

$$|(\mathbf{v},\mathbf{v})| \le \|\mathbf{v}\|\|\mathbf{v}\|.$$

**Proof:**
Let [17]

$$P_2(\sigma) = \|\mathbf{v}+\sigma\mathbf{z}\|^2 = \|\mathbf{v}\|^2+2\sigma(\mathbf{v},\mathbf{z})+\sigma^2\|\mathbf{z}\|^2,$$

denotes a polynomial of degree two. Then $P_2(\sigma) \ge 0, \forall \sigma \in \mathbb{R}$. Thus,

$$\triangle_{P_2(\sigma)} = 4(\mathbf{v},\mathbf{z})^2-4\|\mathbf{v}\|^2\|\mathbf{z}\|^2 \le 0, \Rightarrow (\mathbf{v},\mathbf{z})^2 \le \|\mathbf{v}\|^2\|\mathbf{z}\|^2,$$

which concludes the proof of Corollary 26.

Proof of Corollary 22. By means of Poincaré inequality [17], Schwarz inequality in Lemma 25, Corollary 26 and in view of the prove to Theorem 21,

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\|\mathbf{u}(\cdot,t,t-\tau)\|^2+\tfrac{1}{2}\tfrac{d}{d(t-\tau)}\|\mathbf{u}(\cdot,t,t-\tau)\|^2\\ &\quad+\tfrac{2}{c}\mathbf{D}\|\mathbf{u}(\cdot,t)\|^2\\ &\quad\le 2(\mathscr{F}(\cdot,t,t-\tau),\mathbf{u}(\cdot,t,t-\tau))\\ &\quad\le 2\|\mathscr{F}\|\|\mathbf{u}(\cdot,t,t-\tau)\|,\end{aligned}$$

where, $c>0$, $\mathscr{F} := \mathscr{F}_u-\mathscr{F}_v$. Thus,

$$\left.\begin{aligned}&\tfrac{d}{dt}\|\mathbf{u}(\cdot,t,t-\tau)\|^2+\left(\tfrac{2}{c}\mathbf{D}-\varepsilon\right)\|\mathbf{u}(\cdot,t,t-\tau)\|^2\\ &\quad\le\tfrac{1}{\varepsilon}\|\mathscr{F}\|^2,\\ &\tfrac{d}{d(t-\tau)}\|\mathbf{u}(\cdot,t,t-\tau)\|^2+\left(\tfrac{2}{c}\mathbf{D}-\varepsilon\right)\|\mathbf{u}(\cdot,t,t-\tau)\|^2\\ &\quad\le\tfrac{1}{\varepsilon}\|\mathscr{F}\|^2,\end{aligned}\right\}$$

for some arbitrary sufficiently small $\varepsilon>0$. Let $\eta=\frac{2}{c}\mathbf{D}-\varepsilon>0$ and $\varpi=1/\varepsilon$. Then, equation (2.1) becomes

$$\left.\begin{aligned}&\tfrac{d}{dt}\|\mathbf{u}(\cdot,t,t-\tau)\|^2+\eta\|\mathbf{u}(\cdot,t,t-\tau)\|^2\\ &\quad\le\varpi\|\mathscr{F}\|^2,\\ &\tfrac{d}{d(t-\tau)}\|\mathbf{u}(\cdot,t,t-\tau)\|^2+\eta_\tau\|\mathbf{u}(\cdot,t,t-\tau)\|^2\\ &\quad\le\varpi_\tau\|\mathscr{F}\|^2.\end{aligned}\right\}$$

Applying the Gronwall's inequality [17] to equation in (2.1) the results follows. □

## 2.2 Local stability

Let

$$\mathscr{E} := (n^*,f^*,m^*,\rho^*,E^*,G^*,P^*)$$

denotes the equilibrium point for the dynamics in equation (1.2). Then at the equilibrium point $\mathscr{E}$, the system in equation (1.2) becomes

$$\left.\begin{aligned}&D_n\tfrac{dn}{dx}-\chi_n\tfrac{d}{dx}\left(n\frac{\frac{dE}{dx}}{\sqrt{1+(|\frac{dE}{dx}|/\lambda_E)^2}}\right)\\ &-\chi_n^1 I_s\tfrac{d}{dx}\left(\tfrac{dn}{dx}\frac{\frac{d\rho}{dx}}{\sqrt{1+(|\frac{d\rho}{dx}|/\lambda_\rho)^2}}\right)\\ &\quad+\frac{a_{11}nE^4}{k_E^4+E^4}\left(1-\frac{n}{n_*-a_{12}\rho I_s}\right)\\ &\quad=0\in[0,L_1],\\ &D_f\tfrac{df}{dx}+(-a_{12}G(x)+a_{22})f(x)\\ &\quad=0\in[-L_1,0),\end{aligned}\right\}$$

$$\left.\begin{aligned}&D_m\tfrac{dm}{dx}-\chi_m\tfrac{d}{dx}\left(m\frac{\frac{dG}{dx}}{\sqrt{1+(|\frac{dG}{dx}|/\lambda_G)^2}}\right)\\ &\quad+a_{21}G(x)f(x)+a_{31}m\\ &\quad=0\in[-L_1,0),\\ &D_\rho\tfrac{d\rho}{dx}-a_{41}P(x)n(x)\\ &\quad+(a_{42}f+a_{43}m)\left(1-\frac{\rho}{\rho_*}\right)\\ &\quad=0\in[-L_1,L_1],\end{aligned}\right\}$$

$$\left.\begin{array}{l} D_E\frac{dE}{dx}+I_{\Omega_-}(a_{51}f(x)\\ \quad +a_{52}m(x))-a_{53}E=0\\ \quad \in[-L_1,L_1],\\ D_G\frac{dG}{dx}+a_{61}I_{\Omega_+}n(x)-a_{62}G\\ \quad =0\in[-L_1,L_1],\\ D_P\frac{dP}{dx}+a_{71}I_{\Omega\_}m(x)-a_{72}P\\ \quad =0\in[-L_1,L_1]. \end{array}\right\}$$

Neglecting, the spacial distributions, one obtains $G^*=a_{22}/a_{12}\in[-L_1,0]$, $n^*=a_{62}a_{22}/a_{61}a_{12}\in[0,L_1]$, and $f^*=0$ on $[-L_1,0]$. This yields, $m^*=E^*=P^*=0$, which implies that equation (2.2) reduces to solving

$$\frac{d\rho(x)}{dx}=\int_{-L_1}^{L_1}\sqrt{1+\left|\frac{d\rho}{\lambda_\rho dx}\right|^2}dx.$$

Hence, $\rho^*=L_1\sqrt{1+(L_1)^2}+\ln\left(\frac{L_1+\sqrt{1+L_1^2}}{\sqrt{1+L_1^2}-L_1}\right)^{\frac{1}{2}}$, such that $\sqrt{1+L_1^2}>L_1$. Thus, the following results.

**Theorem 27** *The equilibrium point for system in equation (1.2) is given by*

$$\mathscr{E}=\left(\frac{a_{62}a_{22}}{a_{61}a_{12}},0,0,L_1\sqrt{1+(L_1)^2}\right.$$
$$\left.+\ln\left(\frac{L_1+\sqrt{1+L_1^2}}{\sqrt{1+L_1^2}-L_1}\right)^{\frac{1}{2}},0,\frac{a_{22}}{a_{12}},0\right).$$

To investigate the linearized stability of the system (1.2), we let

$$(n,f,m,\rho,E,G,P)=$$
$$(\bar{n}+n^*,\bar{f}+f^*,\bar{m}+m^*,\bar{\rho}+\rho^*,\bar{E}+E^*,\bar{G}+G^*,\bar{P}+P^*).$$

Substituting into (1.2), and retaining only the linear terms in $n,f,m,\rho,E,G,P$, one finds

$$\left.\begin{array}{l} \frac{\partial\bar{n}}{\partial t}-D_n\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\bar{n}\right)=0, \text{ in } \in[0,L_1],t>0,\\ \frac{\partial\bar{f}}{\partial t}-D_f\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\bar{f}\right)=a_{22}\bar{f}(x,t-\tau)\in[-L_1,0],t>0,\\ \frac{\partial\bar{m}}{\partial t}-D_m\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\bar{m}\right)=a_{31}\bar{m}(x,t)\in[-L_1,0],t>0,\\ \frac{\partial\bar{\rho}}{\partial t}-D_\rho\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\bar{\rho}\right)=a_{42}\bar{f}(x,t)+a_{43}\bar{m}(x,t)\in[-S,S],t>0,\\ \frac{\partial\bar{E}}{\partial t}-D_E\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\bar{E}\right)=-a_{53}\bar{E}(x,t)\in[-L_1,L_1],t>0,\\ \frac{\partial\bar{G}}{\partial t}-D_G\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\bar{G}\right)=a_{61}\bar{n}(x,t-\tau)+a_{62}\bar{G}(x,t)\\ \quad +\frac{a_{62}a_{22}}{a_{61}a_{12}}-\frac{a_{62}a_{22}}{a_{12}}\in[-L_1,L_1],t>0,\\ \frac{\partial\bar{P}}{\partial t}-D_P\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\bar{P}\right)=a_{71}I_{\Omega\_}\bar{m}(x,t-\tau)\\ \quad -a_{72}\bar{P}(x,t)\in[-L_1,L_1], \end{array}\right\}$$

on $(x,t)\in[-\bar{L_1},L_1]\times[-\tau,0]$, where, the characteristic equation is

$$(\lambda-D_n)(\lambda-D_f-a_{22}e^{-\lambda t})(\lambda-D_m-a_{31})(\lambda-D_\rho)$$
$$(\lambda-D_E+a_{53})(\lambda-D_G-a_{62})(\lambda-D_P+a_{72})=0.$$

Hence, the following results.

**Theorem 28** *The dynamics in equation (1.2) are asymptotically stable if $\lambda<D_n,\lambda<D_f+a_{22}e^{-\lambda t}$, $\lambda<D_m+a_{31}$, $\lambda<D_E-a_{53}$, $\lambda<D_G+a_{62}$, $\lambda<D_P-a_{72}$.*

### *2.3 Hopf Bifurcation analysis*

When $\tau\neq 0$, we assume that $\lambda=i\omega$ for $\omega>0$ and $i=\sqrt{-1}$. In view of the eigenvalues, we have

$$i\omega-D_f-a_{22}\exp(i\omega\tau)=i\omega-D_f$$
$$-a_{22}(\cos(\omega\tau)+i\sin(\omega\tau))=0.$$

Separating real and imaginary parts we have

$$\omega+a_{22}\sin(\omega\tau)=0 \text{ and } -D_f-a_{22}\cos(\omega\tau)=0, \quad (2.5)$$

which yields

$$\tau_i=\frac{1}{\omega_0}\cos^{-1}\left(\frac{D_f}{a_{22}}+2i\pi\right),i=0,1,2,3,\ldots.$$

Squaring on both sides of equations in (2.5), we find

$$\omega^2+2a_{22}\omega\sin(\omega\tau)+a_{22}^2\sin^2(\omega\tau)=0,$$
$$D_f^2+2D_fa_{22}\cos(\omega\tau)+a_{22}^2\cos^2(\omega\tau)=0. \quad (2.6)$$

Adding the two equations in equation (2.6) one finds

$$\omega^2+D_f^2+2(D_f\cos(\omega\tau)+\omega\sin(\omega\tau))a_{22}+a_{22}^2=0,$$

which simplifies to

$$\omega^2+D_f^2-2(D_f^2+\omega^2)+a_{22}^2=0,$$
$$\Rightarrow-\omega^2-D_f^2+a_{22}^2=0,\Rightarrow\omega_0=\pm\sqrt{a_{22}^2-D_f^2}.$$

Choosing $\tau_0=\min\{\tau_i\}$, we need to show that

$$\mathscr{R}\left(\frac{d\lambda}{d\tau}\right)\Big|_{\tau=\tau_0}\neq 0.$$

From the eigenvalues we have

$$\mathscr{R}\left(\frac{d\lambda}{d\tau}\right)=\mathscr{R}\left(\frac{d(D_f+a_{22}(\cos(\omega\tau)+i\sin(\omega\tau))}{d\tau}\right)$$
$$=-a_{22}\omega\sin(\omega_0\tau_0),$$

where $\omega_0\tau_0\neq 0$. By summarizing the above analysis, we arrive at the following theorem.

**Theorem 29** *The equilibrium*

$$\mathscr{E}=\left(\frac{a_{62}a_{22}}{a_{61}a_{12}},0,0,L_1\sqrt{1+(L_1)^2}\right.$$
$$\left.+\ln\left(\frac{L_1+\sqrt{1+L_1^2}}{\sqrt{1+L_1^2}-L_1}\right)^{\frac{1}{2}},0,\frac{a_{22}}{a_{12}},0\right),$$

*of the system (1.2) is asymptotically stable for $\tau\in[0,\tau_0)$ and it undergoes Hopf bifurcation at $\tau=\tau_0$.*

### 2.4 Global stability analysis

In this section, one establishes that the equilibrium

$$\mathscr{E} = \left(\frac{a_{62}a_{22}}{a_{61}a_{12}}, 0, 0, L_1\sqrt{1+(L_1)^2} + \ln\left(\frac{L_1+\sqrt{1+L_1^2}}{\sqrt{1+L_1^2}-L_1}\right)^{\frac{1}{2}}, 0, \frac{a_{22}}{a_{12}}, 0\right),$$

is globally asymptotically stable with the method of upper and lower solution [18,19].

Let $\vartheta_E = \frac{E^4}{k_E^4+E^4}$. Then denoting the reaction terms in equation (1.2) by $h_j(n,f,m,\rho,E,G,P)$ for $j=1,2,3,4,5,6,7$, one has

$$\left.\begin{array}{l} h_1 = a_{11}n\vartheta_E\left(1-\frac{n}{n_*-a_{12}\rho I_s}\right) \in \Omega_+,\\ h_2 = (-a_{12}G(x,t-\tau)+a_{22})f(x,t-\tau) \in [-L_1,0],\\ h_3 = a_{21}G(x,t-\tau)f(x,t-\tau)+a_{31}m \in [-L_1,0],\\ h_4 = -a_{41}P(x,t-\tau)n(x,t-\tau)\\ \quad +(a_{42}f+a_{43}m)\left(1-\frac{\rho}{\rho_*}\right) \in [-L_1,L_1],\\ h_5 = I_{\Omega_-}(a_{51}f(x,t-\tau)+a_{52}m(x,t-\tau))\\ \quad -a_{53}E \in [-L_1,L_1],\\ h_6 = a_{61}I_{\Omega_+}n(x,t-\tau)\\ \quad -a_{62}G \in [-L_1,L_1],\\ h_7 = a_{71}I_{\Omega_-}m(x,t-\tau)\\ \quad -a_{72}P \in [-L_1,L_1]. \end{array}\right\} \quad (2.7)$$

Let $S \subset \mathbb{R}_+^5$ such that $S = \{\mathbf{u} \in \mathbb{R}_+^7 : \underline{\mathbf{u}} \le 0 \le \bar{\mathbf{u}}\}$ and $K_j$ be any positive constant satisfying

$$K \ge \max\{K_n, K_f, K_m, K_\rho, K_E, K_G, K_P,\} \ge \max\left\{\frac{-\partial h_j}{\partial u_j} : \mathbf{u} = (n,f,m,\rho,E,G,P) \in S\right\},$$

for $j = 1,2,3,4,5,6,7$. Then the following results hold.

**Lemma 210** *Let*

$$\left.\begin{array}{l} \frac{\partial n}{\partial t} - \nabla\cdot(D_n\nabla n) + \nabla\cdot\chi_n n\frac{\nabla E}{\sqrt{1+(|\nabla E|/\lambda_E)^2}}\\ \quad +\nabla\chi_n^1 I_s n\frac{\nabla\rho}{\sqrt{1+(|\nabla\rho|/\lambda_\rho)^2}} \le K_n \in [0,L_1],\\ \frac{\partial f}{\partial t} - \nabla\cdot(D_f\nabla f) \le K_f \in [-L_1,0],\\ \frac{\partial m}{\partial t} - \nabla\cdot(D_m\nabla m) + \nabla\cdot\left(\chi_m m\frac{\nabla G}{\sqrt{1+(|\nabla G|/\lambda_G)^2}}\right)\\ \quad \le K_m \in [-L_1.0],\\ \frac{\partial\rho}{\partial t} - \nabla\cdot(D_\rho\nabla\rho) \le K_\rho, \in [-L_1,L_1],\\ \frac{\partial E}{\partial t} - \nabla\cdot(D_E\nabla E) \le K_E \in [-L_1,L_1],\\ \frac{\partial G}{\partial t} - \nabla\cdot(D_G\nabla G) \le K_G \in [-L_1,L_1],\\ \frac{\partial P}{\partial t} - \nabla\cdot(D_p\nabla P) \le K_P \in [-L_1,L_1]. \end{array}\right\}$$

*Then*

$$\lim_{t\to\infty} n(x,t) = K_n, \lim_{t\to\infty} f(x,t) = K_f, \lim_{t\to\infty} m(x,t) = K_m, \lim_{t\to\infty}\rho(x,t) = K_\rho,$$

$$\lim_{t\to\infty} E(x,t) = K_E, \lim_{t\to\infty} G(x,t) = K_G, \lim_{t\to\infty} P(x,t) = K_P.$$

**Theorem 211** *If $a_{12}\rho I_s < n_*$, then this implies that the equilibrium $(n_* - a_{12}\rho I_s, 0, 0, 0, \frac{a_{22}}{a_{12}})$ is globally asymptotically stable.*

**Proof:** From the maximum principle of parabolic equations, it is known that for any initial value

$$(n_0(t,x),\ f_0(t,x), m_0(t,x), \rho_0(t,x), E_0(t,x), G_0(t,x), P_0(t,x)) > (0,0,0,0,0)$$

the corresponding non-negative solution

$$(n(t,x), f(t,x), m(t,x), \rho(t,x), E(t,x), G(t,x), P(t,x),)$$

is strictly positive for $t > 0$ . Since $a_{12}\rho I_s < n_*$, then choose $\varepsilon_0 \in (0,1)$. According to Lemma (210) and the comparison principle of parabolic equations, there exists $t_1 > t_0 > 0$ such that, for any $t > t_1$,

$$\left.\begin{array}{l} n(x,t) \le K_n+\varepsilon_0 := \bar{n}(x,t) \in [0,L_1],\\ f(x,t) \le K_f+\varepsilon_0 := \bar{f}(x,t) \in [-L_1,0],\\ m(x,t) \le K_m+\varepsilon_0 := \bar{m}(x,t) \in [-L_1,0],\\ \rho(x,t) \le K_\rho+\varepsilon_0 := \bar{\rho}(x,t) \in [-L_1,L_1],\\ E(x,t) \le K_E+\varepsilon_0 := \bar{E}(x,t) \in [-L_1,L_1],\\ G(x,t) \le K_G+\varepsilon_0 := \bar{G}(x,t) \in [-L_1,L_1],\\ P(x,t) \le K_P+\varepsilon_0 := \bar{P}(x,t) \in [-L_1,L_1], \end{array}\right\} \quad (2.8)$$

and

$$\left.\begin{array}{l} n(x,t) \ge K_n-\varepsilon_0 := \underline{n}(x,t) \in [0,L_1],\\ f(x,t) \ge K_f-\varepsilon_0 := \underline{f}(x,t) \in [-L_1,0],\\ m(x,t) \ge K_m-\varepsilon_0 := \underline{m}(x,t) \in [-L_1,0],\\ \rho(x,t) \ge K_\rho-\varepsilon_0 := \underline{\rho}(x,t) \in [-L_1,L_1],\\ E(x,t) \ge K_E-\varepsilon_0 := \underline{E}(x,t) \in [-L_1,L_1],\\ G(x,t) \ge K_G-\varepsilon_0 := \underline{G}(x,t) \in [-L_1,L_1],\\ P(x,t) \ge K_P-\varepsilon_0 := \underline{P}(x,t) \in [-L_1,L_1], \end{array}\right\} \quad (2.9)$$

Thus, for $t > t_0$, it is possible to obtain

$$\begin{array}{l} \underline{n}(x,t) \le n(x,t) \le \bar{n}(x,t) \in [0,L_1],\\ \underline{f}(x,t) \le f(x,t) \le \bar{f}(x,t) \in [-L_1,0], 0,\\ \underline{m}(x,t) \le m(x,t) \le \bar{m}(x,t) \in [-L_1,0],\\ \underline{\rho}(x,t) \le \rho(x,t) \le \bar{\rho}(x,t) \in [-L_1,L_1],\\ \underline{E}(x,t) \le E(x,t) \le \bar{E}(x,t) \in [-L_1,L_1],\\ \underline{G}(x,t) \le G(x,t) \le \bar{G}(x,t) \in [-L_1,L_1],\\ \underline{P}(x,t) \le P(x,t) \le \bar{P}(x,t) \in [-L_1,L_1]. \end{array}$$

Since $h_j(n,f,m,\rho,E,G,P)$ in equation (2.7) is a $C^1$ function of $n,f,m,\rho,E,G,P$, where $h_1$ is quasi-monotone non-decreasing in $f,m,\rho,E,G,P$, $h_2$ is quasi-monotone non-increasing in $n,m,\rho,E,G,P$, $h_3$ is quasi-monotone non-decreasing in $n,f,\rho,E,G,P$, $h_4$ is mixed quasi-monotone non-increasing in $n,f,m,\rho,E,G,P$, $h_5$ is quasi-monotone non-decreasing in $n,f,m,\rho,G,P$, $h_6$ is quasi-monotone non-decreasing in $n,f,m,\rho,G,P$ and $h_7$ is quasi-monotone non-decreasing in $n,f,m,\rho,E,G$, then by the method of upper and lower solutions the system in (1.2) possesses a unique global non-negative solution $n,f,m,\rho,E,G$, [18]. Thus,

$$\underline{n}, \bar{n}, \underline{f}, \bar{f}, \underline{m}, \bar{m}, \underline{\rho}, \bar{\rho}, \underline{E}, \bar{E}, \underline{G}, \bar{G}, \underline{P}, \bar{P},$$

satisfy

$$\left.\begin{array}{l} a_{11}\bar{n}\bar{\vartheta}_E\left(1-\frac{\bar{n}}{n_*-a_{12}\rho I_s}\right)\geq 0 \\ \quad \geq a_{11}\underline{n}\underline{\vartheta}_E\left(1-\frac{\underline{n}}{n_*-a_{12}\rho I_s}\right)\in[0,L_1], \\ (-a_{12}\underline{G}+a_{22})\bar{f}\geq 0\geq(-a_{12}\bar{G}+a_{22})\underline{f}\in[-L_1,0], \\ a_{21}\bar{G}\bar{f}+a_{31}\bar{m}\geq 0\geq a_{21}\underline{G}\underline{f}+a_{31}\underline{m}\in[-L_1,0], \\ -a_{41}\underline{P}\underline{n}+(a_{42}\underline{f}+a_{43}\underline{m})\left(1-\frac{\bar{\rho}}{\rho_*}\right)\geq 0 \\ \quad \geq -a_{41}\bar{P}\bar{n} \\ \quad +(a_{42}\bar{f}+a_{43}\bar{m})\left(1-\frac{\underline{\rho}}{\rho_*}\right)\in[-L_1,L_1], \end{array}\right\}$$

$$\left.\begin{array}{l} I_{[-L_1,0]}(a_{51}\bar{f}+a_{52}\bar{m})-a_{53}\bar{E}\geq 0 \\ \quad \geq I_{[-L_1,0]}(a_{51}\underline{f}+a_{52}\underline{m}) \\ \quad -a_{53}\underline{E}\in[-L_1,L_1], \\ a_{61}I_{\Omega_+}\bar{n}-a_{62}\bar{G}\geq 0\geq a_{61}I_{[0,L_1]}\underline{n} \\ \quad -a_{62}\underline{G}\in[-L_1,L_1], \\ a_{71}I_{[-L_1,0]}\bar{m}-a_{72}\bar{P}\geq 0 \\ \quad \geq a_{71}I_{[-L_1,0]}\underline{m}-a_{72}\underline{P}\in[-L_1,L_1]. \end{array}\right\}$$

Therefore,

$(\bar{n},\bar{f},\bar{m},\bar{\rho},\bar{E},\bar{G},\bar{P})$, and $(\underline{n},\underline{f},\underline{m},\underline{\rho},\underline{E},\underline{G},\underline{P})$,

are a pair of coupled upper and lower solutions for system (1.2),[20], respectively. Thus, for any

$$(\underline{n},\underline{f},\underline{m},\underline{\rho},\underline{E},\underline{G},\underline{P})\leq(n_1,f_1,m_1,\rho_1,E_1,G_1,P_1)$$

and

$$(n_2,f_2,m_2,\rho_2,E_2,G_2,P_2)\leq(\bar{n},\bar{f},\bar{m},\bar{\rho},\bar{E},\bar{G},\bar{P})$$

we have

$$\left.\begin{array}{l} \Big|a_{11}n_1(\vartheta_E)_1\left(1-\frac{n_1}{n_*-a_{12}\rho I_s}\right) \\ \quad -a_{11}n_2(\vartheta_E)_2\left(1-\frac{n_2}{n_*-a_{12}\rho I_s}\right)\Big| \\ \quad \leq K(|E_1-E_2|+|n_1-n_2|)\in[0,L_1], \\ \\ |(-a_{12}G_1+a_{22})f_1-(-a_{12}G_2+a_{22})f_2)| \\ \quad \leq K(|G_1-G_2|+|f_1-f_2|)\in[-L_1,0], \end{array}\right\}$$

v

$$\left.\begin{array}{l} |a_{21}G_1f_1+a_{31}m_1-(a_{21}G_2f_2+a_{31}m_2)| \\ \quad \leq K(|G_1-G_2|+|f_1-f_2|)\in[-L_1,0], \\ \\ \Big|-a_{41}P_1n_1+(a_{42}f_1+a_{43}m_1)\left(1-\frac{\rho_1}{\rho_*}\right) \\ \quad -(-a_{41}P_1n_1+(a_{42}f_1+a_{43}m_1)\left(1-\frac{\rho_1}{\rho_*}\right))\Big| \\ \leq K(|n_1-n_2|+|f_1-f_2|+|m_1-m_2| \\ \quad +|\rho_1-\rho_2|+|P_1-P_2|) \\ \quad \in[-L_1,L_1], \end{array}\right\}$$

$$\left.\begin{array}{l} \Big|I_{[-L_1,0]}(a_{51}f_1+a_{52}m_1)-a_{53}E_1 \\ \quad -(I_{[-L_1,0]}(a_{51}f_2+a_{52}m_2)-a_{53}E_2)\Big| \\ \quad \leq K(|f_1-f_2|+|m_1-m_2|+|E_1-E_2|) \\ \quad \in[-L_1,L_1], \\ \\ \Big|a_{61}I_{[0,L_1]}n_1-a_{62}G_1 \\ \quad -(a_{61}I_{[0,L_1]}n_2-a_{62}G_2)\Big| \\ \quad \leq K(|n_1-n_2|+|G_1-G_2|)\in[-L_1,L_1], \\ \\ \Big|a_{71}I_{[-L_1,0]}m_1-a_{72}P_1 \\ \quad -(a_{71}I_{[-L_1,0]}m_2-a_{72}P_2)\Big| \\ \quad \leq K(|m_1-m_2|+|P_1-P_2|)\in[-L_1,L_1]. \end{array}\right\}$$

$$\left.\begin{array}{l} \Big|a_{71}I_{[-L_1,0]}m_1-a_{72}P_1 \\ \quad -(a_{71}I_{[-L_1,0]}m_2-a_{72}P_2)\Big| \\ \quad \leq K(|m_1-m_2|+|P_1-P_2|)\in[-L_1,L_1]. \end{array}\right\}$$

Defining two iteration sequences $(\bar{n},\bar{f},\bar{m},\bar{\rho},\bar{E},\bar{G},\bar{P})$ and $(\underline{n},\underline{f},\underline{m},\underline{\rho},\underline{E},\underline{G},\underline{P})$ for $i\geq 1$,

$$\left.\begin{array}{l} \bar{n}^{(i)}=\bar{n}^{(i-1)}+(a_{11}\bar{n}^{(i-1)}(\bar{\vartheta}_E)^{(i-1)} \\ \quad \times\left(1-\frac{\bar{n}^{(i-1)}}{n_*-a_{12}\rho I_s}\right))/K\in[0,L_1], \\ \bar{f}^{(i)}=\bar{f}^{(i-1)} \\ \quad +(-a_{12}\underline{G}^{(i-1)}+a_{22})\bar{f}^{(i-1)})/K\in[-L_1,0], \\ \bar{m}^{(i)}=\bar{m}^{(i-1)} \\ \quad +(a_{21}\bar{G}^{(i-1)}\bar{f}^{(i-1)}+a_{31}\bar{m}^{(i-1)})/K\in\Omega_-, \\ \bar{\rho}^{(i)}=\bar{\rho}^{(i-1)}+(-a_{41}\underline{P}^{(i-1)}\underline{n}^{(i-1)}+(a_{42}\underline{f}^{(i-1)}+a_{43}\underline{m}^{(i-1)}) \\ \quad \times\left(1-\frac{\bar{\rho}^{(i-1)}}{\rho_*}\right))/K\in[-L_1,L_1], \\ \bar{E}^{(i)}=\bar{E}^{(i-1)}+(I_{[-L_1,0]}(a_{51}\bar{f}^{(i-1)}+a_{52}\bar{m}^{(i-1)}) \\ \quad -a_{53}\bar{E}^{(i-1)})/K\in[-L_1,L_1], \\ \bar{G}^{(i)}=\bar{G}^{(i-1)}+(a_{61}I_{[0,L_1]}\bar{n}^{(i-1)} \\ \quad -a_{62}\bar{G}^{(i-1)})/K\in[-L_1,L_1], \\ \bar{P}^{(i)}=\bar{P}^{(i-1)}+(a_{71}I_{[-L_1,0]}\bar{m}^{(i-1)}-a_{72}\bar{P}^{(i-1)})/K\in[-L_1,L_1], \\ \underline{n}^{(i)}=\underline{n}^{(i-1)}+(a_{11}\underline{n}^{(i-1)}\underline{\vartheta}_E^{(i-1)}\left(1-\frac{\underline{n}^{(i-1)}}{n_*-a_{12}\rho I_s}\right))/K\in[0,L_1], \\ \underline{f}^{(i)}=\underline{f}^{(i-1)}+((-a_{12}\bar{G}+a_{22})\underline{f})/K\in[-L_1,0], \\ \underline{m}^{(i)}=\underline{m}^{(i-1)}+(a_{21}\underline{G}^{(i-1)}\underline{f}^{(i-1)}+a_{31}\underline{m}^{(i-1)})/K\in[-L_1,0], \\ \underline{\rho}^{(i)}=\underline{\rho}^{(i-1)}+(-a_{41}\bar{P}^{(i-1)}\bar{n}^{(i-1)}+(a_{42}\bar{f}^{(i-1)}+a_{43}\bar{m}^{(i-1)}) \\ \quad \times\left(1-\frac{\underline{\rho}^{(i-1)}}{\rho_*}\right))/K\in[-L_1,L_1], \\ \underline{E}^{(i)}=\underline{E}^{(i-1)}+(I_{[-L_1,0]}(a_{51}\underline{f}^{(i-1)}+a_{52}\underline{m}^{(i-1)}) \\ \quad -a_{53}\underline{E}^{(i-1)})/K\in[-L_1,L_1], \\ \underline{G}^{(i)}=\underline{G}^{(i-1)}+(a_{61}I_{[0,L_1]}\underline{n}^{(i-1)}-a_{62}\underline{G}^{(i-1)})/K\in[-L_1,L_1], \\ \bar{P}^{(i)}=\bar{P}^{(i-1)}+(a_{71}I_{[-L_1,0]}\underline{m}^{(i-1)}-a_{72}\underline{P}^{(i-1)})/K\in[-L_1,L_1], \end{array}\right\}$$

where,

$$(\bar{n}^{(0)},\bar{f}^{(0)},\bar{m}^{(0)},\bar{\rho}^{(0)},\bar{E}^{(0)},\bar{G}^{(0)})=(\bar{n},\bar{f},\bar{m},\bar{E},\bar{G},\bar{P})$$

and

$$(\underline{n}^{(0)},\underline{f}^{(0)},\underline{m}^{(0)},\underline{\rho}^{(0)},\underline{E}^{(0)},\underline{G}^{(0)},\underline{P}^{(0)}) \\ =(\underline{n},\underline{f},\underline{m},\underline{\rho},\underline{E},\underline{G},\underline{P}).$$

Thus, for $i \geq 1$,

$$\begin{array}{l}(\underline{n},\underline{f},\underline{m},\underline{\rho},\underline{E},\underline{G},\underline{P}) \leq (\underline{n}^{(i)},\underline{f}^{(i)},\underline{m}^{(i)},\underline{\rho}^{(i)},\underline{E}^{(i)},\underline{G}^{(i)},\underline{P}^{(i)})\\ \leq (\underline{n}^{(i+1)},\underline{f}^{(i+1)},\underline{m}^{(i+1)},\underline{\rho}^{(i+1)},\underline{E}^{(i+1)},\underline{G}^{(i+1)},\underline{P}^{(i+1)})\\ \leq (\bar{n}^{(i+1)},\bar{f}^{(i+1)},\bar{m}^{(i+1)},\bar{\rho}^{(i+1)},\bar{E}^{(i+1)},\bar{G}^{(i+1)}\bar{P}^{(i+1)})\\ \leq (\bar{n}^{(i)},\bar{f}^{(i)},\bar{m}^{(i)},\bar{\rho}^{(i)},\bar{E}^{(i)},\bar{G}^{(i)},\bar{P}^{(i)})\\ \leq (\bar{n},\bar{f},\bar{m},\bar{\rho},\bar{E},\bar{G},\bar{P}),\end{array}$$

and thus, there exist

$$(\tilde{n}^{(0)},\tilde{f}^{(0)},\tilde{m}^{(0)},\tilde{\rho}^{(0)},\tilde{E}^{(0)},\tilde{G}^{(0)},\tilde{P}^{(0)}) > (0,0,0,0,0)$$

and

$$(\hat{n}^{(0)},\hat{f}^{(0)},\hat{m}^{(0)},,\hat{\rho}^{(0)},\hat{E}^{(0)},\hat{G}^{(0)},\hat{P}^{(0)}) > (0,0,0,0,0)$$

such that

$$\begin{array}{c}\lim_{i\to\infty}\bar{n}=\tilde{n},\\ \lim_{i\to\infty}\bar{f}=\tilde{f},\\ \lim_{i\to\infty}\bar{m}=\tilde{m},\\ \lim_{i\to\infty}\bar{\rho}=\tilde{\rho},\\ \lim_{i\to\infty}\bar{E}=\tilde{E},\\ \lim_{i\to\infty}\bar{G}=\tilde{G},\\ \lim_{i\to\infty}\bar{P}=\tilde{P}\end{array}$$

and

$$\begin{array}{l}\lim_{i\to\infty}\underline{n}=\hat{n},\ \lim_{i\to\infty}\underline{f}=\hat{f},\\ \lim_{i\to\infty}\underline{m}=\hat{m},\ \lim_{i\to\infty}\underline{\rho}=\hat{\rho},\\ \lim_{i\to\infty}\underline{E}=\hat{E},\ \lim_{i\to\infty}\underline{G}=\hat{G},\\ \lim_{i\to\infty}\underline{P}=\hat{P},\end{array}$$

and

$$\left.\begin{array}{l}a_{11}\tilde{n}\tilde{\vartheta}_E\left(1-\frac{\tilde{n}}{n_*-a_{12}\rho I_s}\right)=0\\ a_{11}\hat{n}\hat{\vartheta}_E\left(1-\frac{\hat{n}}{n_*-a_{12}\rho I_s}\right)=0\in[0,L_1],\\ (-a_{12}\hat{G}+a_{22})\tilde{f}=0,\\ (-a_{12}\tilde{G}+a_{22})\hat{f}=0\in[-L_1,0],\\ a_{21}\tilde{G}\tilde{f}+a_{31}\tilde{m}=0,\\ a_{21}\hat{G}\hat{f}+a_{31}\hat{m}=0\in[-L_1,0],\\ -a_{41}\hat{P}\hat{n}+(a_{42}\hat{f}+a_{43}\hat{m})\left(1-\frac{\tilde{\rho}}{\rho_*}\right)=0\\ -a_{41}\tilde{P}\tilde{n}+(a_{42}\tilde{f}+a_{43}\tilde{m})\left(1-\frac{\hat{\rho}}{\rho_*}\right)=0\in[-L_1,L_1],\\ I_{[-L_1,0]}(a_{51}\tilde{f}+a_{52}\tilde{m})-a_{53}\tilde{E}=0,\\ I_{[-L_1,0]}(a_{51}\hat{f}+a_{52}\hat{m})-a_{53}\hat{E}=0\in[-L_1,L_1],\\ a_{61}I_{[0,L_1]}\tilde{n}-a_{62}\tilde{G}=0,\\ a_{61}I_{[0,L_1]}\hat{n}-a_{62}\hat{G}=0\in[-L_1,L_1],\\ a_{71}I_{[-L_1,0]}\tilde{m}-a_{72}\tilde{P}=0,\\ a_{71}I_{[-L_1,0]}\tilde{m}-a_{72}\tilde{P}\in[-L_1,L_1],\end{array}\right\}$$

Since,

$$\mathscr{E}=\left(\frac{a_{62}a_{22}}{a_{61}a_{12}},0,0,L_1\sqrt{1+(L_1)^2}\right.$$
$$\left.+\ln\left(\frac{L_1+\sqrt{1+L_1^2}}{\sqrt{1+L_1^2}-L_1}\right)^{\frac{1}{2}},0,\frac{a_{22}}{a_{12}},0\right),$$

is the unique semi-positive constant equilibrium of system (1.2), it must hold for

$$(\tilde{n},\tilde{f},\tilde{m},\tilde{\rho},\tilde{E},\tilde{G},\tilde{P})=(\hat{n},\hat{f},\hat{m},\hat{\rho},\hat{E},\hat{G},\hat{P})$$
$$=(n_*-a_{12}\rho I_s,0,0,0,\tfrac{a_{22}}{a_{12}}).$$

Thus, by [18,19], the solution $(n(x,t),f(x,t),m(x,t),\rho(x,t),E(x,t),G(x,t),P(x,t))$ of system (1.2) satisfies

$$\begin{array}{l}\lim_{t\to\infty}n(x,t)=n^*,\ \lim_{t\to\infty}f(x,t)=f^*,\\ \lim_{t\to\infty}m(x,t)=m^*,\ \lim_{t\to\infty}\rho(x,t)=\rho^*\\ \lim_{t\to\infty}E(x,t)=E^*,\\ \lim_{t\to\infty}G(x,t)=G^*\ \lim_{t\to\infty}P(x,t)=P^*,\end{array}$$

which concludes the prove. □

## 3 Derivation and analysis of the numerical method

The derivation of the fitted numerical method for solving the system in equation (1.2) is as follows. We determine an approximation to the derivatives for the functions

$$n(t,x),f(x,t),m(x,t),\rho(x,t),E(x,t),G(x,t),P(x,t),$$

with respect to the spatial variable $x$.

Let $S_x$ be a positive integer. Discretize the interval $[-L/2,L/2]$ through the points

$$\begin{array}{l}-L/2=x_0<x_1<x_2<\cdots<x_{s-1}<x_s\\ \quad<x_{s+1}\cdots<x_{S_x-2}<x_{S_x-1}<x_{S_x}=L/2,\end{array}$$

where, the step-size $\Delta x = x_{j+1}-x_j = 1/S_x$, $j=0,1,\ldots,S_x$. Let

$$\mathscr{N}_j(t),\mathscr{F}_j(t),\mathscr{M}_j(t),\mathscr{R}_j(t),\mathscr{E}_j(t),\mathscr{G}_j(t),\mathscr{P}_j(t),$$

denote the numerical approximations for $n(t,x),f(x,t),m(x,t),\rho(x,t),E(x,t),G(x,t),P(x,t)$. Then the spatial derivatives in the system in equation (1.2) are approximated as follows

$$\left.\begin{array}{l}\frac{\partial}{\partial x}\left(D_n\frac{\partial n}{\partial x}-\chi_n n\frac{\frac{\partial E}{\partial x}}{\sqrt{1+(\frac{\partial E}{\partial x}/\lambda_E)^2}}\right.\\ \quad\left.-\chi_n^1 I_s n\frac{\frac{\partial\rho}{\partial x}}{\sqrt{1+(\frac{\partial\rho}{\partial x}/\lambda_\rho)^2}}\right)(t_i,x_j)\\ \approx D_n\frac{\mathscr{N}_{j+1}-2\mathscr{N}_j+\mathscr{N}_{j-1}}{\phi_n^2}\\ -\chi_n(D_x^-\mathscr{N}_j)\frac{(D_x^-\mathscr{E}_j)}{\sqrt{1+\left(\frac{D_x^-\mathscr{E}_j}{\lambda_E}\right)^2}}\\ \quad-\chi_n\mathscr{N}_j\frac{D_x^+(D_x^-\mathscr{E}_j)}{\left(1+\left(\frac{D_x^-\mathscr{E}_j}{\lambda_E}\right)^2\right)^{3/2}},\\ -\chi_n(D_x^-\mathscr{N}_j)\frac{(D_x^-\mathscr{R}_j)}{\sqrt{1+\left(\frac{D_x^-\mathscr{R}_j}{\lambda_\rho}\right)^2}}\\ \quad-\chi_n^1 I_s\mathscr{N}_j\frac{D_x^+(D_x^-\mathscr{R}_j)}{\left(1+\left(\frac{D_x^-\mathscr{R}_j}{\lambda_\rho}\right)^2\right)^{3/2}},\end{array}\right\}$$

$$\left.\begin{array}{l}\frac{\partial}{\partial x}\left(D_f\frac{\partial f}{\partial x}\right)(t_i,x_j)\approx D_f\frac{\mathscr{F}_{j+1}-2\mathscr{F}_j+\mathscr{F}_{j-1}}{\phi_f^2},\\ \frac{\partial}{\partial x}\left(D_m\frac{\partial m}{\partial x}-\chi_m m\frac{\frac{\partial G}{\partial x}}{\sqrt{1+(\frac{\partial G}{\partial x}/\lambda_G)^2}}\right)(t_i,x_j)\\ \approx D_m\frac{\mathscr{M}_{j+1}-2\mathscr{M}_j+\mathscr{M}_{j-1}}{\phi_m^2}\\ -\chi_m(D_x^-\mathscr{M}_j)\frac{(D_x^-\mathscr{G}_j)}{\sqrt{1+\left(\frac{D_x^-\mathscr{G}_j}{\lambda_G}\right)^2}}\\ -\chi_m\mathscr{M}_j\frac{D_x^+(D_x^-\mathscr{G}_j)}{\left(1+\left(\frac{D_x^-\mathscr{G}_j}{\lambda_G}\right)^2\right)^{3/2}},\end{array}\right\}$$

$$\left.\begin{array}{l}\frac{\partial}{\partial x}\left(D_\rho\frac{\partial \rho}{\partial x}\right)(t_i,x_j)\approx D_\rho\frac{\mathscr{R}_{j+1}-2\mathscr{R}_j+\mathscr{R}_{j-1}}{(\Delta t)^2},\\ \frac{\partial}{\partial x}\left(D_E\frac{\partial E}{\partial x}\right)(t_i,x_j)\approx D_E\frac{\mathscr{E}_{j+1}-2\mathscr{E}_j+\mathscr{E}_{j-1}}{\phi_E^2},\\ \frac{\partial}{\partial x}\left(D_G\frac{\partial G}{\partial x}\right)(t_i,x_j)\approx D_G\frac{\mathscr{G}_{j+1}-2\mathscr{G}_j+\mathscr{G}_{j-1}}{\phi_G^2},\end{array}\right\}$$

where,

$$D^+(\cdot)_j:=\frac{(\cdot)_{j+1}-(\cdot)_j}{\Delta x},\ D^-(\cdot)_j:=\frac{(\cdot)_j-(\cdot)_{j-1}}{\Delta x},$$

and the denominator functions

$$\begin{array}{l}\phi_n^2:=\frac{D_n\Delta x}{\chi_n}\left[\exp(\frac{\chi_n\Delta x}{D_n})-1\right],\\ \phi_f^2:=\frac{4}{\rho_f^2}\sin\left(\frac{\rho_f\Delta x}{2}\right)^2,,\ \ \rho_f:=\sqrt{\frac{a_{22}}{D_f}},\\ \phi_m^2:=\frac{D_m\Delta x}{\chi_m}\left[\exp(\frac{\chi_m\Delta x}{D_m})-1\right],\\ \phi_E^2:=\frac{4}{\rho_e^2}\sinh\left(\frac{\rho_e\Delta x}{2}\right)^2,\ \ \rho_e:=\sqrt{\frac{a_{53}}{D_e}},\\ \phi_G^2:=\frac{4}{\rho_g^2}\sinh\left(\frac{\rho_g\Delta x}{2}\right)^2,\ \ \rho_g:=\sqrt{\frac{a_{62}}{Dg}},\\ \phi_P^2:=\frac{4}{\rho_p^2}\sinh\left(\frac{\rho_p\Delta x}{2}\right)^2,\ \ \rho_p:=\sqrt{\frac{a_{72}}{D_p}}.\end{array}$$

Let $S_t$ be a positive integer such that $\Delta t=1/S_t$ where $0<t<S_t$. Then discretizing the time interval $[0,T]$, through the points

$$0=t_0<t_1<\cdots<t_{S_t}=T,$$

$$t_{i+1}-t_i=\Delta t,\ i=0,1,\ldots,(t_{S_t}-1).$$

We approximate the time derivative at $t_i$ by

$$\left.\begin{array}{l}\frac{\partial n}{\partial t}(x_j,t_i)\approx\frac{\mathscr{N}_{j+1}^{i+1}-\mathscr{N}_j^i}{\Delta t},\ \frac{\partial f}{\partial t}(x_j,t_i)\\ \approx\frac{\mathscr{F}_{j+1}^{i+1}-\mathscr{F}_j^i}{\psi_f},\ \frac{\partial m}{\partial t}(x_j,t_i)\approx\frac{\mathscr{M}_{j+1}^{i+1}-\mathscr{M}_j^i}{\Delta t},\end{array}\right\}$$

$$\left.\begin{array}{l}\frac{\partial \rho}{\partial t}(x_j,t_i)\approx\frac{\mathscr{R}_{j+1}^{i+1}-\mathscr{R}_j^i}{\Delta t},\ \frac{\partial E}{\partial t}(x_j,t_i)\\ \approx\frac{\mathscr{E}_{j+1}^{i+1}-\mathscr{E}_j^i}{\psi_E},\ \frac{\partial G}{\partial t}(x_j,t_i)\approx\frac{\mathscr{G}_{j+1}^{i+1}-\mathscr{G}_j^i}{\psi_G},\\ \frac{\partial P}{\partial t}(x_j,t_i)\approx\frac{\mathscr{P}_{j+1}^{i+1}-\mathscr{P}_j^i}{\psi_P},\end{array}\right\}$$

where,

$$\begin{array}{l}\psi_f=\psi_w(\Delta t)=(1-\exp(-a_{22}\Delta t))/a_{22},\\ \psi_E=(1-\exp(-a_{53}\Delta t))/a_{43},\\ \psi_G=(1-\exp(-a_{62}\Delta t))/a_{62},\\ \psi_P=(1-\exp(-a_{72}\Delta t))/a_{72},\end{array}$$

where, one can see that

$$\begin{array}{l}\phi_n\to\Delta x,\phi_f\to\Delta x,\psi_f\to\Delta t,\\ \phi_m\to\Delta x,\phi_E\to\Delta x,\psi_E\to\Delta t,\\ \phi_G\to\Delta x,\psi_G\to\Delta t,\\ \phi_P\to\Delta x,\psi_P\to\Delta t\\ (\Delta t,\Delta x)\to(0,0).\end{array}$$

The denominator functions in equations (3) and (3) are used explicitly to remove the inherent stiffness in the central finite derivatives parts and can be derived by using the theory of nonstandard finite difference methods, see, e.g., [21,22,23] and references therein. Combining the equation (3) for the spatial derivatives with equation (3) for time derivatives, we obtain

$$\left.\frac{\mathscr{N}_j^{i+1}-\mathscr{N}_j^i}{\Delta t}-D_n\frac{\mathscr{N}_{j+1}^{i+1}-2\mathscr{N}_j^{i+1}+\mathscr{N}_{j-1}^{i+1}}{\phi_n^2}\right\}$$

$$\left.=-\chi_n(D_x^-n_j^i)\frac{(D_x^-\mathscr{E}_j^i)}{\sqrt{1+\left(\frac{D_x^-\mathscr{E}_j^i}{\lambda_E}\right)^2}}\right\}$$

$$\left.\begin{array}{l}-\chi_n\mathscr{N}_j^i\frac{D_x^+(D_x^-\mathscr{E}_j^i)}{\left(1+\left(\frac{D_x^-\mathscr{E}_j^i}{\lambda_E}\right)^2\right)^{3/2}}-\chi_n^1 I_s\mathscr{N}_j^i\frac{D_x^+(D_x^-\mathscr{R}_j^i)}{\left(1+\left(\frac{D_x^-\mathscr{R}_j^i}{\lambda_\rho}\right)^2\right)^{3/2}}\\ +\frac{a_{11}(\mathscr{E}^4)_j^i}{k_E^4+(\mathscr{E}^4)_j^i\mathscr{N}_j^i}(1-\frac{\mathscr{N}_j^i}{n_*-a_{12}\rho I_s}),x\in[x_s,L/2],\end{array}\right\}$$

$$\left.\begin{array}{l}\frac{\mathscr{F}_j^{i+1}-\mathscr{F}_j^i}{\psi_f}-D_f\frac{\mathscr{F}_{j+1}^{i+1}-2\mathscr{F}_j^{i+1}+\mathscr{F}_{j-1}^{i+1}}{\phi_f^2}=-a_{21}(\mathscr{H}_G)_j^i(\mathscr{H}_f)_j^i\\ +a_{22}(\mathscr{H}_f)_j^i,x\in[-\frac{L}{2},x_s],\\ \frac{\mathscr{M}_j^{i+1}-\mathscr{M}_j^i}{\Delta t}-D_m\frac{\mathscr{M}_{j+1}^{i+1}-2\mathscr{M}_j^{i+1}+\mathscr{M}_{j-1}^{i+1}}{\phi_m^2}\\ =-\chi_m(D_x^-\mathscr{M}_j^i)\frac{(D_x^-\mathscr{G}_j^i)}{\sqrt{1+\left(\frac{D_x^-\mathscr{G}_j^i}{\lambda_G}\right)^2}}\\ -\chi_m\mathscr{M}_j^i\frac{D_x^+(D_x^-\mathscr{G}_j^i)}{\left(1+\left(\frac{D_x^-\mathscr{G}_j^i}{\lambda_G}\right)^2\right)^{3/2}}\\ +a_{21}(\mathscr{H}_G)_j^i(\mathscr{H}_f)_j^i+a_{31}\mathscr{M}_j^i,\\ x\in[-\frac{L}{2},x_s],\end{array}\right\}$$

$$\left.\begin{array}{l}\frac{\mathscr{R}_j^{i+1}-\mathscr{R}_j^i}{\Delta t}-D_\rho\frac{\mathscr{R}_{j+1}^{i+1}-2\mathscr{R}_j^{i+1}+\mathscr{R}_{j-1}^{i+1}}{\phi_\rho^2}=-a_{41}(\mathscr{H}_p)_j^i(\mathscr{H}_n)_j^i\\ +(a_{42}\mathscr{F}_j^i+a_{43}\mathscr{M}_j^i)(1-\frac{\mathscr{R}_j^i}{\rho_*}),x\in[-\frac{L}{2},\frac{L}{2}],\\ \frac{\mathscr{E}_j^{i+1}-\mathscr{E}_j^i}{\psi_E}-D_E\frac{\mathscr{E}_{j+1}^{i+1}-2\mathscr{E}_j^{i+1}+\mathscr{E}_{j-1}^{i+1}}{\phi_E^2}\\ =I_{\Omega\_}a_{51}(\mathscr{H}_f)_j^i+a_{52}(\mathscr{H}_m)_j^i\\ -a_{53}\mathscr{E}_j^i,x\in[-\frac{L}{2},\frac{L}{2}],\end{array}\right\}$$

$$\left.\begin{aligned}
&\frac{\mathscr{G}_j^{i+1}-\mathscr{G}_j^i}{\psi_G}-D_G\frac{\mathscr{G}_{j+1}^{i+1}-2\mathscr{G}_j^{i+1}+\mathscr{G}_{j-1}^{i+1}}{\phi_G^2}\\
&\quad=I_{\Omega_+}a_{61}(\mathscr{H}_n)_j^i-a_{62}\mathscr{G}_j^i,\\
&\quad x\in[-\tfrac{L}{2},\tfrac{L}{2}],\\
&\frac{\mathscr{P}_j^{i+1}-\mathscr{P}_j^i}{\psi_P}-D_P\frac{\mathscr{P}_{j+1}^{i+1}-2\mathscr{P}_j^{i+1}+\mathscr{P}_{j-1}^{i+1}}{\phi_P^2}\\
&\quad=I_{\Omega_-}a_{71}(\mathscr{H}_m)_j^i-a_{72}\mathscr{P}_j^i,\\
&\quad x\in[-\tfrac{L}{2},\tfrac{L}{2}],
\end{aligned}\right\}$$

with the terminal conditions

$$\left.\begin{aligned}
&\mathscr{N}_{x_s-1}^i=\mathscr{N}_{x_s+1}^i-2\Delta x\chi_n\mathscr{N}_{x_s}^i\left(\frac{\mathscr{E}_{x_s+1}^i-\mathscr{E}_{x_s-1}^i}{2\Delta x\sqrt{1+\left(\frac{\mathscr{E}_{x_s+1}^i-\mathscr{E}_{x_s-1}^i}{2\Delta x\lambda_E}\right)^2}}\right)\\
&-2\Delta x\chi_n^1\mathscr{N}_{x_s}^i\left(\frac{\mathscr{R}_{x_s+1}^i-\mathscr{R}_{x_s-1}^i}{2\Delta x\sqrt{1+\left(\frac{\mathscr{R}_{x_s+1}^i-\mathscr{R}_{x_s-1}^i}{2\Delta x\lambda_\rho}\right)^2}}\right),\\
&\mathscr{F}^i_{\frac{-L}{2}+1}=\mathscr{F}^i_{\frac{-L}{2}-1},\ \mathscr{M}^i_{\frac{-L}{2}-1}\\
&\quad=\mathscr{M}^i_{\frac{-L}{2}+1}-2\Delta x\chi_m\mathscr{M}^i_{\frac{-L}{2}}\left(\frac{\mathscr{G}^i_{\frac{-L}{2}+1}-\mathscr{G}^i_{\frac{-L}{2}-1}}{2\Delta x\sqrt{1+\left(\frac{\mathscr{G}^i_{\frac{-L}{2}+1}-\mathscr{G}^i_{\frac{-L}{2}-1}}{2\Delta x\lambda_G}\right)^2}}\right),
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&\mathscr{R}^i_{-\frac{L}{2}+1}=\mathscr{R}^i_{-\frac{L}{2}-1},\mathscr{E}^i_{\frac{-L}{2}-1}=(\mathscr{E}^-)^i_{\frac{-L}{2}+1}(1+2\Delta x\gamma),\\
&\mathscr{G}^i_{\frac{-L}{2}-1}=(\mathscr{G}^-)^i_{\frac{-L}{2}+1}(1+2\Delta x\gamma),\mathscr{P}^i_{\frac{-L}{2}-1}=(\mathscr{P}^-)^i_{\frac{-L}{2}+1}(1+2\Delta x\gamma),\\
&\mathscr{N}^0_{x_j}=\tfrac{1}{2}\left(1+\tanh\left(-\tfrac{1}{\varepsilon}(0.8-x_j)\right)\right),\mathscr{F}^0_{x_j}\\
&\quad=0.143\tfrac{1}{2}\left(1+\tanh\left(-\tfrac{1}{\varepsilon}(x-0.2)\right)\right),\\
&\mathscr{R}^0_{x_j}=1.0,\ \mathscr{M}^0_{x_j}=0.00,\ \mathscr{E}^0_{x_j}=\mathscr{G}^0_{x_j}=\mathscr{P}^0_{x_j}=1.00,
\end{aligned}\right\}$$

where, the no-flux boundary conditions are discretised by means of the central finite difference [24], $j=-L/2,2,\ldots,L/2-1,\ i=0,1,\ldots,T-1$ and

$$\begin{aligned}
&(\mathscr{H}_n)_j^i\approx N(t_i-\tau,x_j),\ (\mathscr{H}_f)_j^i\\
&\qquad\approx F(t_i-\tau,x_j), &(3.10)\\
&(\mathscr{H}_G)_j^i\approx G(t_i-\tau,x_j),\\
&(\mathscr{H}_m)_j^i\approx M(t_i-\tau,x_j),\\
&(\mathscr{H}_P)_j^i\approx P(t_i-\tau,x_j), &(3.11)
\end{aligned}$$

are denoting the history functions corresponding to $n,f,m,G,P$. The system in equation (3) can further be simplified as

$$\left.\begin{aligned}
&-\frac{D_n}{\phi_n^2}\mathscr{N}_{j-1}^{i+1}\\
&+\left(\frac{1}{\Delta t}+\frac{2D_n}{\phi_n^2}\right)\mathscr{N}_j^{i+1}-\frac{D_n}{\phi_n^2}\mathscr{N}_{j+1}^{i+1}\\
&=-\chi_n(D_x^-n_j^i)\frac{(D_x^-\mathscr{E}_j^i)}{\sqrt{1+\left(\frac{D_x^-\mathscr{E}_j^i}{\lambda_E}\right)^2}}\\
&-\chi_n\mathscr{N}_j^i\frac{D_x^+(D_x^-\mathscr{E}_j^i)}{\left(1+\left(\frac{D_x^-\mathscr{E}_j^i}{\lambda_E}\right)^2\right)^{3/2}}\\
&-\chi_n^1I_s\mathscr{N}_j^i\frac{D_x^+(D_x^-\mathscr{R}_j^i)}{\left(1+\left(\frac{D_x^-\mathscr{R}_j^i}{\lambda_\rho}\right)^2\right)^{3/2}}\\
&+a_{11}\frac{(\mathscr{E}^4)_j^i}{k_E^4+(\mathscr{E}^4)_j^i}\mathscr{N}_j^i\left(1-\frac{\mathscr{N}_j^i}{n_*-a_{12}\rho I_s}\right)+\frac{\mathscr{N}_j^i}{\Delta t},
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&-\frac{D_f}{\phi_f^2}\mathscr{F}_{j-1}^{i+1}+\left(\frac{1}{\psi_f}+\frac{2D_f}{\phi_f^2}\right)\mathscr{F}_j^{i+1}-\frac{D_f}{\phi_f^2}\mathscr{F}_{j+1}^{i+1}\\
&=-a_{21}(\mathscr{H}_G)_j^i(\mathscr{H}_f)_j^i+a_{22}(\mathscr{H}_f)_j^i+\frac{\mathscr{F}_j^i}{\psi_f},\\
&-\frac{D_m}{\phi_m^2}\mathscr{M}_{j-1}^{i+1}+\left(\frac{1}{\Delta t}+\frac{2D_m}{\phi_m^2}\right)\mathscr{M}_j^{i+1}-\frac{D_m}{\phi_m^2}\mathscr{M}_{j+1}^{i+1}\\
&=-\chi_m(D_x^-\mathscr{M}_j^i)\frac{(D_x^-\mathscr{G}_j^i)}{\sqrt{1+\left(\frac{D_x^-\mathscr{G}_j^i}{\lambda_G}\right)^2}}\\
&-\chi_m\mathscr{M}_j^i\frac{D_x^+(D_x^-\mathscr{G}_j^i)}{\left(1+\left(\frac{D_x^-\mathscr{G}_j^i}{\lambda_G}\right)^2\right)^{3/2}}\\
&+a_{21}(\mathscr{H}_G)_j^i(\mathscr{H}_f)_j^i+a_{31}\mathscr{M}_j^i+\frac{\mathscr{M}_j^i}{\Delta t},
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&-\frac{D_\rho}{(\Delta x)^2}\mathscr{R}_{j-1}^{i+1}+\left(\frac{1}{(\Delta t)}+\frac{2D_\rho}{(\Delta x)^2}\right)\mathscr{R}_j^{i+1}-\frac{D_\rho}{(\Delta x)^2}\mathscr{R}_{j+1}^{i+1}\\
&=-a_{41}(\mathscr{H}_P)_j^i(\mathscr{H}_n)_j^i+(a_{42}\mathscr{F}_j^i\\
&+a_{43}\mathscr{M}_j^i)\left(1-\frac{\mathscr{R}}{\rho_*}\right)+\frac{\mathscr{R}_j^i}{(\Delta t)},\\
&-\frac{D_E}{\phi_E^2}\mathscr{E}_{j-1}^{i+1}+\left(\frac{1}{\psi_E}+\frac{2D_E}{\phi_E^2}\right)\mathscr{E}_j^{i+1}-\frac{D_E}{\phi_E^2}\mathscr{E}_{j+1}^{i+1}\\
&=I_{\Omega_-}(a_{51}(\mathscr{H}_f)_j^i+a_{52}(\mathscr{H}_m)_j^i)-a_{62}\mathscr{E}_j^i+\frac{\mathscr{E}_j^i}{\psi_E},
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&-\frac{D_G}{\phi_G^2}\mathscr{G}_{j-1}^{i+1}+\left(\frac{1}{\psi_G}+\frac{2D_G}{\phi_G^2}\right)\mathscr{G}_j^{i+1}-\frac{D_G}{\phi_G^2}\mathscr{G}_{j+1}^{i+1}\\
&=a_{61}I_{\Omega_+}(\mathscr{H}_n)_j^i-a_{62}\mathscr{G}_j^i+\frac{\mathscr{G}_j^i}{\psi_G},\\
&-\frac{D_P}{\phi_P^2}\mathscr{P}_{j-1}^{i+1}+\left(\frac{1}{\psi_P}+\frac{2D_P}{\phi_P^2}\right)\mathscr{P}_j^{i+1}-\frac{D_P}{\phi_P^2}\mathscr{P}_{j+1}^{i+1}\\
&=a_{71}I_{\Omega_-}(\mathscr{H}_m)_j^i-a_{72}\mathscr{P}_j^i+\frac{\mathscr{P}_j^i}{\psi_P},
\end{aligned}\right\}$$

which can be written as a tridiagonal system given by

$$\left.\begin{aligned}
&A_n\mathscr{N}_j^{i+1}\\
&=-\chi_n(D_x^-n_j^i)\frac{(D_x^-\mathscr{E}_j^i)}{\sqrt{1+\left(\frac{D_x^-\mathscr{E}_j^i}{\lambda_E}\right)^2}}-\chi_n\mathscr{N}_j^i\frac{D_x^+(D_x^-\mathscr{E}_j^i)}{\left(1+\left(\frac{D_x^-\mathscr{E}_j^i}{\lambda_E}\right)^2\right)^{3/2}}\\
&-\chi_n^1I_s\mathscr{N}_j^i\frac{D_x^+(D_x^-\mathscr{R}_j^i)}{\left(1+\left(\frac{D_x^-\mathscr{R}_j^i}{\lambda_\rho}\right)^2\right)^{3/2}}\\
&+a_{11}\frac{(\mathscr{E}^4)_j^i}{k_E^4+(\mathscr{E}^4)_j^i}\mathscr{N}_j^i\left(1-\frac{\mathscr{N}_j^i}{n_*-a_{12}\rho I_s}\right)+\frac{\mathscr{N}_j^i}{\Delta t},
\end{aligned}\right\}$$

$$\left.\begin{aligned} A_f\mathscr{F}_j^{i+1} &= -a_{21}(\mathscr{H}_G)_j^i(\mathscr{H}_f)_j^i + a_{22}(\mathscr{H}_f)_j^i + \frac{\mathscr{F}_j^i}{\psi_f},\\ A_m\mathscr{M}_j^{i+1} &= -\chi_m(D_x^-\mathscr{M}_j^i)\frac{(D_x^-\mathscr{G}_j^i)}{\sqrt{1+\left(\frac{D_x^-\mathscr{G}_j^i}{\lambda_G}\right)^2}}\\ &\quad -\chi_m\mathscr{M}_j^i\frac{D_x^+(D_x^-\mathscr{G}_j^i)}{\left(1+\left(\frac{D_x^-\mathscr{G}_j^i}{\lambda_G}\right)^2\right)^{3/2}}\\ &\quad +a_{21}(\mathscr{H}_G)_j^i(\mathscr{H}_f)_j^i + a_{31}\mathscr{M}_j^i + \frac{\mathscr{M}_j^i}{\Delta t}, \end{aligned}\right\}$$

$$\left.\begin{aligned} A_\rho\mathscr{R}_j^{i+1} &= -a_{41}(\mathscr{H}_P)_j^i(\mathscr{H}_n)_j^i + (a_{42}\mathscr{F}_j^i + a_{43}\mathscr{M}_j^i)\\ &\quad \times(1-\frac{\mathscr{R}}{\rho_*}) + \frac{\mathscr{R}_j^i}{(\Delta t)},\\ A_E\mathscr{E}_j^{i+1} &= I_{\Omega_-}(a_{51}(\mathscr{H}_f)_j^i + a_{52}(\mathscr{H}_m)_j^i)\\ &\quad -a_{62}\mathscr{E}_j^i + \frac{\mathscr{E}_j^i}{\psi_E}, \end{aligned}\right\}$$

$$\left.\begin{aligned} A_G\mathscr{G}_j^{i+1} &= a_{61}I_{\Omega_+}(\mathscr{H}_n)_j^i - a_{62}\mathscr{G}_j^i\\ &\quad + \frac{\mathscr{G}_j^i}{\psi_G}, A_P\mathscr{P}_j^{i+1}\\ &= a_{71}I_{\Omega_-}(\mathscr{H}_m)_j^i - a_{72}\mathscr{P}_j^i + \frac{\mathscr{P}_j^i}{\psi_P}, \end{aligned}\right\}$$

where,

$$\left.\begin{aligned} A_n &= \left(-\frac{D_n}{\phi_n^2}, \frac{1}{\Delta t} + \frac{2D_n}{\phi_n^2}, -\frac{D_n}{\phi_n^2}\right),\\ A_f &= \left(-\frac{D_f}{\phi_f^2}, \frac{1}{\psi_f} + \frac{2D_f}{\phi_f^2}, -\frac{D_f}{\phi_f^2}\right),\\ A_m &= \left(-\frac{D_m}{\phi_m^2}, \frac{1}{\Delta t} + \frac{2D_m}{\phi_m^2}, -\frac{D_m}{\phi_m^2}\right),\\ A_\rho &= \left(-\frac{D_\rho}{(\Delta x)^2}, \frac{1}{\Delta t} + \frac{2D_\rho}{(\Delta x)^2}, -\frac{D_\rho}{(\Delta x)^2}\right), \end{aligned}\right\}$$

$$\left.\begin{aligned} A_E &= \left(-\frac{D_E}{\phi_E^2}, \frac{1}{\psi_E} + \frac{2D_E}{\phi_E^2}, -\frac{D_E}{\phi_E^2}\right)\\ ,A_G &= \left(-\frac{D_G}{\phi_G^2}, \frac{1}{\psi_G} + \frac{2D_G}{\phi_G^2}, -\frac{D_G}{\phi_G^2}\right)\\ A_P &= \left(-\frac{D_P}{\phi_P^2}, \frac{1}{\psi_P} + \frac{2D_P}{\phi_P^2}, -\frac{D_P}{\phi_P^2}\right). \end{aligned}\right\}$$

On the interval $[0,\tau]$, the delayed arguments $t_n - \tau$ belong to $[-\tau, 0]$, and therefore, the delayed variables in equation (3) are evaluated directly from the history functions

$$n^0(t,x), f^0(t,x), m^0(t,x), G^0(t,x), P^0(t,x),$$

as

$$\begin{aligned} &(\mathscr{H}_n)_j^i \approx n^0(t_i-\tau,x_j),\ (\mathscr{H}_f)_j^i \approx f^0(t_i-\tau,x_j),\\ &(\mathscr{H}_m)_j^i \approx m^0(t_i-\tau,x_j),\\ &(\mathscr{H}_G)_j^i \approx G^0(t_i-\tau,x_j),\\ &(\mathscr{H}_P)_j^i \approx P^0(t_i-\tau,x_j), \end{aligned} \tag{3.12}$$

and equation (3) becomes

$$\left.\begin{aligned} A_n\mathscr{N}_j^{i+1} &= -\chi_n(D_x^- n_j^i)\frac{(D_x^-\mathscr{E}_j^i)}{\sqrt{1+\left(\frac{D_x^-\mathscr{E}_j^i}{\lambda_E}\right)^2}}\\ &\quad -\chi_n\mathscr{N}_j^i\frac{D_x^+(D_x^-\mathscr{E}_j^i)}{\left(1+\left(\frac{D_x^-\mathscr{E}_j^i}{\lambda_E}\right)^2\right)^{3/2}}\\ &\quad -\chi_n^1 I_s\mathscr{N}_j^i\frac{D_x^+(D_x^-\mathscr{R}_j^i)}{\left(1+\left(\frac{D_x^-\mathscr{R}_j^i}{\lambda_\rho}\right)^2\right)^{3/2}}\\ &\quad +a_{11}\frac{(\mathscr{E}^4)_j^i}{k_E^4+(\mathscr{E}^4)_j^i}\mathscr{N}_j^i(1-\frac{\mathscr{N}_j^i}{n_*-a_{12}\rho I_s}) + \frac{\mathscr{N}_j^i}{\Delta t}, \end{aligned}\right\}$$

$$\left.\begin{aligned} A_f\mathscr{F}_j^{i+1} &= -a_{21}G^0(t_i-\tau,x_j)f^0(t_i-\tau,x_j)\\ &\quad +a_{22}f^0(t_i-\tau,x_j) + \frac{\mathscr{F}_j^i}{\psi_f},\\ A_m\mathscr{M}_j^{i+1} &= -\chi_m(D_x^-\mathscr{M}_j^i)\frac{(D_x^-\mathscr{G}_j^i)}{\sqrt{1+\left(\frac{D_x^-\mathscr{G}_j^i}{\lambda_G}\right)^2}}\\ &\quad -\chi_m\mathscr{M}_j^i\frac{D_x^+(D_x^-\mathscr{G}_j^i)}{\left(1+\left(\frac{D_x^-\mathscr{G}_j^i}{\lambda_G}\right)^2\right)^{3/2}}\\ &\quad +a_{21}G^0(t_i-\tau,x_j)f^0(t_i-\tau,x_j)\\ &\quad +a_{31}\mathscr{M}_j^i + \frac{\mathscr{M}_j^i}{\Delta t}, \end{aligned}\right\}$$

$$\left.\begin{aligned} A_\rho\mathscr{R}_j^{i+1} &= -a_{41}P^0(t_i-\tau,x_j)n^0(t_i-\tau,x_j)\\ &\quad +(a_{42}\mathscr{F}_j^i + a_{43}\mathscr{M}_j^i)(1-\frac{\mathscr{R}}{\rho_*}) + \frac{\mathscr{R}_j^i}{(\Delta t)},\\ A_E\mathscr{E}_j^{i+1} &= I_{\Omega_-}(a_{51}f^0(t_i-\tau,x_j)\\ &\quad +a_{52}m^0(t_i-\tau,x_j)) - a_{62}\mathscr{E}_j^i + \frac{\mathscr{E}_j^i}{\psi_E},\\ A_G\mathscr{G}_j^{i+1} &= a_{61}I_{\Omega_+}n^0(t_i-\tau,x_j) - a_{62}\mathscr{G}_j^i + \frac{\mathscr{G}_j^i}{\psi_G},\\ A_P\mathscr{P}_j^{i+1} &= a_{71}I_{\Omega_-}m^0(t_i-\tau,x_j) - a_{72}\mathscr{P}_j^i + \frac{\mathscr{P}_j^i}{\psi_P}. \end{aligned}\right\}$$

Let $s$ denotes the largest integer such that $\tau_s \le \tau$. Then using the system in equation (3), one can compute $\mathscr{N}_j^i, \mathscr{F}_j^i, \mathscr{M}_j^i, \mathscr{R}_j^i, \mathscr{E}_j^i, \mathscr{G}^i j\mathscr{P}_j^i$, for $1 \le i \le s$. Up to this stage, one interpolates the data

$$\begin{aligned} &(t_0,\mathscr{N}_j^0),\ (t_1,\mathscr{N}_j^1),\ldots,(t_s,\mathscr{N}_j^s),\ (t_0,\mathscr{F}_j^0),\ (t_1,\mathscr{F}_j^1),\ldots,(t_s,\mathscr{F}_j^s),\\ &(t_0,\mathscr{M}_j^0),\ (t_1,\mathscr{M}_j^1),\ldots,(t_s,\mathscr{M}_j^s),\ (t_0,\mathscr{R}_j^0),\ (t_1,\mathscr{R}_j^1),\ldots,(t_s,\mathscr{R}_j^s),\\ &(t_0,\mathscr{E}_j^0),\ (t_1,\mathscr{E}_j^1),\ldots,(t_s,\mathscr{E}_j^s),\ (t_0,\mathscr{G}_j^0),\ (t_1,\mathscr{G}_j^1),\ldots,(t_s,\mathscr{G}_j^s),\\ &(t_0,\mathscr{P}_j^0),\ (t_1,\mathscr{P}_j^1),\ldots,(t_s,\mathscr{P}_j^s), \end{aligned}$$

using an interpolating cubic Hermite spline $\varphi_j(t)$ ([24]). Then

$$\begin{aligned} \mathscr{N}_j^i &= \varphi_n(t_i,x_j),\ \mathscr{F}_j^i = \varphi_f(t_i,x_j),\ \mathscr{M}_j^i\\ &= \varphi_m(t_i,x_j),\ \mathscr{R}_j^i = \varphi_\rho(t_i,x_j),\ \mathscr{E}_j^i = \varphi_E(t_i,x_j),\\ \mathscr{G}_j^i &= \varphi_G(t_i,x_j),\ \mathscr{P}_j^i = \varphi_P(t_i,x_j), \end{aligned}$$

for all $i = 0,1,\ldots,s$ and $j = -L/2, 2, \ldots, L/2-1$.

For $i = s+1, s+2, \ldots, T-1$, when we move from level $i$ to level $i+1$ we extend the definitions of the cubic Hermite spline $\varphi_j(t)$ to the point

$$\begin{aligned} &(t_i+\Delta t, (\mathscr{H}_n)_j^i, t_i+\Delta t, (\mathscr{H}_f)_j^i, t_i+\Delta t, (\mathscr{H}_m)_j^i, t_i\\ &\quad +\Delta t, (\mathscr{H}_G)_j^i), t_i+\Delta t, (\mathscr{H}_P)_j^i). \end{aligned}$$

Then, the history terms $(\mathscr{H}_n)^i_j, (\mathscr{H}_f)^i_j, (\mathscr{H}_m)^i_j, (\mathscr{H}_G)^i_j, (\mathscr{H}_P)^i_j$ can be approximated by the functions $(\varphi_n)_j(t_i-\tau), (\varphi_f)_j(t_i-\tau), (\varphi_m)_j(t_i-\tau), (\varphi_G)_j(t_i-\tau), (\varphi_P)_j(t_i-\tau)$ for $i \geq s$. This implies that,

$$
\begin{aligned}
&(\mathscr{H}_n)^i_j \approx (\varphi_n)_j(t_i-\tau),\ (\mathscr{H}_f)^i_j \approx (\varphi_f)_j(t_i-\tau),\\
&(\mathscr{H}_m)^i_j \approx (\varphi_m)_j(t_i-\tau),\\
&(\mathscr{H}_G)^i_j \approx (\varphi_G)_j(t_i-\tau),\\
&(\mathscr{H}_P)^i_j \approx (\varphi_P)_j(t_i-\tau), \qquad (3.13)
\end{aligned}
$$

and equation (3) becomes

$$
\left.
\begin{aligned}
A_n\mathscr{N}^{i+1}_j &= -\chi_n(D^-_x n^i_j)\frac{(D^-_x\mathscr{E}^i_j)}{\sqrt{1+\left(\frac{D^-_x\mathscr{E}^i_j}{\lambda_E}\right)^2}}\\
&-\chi_n\mathscr{N}^i_j\frac{D^+_x(D^-_x\mathscr{E}^i_j)}{\left(1+\left(\frac{D^-_x\mathscr{E}^i_j}{\lambda_E}\right)^2\right)^{3/2}}\\
&-\chi^1_n I_s\mathscr{N}^i_j\frac{D^+_x(D^-_x\mathscr{R}^i_j)}{\left(1+\left(\frac{D^-_x\mathscr{R}^i_j}{\lambda_\rho}\right)^2\right)^{3/2}}\\
&+a_{11}\frac{(\mathscr{E}^4)^i_j}{k^4_E+(\mathscr{E}^4)^i_j}\mathscr{N}^i_j\left(1-\frac{\mathscr{N}^i_j}{n_*-a_{12}\rho I_s}\right)+\frac{\mathscr{N}^i_j}{\Delta t},\\
A_f\mathscr{F}^{i+1}_j &= -a_{21}(\varphi_G)_j(t_i-\tau)(\varphi_f)_j(t_i-\tau)\\
&+a_{22}(\varphi_f)_j(t_i-\tau)+\frac{\mathscr{F}^i_j}{\psi_f},\\
A_m\mathscr{M}^{i+1}_j &= -\chi_m(D^-_x\mathscr{M}^i_j)\frac{(D^-_x\mathscr{G}^i_j)}{\sqrt{1+\left(\frac{D^-_x\mathscr{G}^i_j}{\lambda_G}\right)^2}}\\
&-\chi_m\mathscr{M}^i_j\frac{D^+_x(D^-_x\mathscr{G}^i_j)}{\left(1+\left(\frac{D^-_x\mathscr{G}^i_j}{\lambda_G}\right)^2\right)^{3/2}}\\
&+a_{21}(\varphi_G)_j(t_i-\tau)(\varphi_f)_j(t_i-\tau)\\
&+a_{31}\mathscr{M}^i_j+\frac{\mathscr{M}^i_j}{\Delta t},\\
A_\rho\mathscr{R}^{i+1}_j &= -a_{41}(\varphi_P)_j(t_i-\tau)(\varphi_n)_j(t_i-\tau)\\
&+(a_{42}\mathscr{F}^i_j+a_{43}\mathscr{M}^i_j)\left(1-\frac{\mathscr{R}}{\rho_*}\right)+\frac{\mathscr{R}^i_j}{(\Delta t)},\\
A_E\mathscr{E}^{i+1}_j &= I_{\Omega_-}(a_{51}(\varphi_f)_j(t_i-\tau)\\
&+a_{52}(\varphi_m)_j(t_i-\tau))-a_{62}\mathscr{E}^i_j+\frac{\mathscr{E}^i_j}{\psi_E},\\
A_G\mathscr{G}^{i+1}_j &= a_{61}I_{\Omega_+}(\varphi_n)_j(t_i-\tau)-a_{62}\mathscr{G}^i_j+\frac{\mathscr{G}^i_j}{\psi_G},\\
A_P\mathscr{P}^{i+1}_j &= a_{71}I_{\Omega_-}(\varphi_m)_j(t_i-\tau)-a_{72}\mathscr{P}^i_j+\frac{\mathscr{P}^i_j}{\psi_P},
\end{aligned}
\right\}
$$

where,

$$
\begin{aligned}
\varphi_n(t_i-\tau) &= [(\mathscr{H}_n)^i_1,(\mathscr{H}_n)^i_2\ldots,(\mathscr{H}_n)^i_{\frac{L}{2}-1}]',\ \varphi_f(t_i-\tau)\\
&= [(\mathscr{H}_f)^i_{\frac{-L}{2}},(\mathscr{H}_f)^i_{\frac{-L}{2}+1}\ldots,(\mathscr{H}_f)^i_{x_0-1}]',\\
\varphi_m(t_i-\tau) &= [(\mathscr{H}_m)^i_{\frac{-L}{2}},(\mathscr{H}_m)^i_{\frac{-L}{2}+1}\ldots,(\mathscr{H}_m)^i_{x_0-1}]',\\
\varphi_G(t_i-\tau) &= [\mathscr{G}^i_{\frac{-L}{2}},\mathscr{G}^i_{\frac{-L}{2}+1}\ldots,\mathscr{G}^i_{\frac{L}{2}-1}]',\\
\varphi_P(t_i-\tau) &= [(\mathscr{H}_P)^i_{\frac{-L}{2}},(\mathscr{H}_P)^i_{\frac{-L}{2}+1}\ldots,(\mathscr{H}_P)^i_{\frac{L}{2}-1}]'.
\end{aligned}
$$

The FOFDM can be rewriting as a system of equations

$$
\left.
\begin{aligned}
&A_n\mathscr{N}=F_n, A_f\mathscr{F}=F_f, A_m\mathscr{M}=F_m,\\
&A_\rho\mathscr{R}=F_\rho, A_E\mathscr{E}=F_E, A_G\mathscr{G}=F_G,\\
&A_P\mathscr{P}=F_P.
\end{aligned}
\right\}
$$

Let the functions

$$n(x,t), f(x,t), m(x,t), E(x,t), G(x,t), P(x,t),$$

and their partial derivatives with respect to both $t$ and $x$ be smooth such that they satisfy

$$
\begin{aligned}
&\left|\frac{\partial^{i+j}n(t,x)}{\partial t^i x^j}\right| \leq \Upsilon_n, \left|\frac{\partial^{i+j}f(t,x)}{\partial t^i x^j}\right| \leq \Upsilon_f,\\
&\left|\frac{\partial^{i+j}m(t,x)}{\partial t^i x^j}\right| \leq \Upsilon_m, \left|\frac{\partial^{i+j}E(t,x)}{\partial t^i x^j}\right| \leq \Upsilon_E,\\
&\left|\frac{\partial^{i+j}G(t,x)}{\partial t^i x^j}\right| \leq \Upsilon_G, \left|\frac{\partial^{i+j}P(t,x)}{\partial t^i x^j}\right| \leq \Upsilon_P, \qquad (3.14)\\
&\forall i,j \geq 0, \qquad (3.15)
\end{aligned}
$$

where,

$$\Upsilon_n, \Upsilon_f, \Upsilon_m, , \Upsilon_E, \Upsilon_G, \Upsilon_P,$$

are constant that are independent of the time and space step-sizes. Then in view of the FOFDM one can see that the truncation errors $\varsigma_n, \varsigma_f, \varsigma_m, \varsigma_\rho, \varsigma_E, \varsigma_G, \varsigma_P$, are given by

$$
\left.
\begin{aligned}
(\varsigma_n)^i_j &= (A_n n)^i_j-(F_n)^i_j = (A_n(n-\mathscr{N}))^i_j,\\
(\varsigma_f)^i_j &= (A_f f)^i_j-(F_f)^i_j = (A_f(f-\mathscr{F}))^i_j,\\
(\varsigma_m)^i_j &= (A_m m)^i_j-(F_m)^i_j = (A_m(m-\mathscr{M}))^i_j,\\
(\varsigma_\rho)^i_j &= (A_\rho\rho)^i_j-(F_\rho)^i_j = (A_\rho(\rho-\mathscr{R}))^i_j,\\
(\varsigma_E)^i_j &= (A_E E)^i_j-(F_E)^i_j = (A_E(E-\mathscr{E}))^i_j,\\
(\varsigma_G)^i_j &= (A_G Gn)^i_j-(F_G)^i_j = (A_G(G-\mathscr{G}))^i_j,\\
(\varsigma_P)^i_j &= (A_P P)^i_j-(F_P)^i_j = (A_P(P-\mathscr{P}))^i_j.
\end{aligned}
\right\}
$$

Therefore,

$$
\left.
\begin{aligned}
&\max_{i,j}|n^i_j-\mathscr{N}^i_j| \leq ||A^{-1}_n||\max_{i,j}|(\varsigma_n)^i_j|,\\
&\max_{i,j}|f^i_j-\mathscr{F}^i_j| \leq ||A^{-1}_f||\max_{i,j}|(\varsigma_f)^i_j|,\\
&\max_{i,j}|m^i_j-\mathscr{M}^i_j| \leq ||A^{-1}_m||\max_{i,j}|(\varsigma_m)^i_j|,\\
&\max_{i,j}|\rho^i_j-\mathscr{R}^i_j| \leq ||A^{-1}_\rho||\max_{i,j}|(\varsigma_\rho)^i_j|,\\
&\max_{i,j}|E^i_j-\mathscr{E}^i_j| \leq ||A^{-1}_E||\max_{i,j}|(\varsigma_E)^i_j|,\\
&\max_{i,j}|G^i_j-\mathscr{G}^i_j| \leq ||A^{-1}_G||\max_{i,j}|(\varsigma_G)^i_j|,\\
&\max_{i,j}|P^i_j-\mathscr{P}^i_j| \leq ||A^{-1}_P||\max_{i,j}|(\varsigma_P)^i_j|,
\end{aligned}
\right\}
$$

where,

$$
\left.
\begin{aligned}
&(\varsigma_n)^i_j \leq \frac{(\Delta t)}{2}|n_{tt}(\xi)|-D_n\frac{(\Delta x)^2}{12}|n_{xxxx}(\zeta)|, x\in[x_s,L/2],\\
&(\varsigma_f)^i_j \leq \frac{(\Delta t)}{2}|f_{tt}(\xi)|-D_f\frac{(\Delta x)^2}{12}|f_{xxxx}(\zeta)|, x\in[-\tfrac{L}{2},x_s],\\
&(\varsigma_m)^i_j \leq \frac{(\Delta t)}{2}|m_{tt}(\xi)|-D_m\frac{(\Delta x)^2}{12}|m_{xxxx}(\zeta)|, x\in[-\tfrac{L}{2},x_s],\\
&(\varsigma_\rho)^i_j \leq \frac{(\Delta t)}{2}|\rho_{tt}(\xi)|-D_\rho\frac{(\Delta x)^2}{12}|\rho_{xxxx}(\zeta)|, x\in[-\tfrac{L-x^*}{2},\tfrac{L-x^*}{2}],\\
&(\varsigma_E)^i_j \leq \frac{(\Delta t)}{2}|E_{tt}(\xi)|-D_E\frac{(\Delta x)^2}{12}|E_{xxxx}(\zeta)|, x\in[-\tfrac{L}{2},\tfrac{L}{2}],\\
&(\varsigma_G)^i_j \leq \frac{(\Delta t)}{2}|G_{tt}(\xi)|-D_G\frac{(\Delta x)^2}{12}|G_{xxxx}(\zeta)|, x\in[-\tfrac{L}{2},\tfrac{L}{2}],\\
&(\varsigma_P)^i_j \leq \frac{(\Delta t)}{2}|P_{tt}(\xi)|-D_P\frac{(\Delta x)^2}{12}|P_{xxxx}(\zeta)|, x\in[-\tfrac{L}{2},\tfrac{L}{2}],
\end{aligned}
\right\}
$$

for $t_{i-1} \le \xi \le t_{i+1}$ and $x_{j-1} \le \zeta \le x_{j+1}$. In view of inequalities in (3.14) we see that the inequalities in (3) is equivalent to

$$\left.\begin{array}{l}(\varsigma_n)^i_j \le \left(\frac{(\Delta t)}{2} - D_n\frac{(\Delta x)^2}{12}\right)\Upsilon_n,\ x \in [x_s, L/2],\\ (\varsigma_f)^i_j \le \left(\frac{(\Delta t)}{2} - D_f\frac{(\Delta x)^2}{12}\right)\Upsilon_f,\ x \in [-\frac{L}{2}, x_s],\\ (\varsigma_m)^i_j \le \left(\frac{(\Delta t)}{2} - D_m\frac{(\Delta x)^2}{12}\right)\Upsilon_m,\ x \in [-\frac{L}{2}, x_s],\\ (\varsigma_\rho)^i_j \le \left(\frac{(\Delta t)}{2} - D_\rho\frac{(\Delta x)^2}{12}\right)\Upsilon_\rho,\ x \in [-\frac{L-x^*}{2}, \frac{L-x^*}{2}],\\ (\varsigma_E)^i_j \le \left(\frac{(\Delta t)}{2} - D_E\frac{(\Delta x)^2}{12}\right)\Upsilon_E,\ x \in [-\frac{L}{2}, \frac{L}{2}],\\ (\varsigma_G)^i_j \le \left(\frac{(\Delta t)}{2} - D_G\frac{(\Delta x)^2}{12}\right)\Upsilon_G, x \in [-\frac{L}{2}, \frac{L}{2}],\\ (\varsigma_P)^i_j \le \left(\frac{(\Delta t)}{2} - D_P\frac{(\Delta x)^2}{12}\right)\Upsilon_P, x \in [-\frac{L}{2}, \frac{L}{2}],\end{array}\right\}$$

for $t_{i-1} \le \xi \le t_{i+1}$ and $x_{j-1} \le \zeta \le x_{j+1}$. Moreover, by a result in [25], we have

$$\begin{array}{l}||A_n^{-1}|| \le \Xi_n, ||A_f^{-1}|| \le \Xi_f,\\ ||A_m^{-1}|| \le \Xi_m, ||A_\rho^{-1}|| \le \Xi_\rho,\\ ||A_E^{-1}|| \le \Xi_E,\\ ||A_G^{-1}|| \le \Xi_G, ||A_P^{-1}|| \le \Xi_P.\end{array} \quad (3.16)$$

Using (3) and (3.16) in (3), we obtain the following results.

**Theorem 31** *Let*

$$F_n(x,t), F_f(x,t), F_m(x,t), F_\rho(x,t), F_E(x,t), F_G(x,t), F_P(x,t),$$

*be sufficiently smooth functions so that* $n(x,t), f(x,t), m(x,t), \rho(x,t), E(x,t), G(x,t), P(x,t) \in C^\infty([-L,L] \times [0,T])$. *Let* $(\mathcal{N}^i_j, \mathcal{F}^i_j, \mathcal{M}^i_j, \mathcal{R}^i_j, \mathcal{E}^i_j, \mathcal{G}^i_j, \mathcal{P}^i_j,)$, $j = 1,2,\ldots L, i = 1,2,\ldots T$ *be the approximate solutions obtained using the FOFDM with* $\mathcal{N}^0_j = n^0_j, \mathcal{F}^0_j = f^0_j, \mathcal{M}^0_j = m^0_j, \mathcal{R}^0_j = \rho^0_j, \mathcal{E}^0_j = E^0_j, \mathcal{G}^0_j = G^0_j, \mathcal{P}^0_j = P^0_j$,. *Then there exists* $\Xi_n, \Xi_f, \Xi_m, \Xi_\rho, \Xi_E, \Xi_G, \Xi_P$ *independent of the step sizes* $\Delta t$ *and* $\Delta x$ *such that*

$$\left.\begin{array}{l}\max_{i,j}|n^i_j - \mathcal{N}^i_j| \le \Xi_n[\frac{(\Delta t)}{2} - D_n\frac{(\Delta x)^2}{12}]\Upsilon_n,\\ \max_{i,j}|f^i_j - \mathcal{F}^i_j| \le \Xi_f[\frac{(\Delta t)}{2} - D_f\frac{(\Delta x)^2}{12}(\zeta)]\Upsilon_f,\\ \max_{i,j}|m^i_j - \mathcal{M}^i_j| \le \Xi_m[\frac{(\Delta t)}{2} - D_m\frac{(\Delta x)^2}{12}]\Upsilon_m,\\ \max_{i,j}|\rho^i_j - \mathcal{R}^i_j| \le \Xi_\rho[\frac{(\Delta t)}{2} - D_\rho\frac{(\Delta x)^2}{12}]\Upsilon_\rho,\\ \max_{i,j}|E^i_j - \mathcal{E}^i_j| \le \Xi_E[\frac{(\Delta t)}{2} - D_E\frac{(\Delta x)^2}{12}]\Upsilon_E,\\ \max_{i,j}|G^i_j - \mathcal{G}^i_j| \le \Xi_G[\frac{(\Delta t)}{2} - D_G\frac{(\Delta x)^2}{12}]\Upsilon_G,\\ \max_{i,j}|P^i_j - \mathcal{P}^i_j| \le \Xi_P[\frac{(\Delta t)}{2} - D_P\frac{(\Delta x)^2}{12}]\Upsilon_P.\end{array}\right\} \quad (3.17)$$

## 4 Numerical results and discussions

Setting $S_x = S_t = 50$, times $t = 5$ and $t = 20$ and using the parameter values in Table 2, for $L = 5$ and $T = 1$, the numerical results without a delay term for the dynamics in equation (1.2) are presented in Figures 1-4,5-7,8-11,12-14 whereas in Figures 15- 18,22-25,26-28 we present the numerical results with a delay term. Thus, for the dynamics with a delay term, the results are presented for times $t = 5; 20$ and delay term $\tau = 5; 15$ in Figures 15,19 and Figures 22, 26, respectively. In Figures 1 and 5, the density of the transformed epithelia cells are steadily rising to their steady state within their compartment. Similar phenomena is also observed on the behaviour of the density of fibroblasts, whereas, for the density of the myfobroblasts, a slight growth of the density of the cells is noted, which suddenly increases near the end of their prescribed compartment. This is due to the transformation of the fibroblasts cells into myfibroblasts cells. For the concentration of the extracellular matrix, a very small growth of the concentration which is being degraded by the density of the transformed epithelia cells and its secretion is notable, whereas the behaviour of fibroblasts give rise to the behaviour of the density of the concentration of epidermal growth factor, which they secretes. The density of the transformed fibroblasts cells are influenced by the behaviour of the concentration of the matrix metalloproteinase to certain extend, which once more enhanced by the concentration of the epidermal growth factor molecules. Another interesting phenomena is seen on the behaviour of the concentration of the transformed growth factor molecules, which is attributed by the density of the transformed epithelial cells. These phenomena are exactly the same as in Figures 15 and 19.

Figures 8 and 12 present similar results as we see in Figures 1 and 5, except that the behaviour of the concentration of matrix metalloproteinase has increased quite a great deal. Interestingly, the sinusoidal behaviour for the density of transformed epithelial cells at an initial stage is notable, just before the density rises to its steady state. The effects of the delay term in the behaviour of the concentration of the transformed growth factor is notable, whereas the degradation of the extracellular matrix, behaviours of the concentration of epidermal growth factor and matrix metalloproteinase are presented in a manner, which one can deduce a relationship between the concentration of epidermal growth factor with that of the concentration of matrix metalloproteinase.

**Table 2:** Parameter values used for the invasion essay model [1]

| | |
|---|---|
| $D_n = 3.6 \times 10^{-4}$ | $D_f = 6.12 \times 10^{-5}$ |
| $D_e = 5.98 \times 10^{-1}$ | $D_g = 3.6 \times 10^{-1}$ |
| $\lambda_E = \lambda_G = \lambda_\rho\rho_\star = a_{12} = 1.00$ | $a_{11} = 0.69$ |
| $k_E = 3.32$ | $\kappa = 2.88 \times 10^3$ |
| $a_{62} = 2.89 \times 10^{-2}$ | $a_{71} = 3732$ |
| $a_{43} = 0.518$ | $n_\star = 2.88 \times 10^3$ |
| $a_{31} = 4.53 \times 10^{-3}$ | $r_f = 100.0$ |
| $\chi_m = 3.96 \times 10^{-6}$ | $a_{52} = 2.89 \times 10^{-2}$ |
| | |
| $D_m = 6.12 \times 10^{-4}$ | $D_\rho = 5.12 \times 10^{-4}$ |
| $D_p = 3.6 \times 10^{-1}$ | $\chi_n = 3.6 \times 10^{-8}$ |
| $a_{22} = 1.58 \times 10^{-2}$ | $a_{51} = 2.03 \times 10^{-1}$ |
| $B = 5.00$ | $a_{53} = 2.89 \times 10^{-2}$ |
| $a_{72} = 0.259$ | $a_{41} = 3732$ |
| $\varepsilon = 0.1$ | $a_{21} = 2.61 \times 10^{-2}$ |
| $\gamma = 0.1$ | $\chi_n^1 = 1.8 \times 10^{-4}$ |
| $a_{61} = 2.03 \times 10^{-1}$ | $a_{42} = 0.259$ |

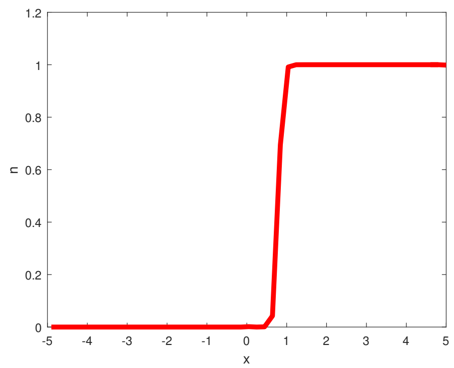


**Fig. 1:** Numerical solution of the system in (1.2) without delay at time (t) = 5: (a) Behaviour of Transformed Epithelial cells (TECs)

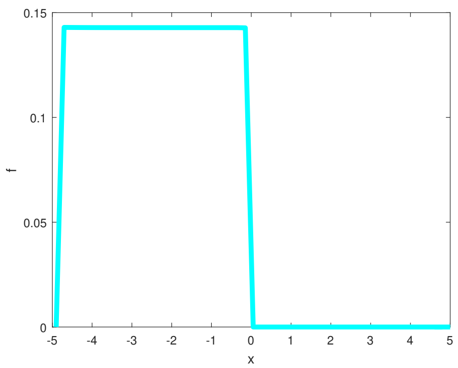


**Fig. 2:** Numerical solution of the system in (1.2) without delay at time (t) = 5: (b)Behaviour of Fibroblasts cells

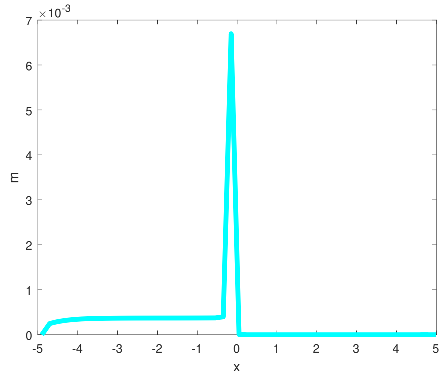


**Fig. 3:** Numerical solution of the system in (1.2) without delay at time (t) = 5: (c) Behaviour of Myfibroblasts cells

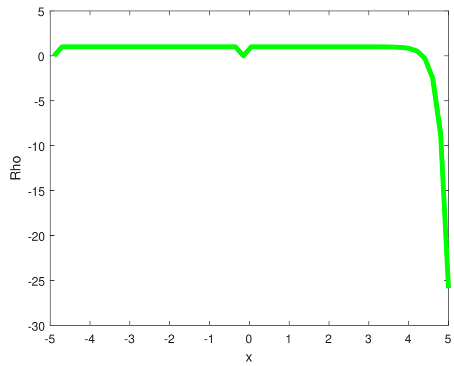


**Fig. 4:** Numerical solution of the system in (1.2) without delay at time (t) = 5: (d) Behaviour of the concentration of Extracellular Matrix (ECM)

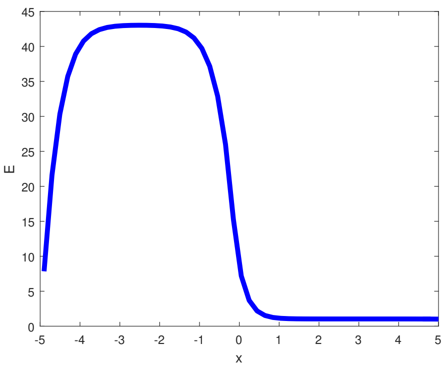


**Fig. 5:** Numerical solution of the system in (1.2) without delay at time (t) = 5:(a)Behaviour of the concentration of Epidermal Growth Factor molecules (EGF)

## 5 Conclusion

In this paper, we extended the model modeling the interaction between transformed epithelial cells (TECs), fibroblasts, myofibroblasts, transformed growth factor (TGF$-\beta$), and epithelial growth factor (EGF), in silico, in a setup which mimics experiments in a tumor chamber invasion assay, where a semi-permeable membrane, (which allows EGF, TGF$-\beta$ and Matrix Metalloproteinase (MMP) to cross it) coated by extra-cellular matrix (ECM) is placed between two chambers, one containing TECs and another containing fibroblasts and myofibroblasts. Our focus was to incorporate some of the crucial transformations ought to take take place during the interaction of the experiment proposed in [1]. The incorporation of some transformations, led the original model to be transformed to a system of non-linear delay parabolic partial differential equations. The establishment for existence of

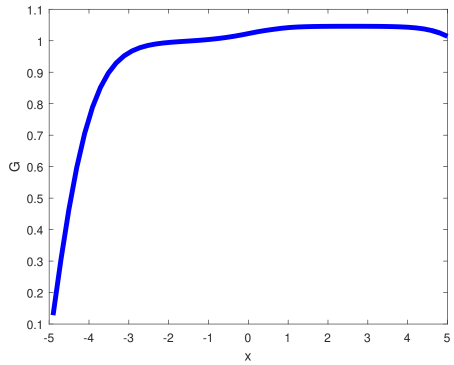


**Fig. 6:** Numerical solution of the system in (1.2) without delay at time (t) = 5:(b)Behaviour of the concentration of Transformed Growth Factor molecules (TGF-$\beta$)

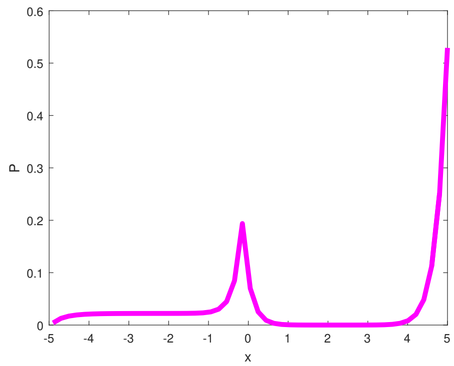


**Fig. 7:** Numerical solution of the system in (1.2) without delay at time (t) = 5: (c) Behaviour of the concentration of Matrix MetalloProteinase (MMP)

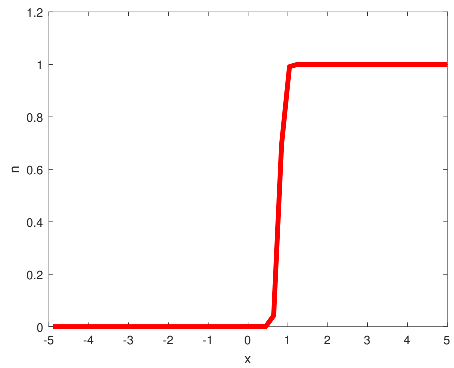


**Fig. 8:** Numerical solution of the system in (1.2) without delay at time (t) = 20: (a) Behaviour of Transformed Epithelial cells (TECs)

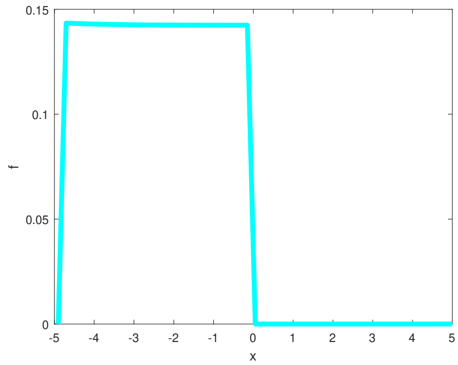


**Fig. 9:** Numerical solution of the system in (1.2) without delay at time (t) = 20: (b)Behaviour of Fibroblasts cells

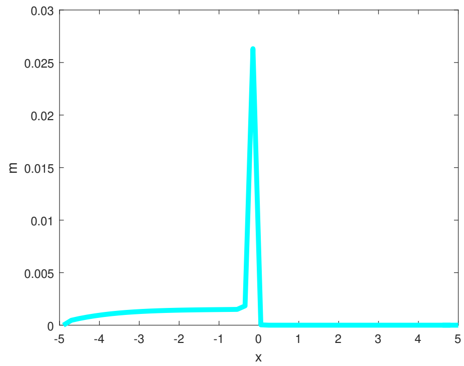


**Fig. 10:** Numerical solution of the system in (1.2) without delay at time (t) = 20: (c) Behaviour of Myfibroblasts cells

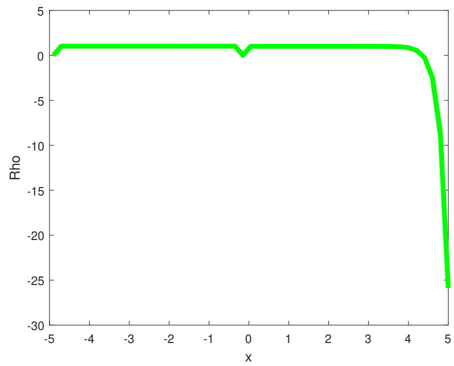


**Fig. 11:** Numerical solution of the system in (1.2) without delay at time (t) = 20:(d) Behaviour of the concentration of Extracellular Matrix (ECM)

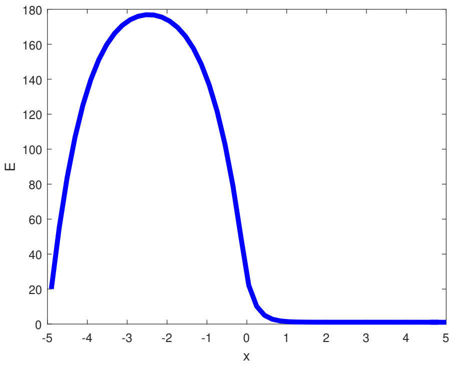


**Fig. 12:** Numerical solution of the system in (1.2) without delay at time (t) = 20: (a) Behaviour of the concentration of Epidermal Growth Factor molecules (EGF)

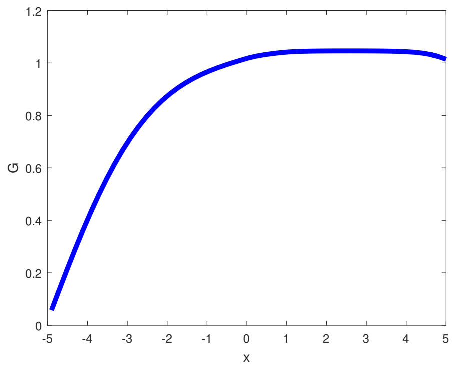


**Fig. 13:** Numerical solution of the system in (1.2) without delay at time (t) = 20: (b) Behaviour of the concentration of Transformed Growth Factor molecules (TGF-$\beta$)

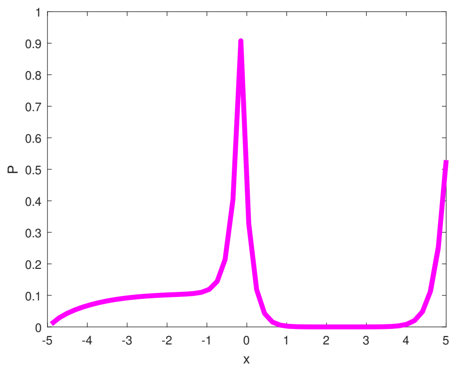


**Fig. 14:** Numerical solution of the system in (1.2) without delay at time (t) = 20: (c) Behaviour of the concentration of Matrix MetalloProteinase (MMP)

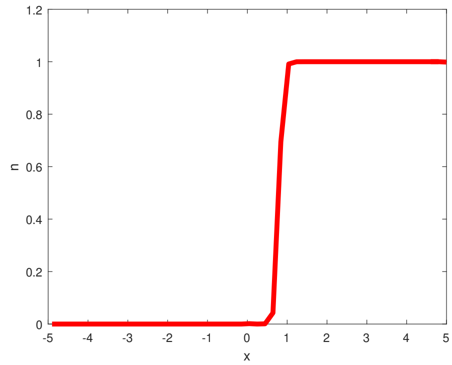


**Fig. 15:** Numerical solution of the system in (1.2) with delay $\tau = 5$ and at time (t) = 5:(a) Behaviour of Transformed Epithelial cells (TECs)

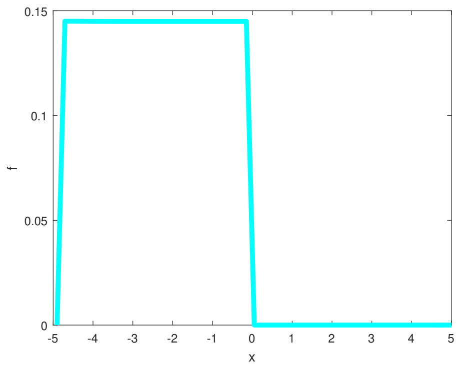


**Fig. 16:** Numerical solution of the system in (1.2) with delay $\tau = 5$ and at time (t) = 5: (b)Behaviour of Fibroblasts cells

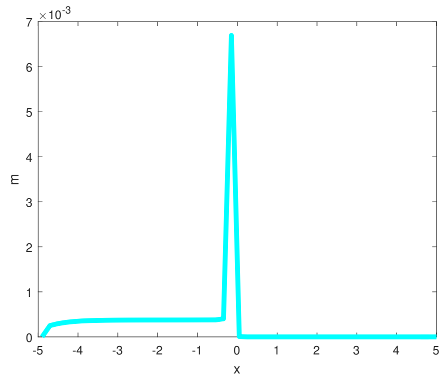


**Fig. 17:** Numerical solution of the system in (1.2) with delay $\tau = 5$ and at time (t) = 5: (c) Behaviour of Myfibroblasts cells

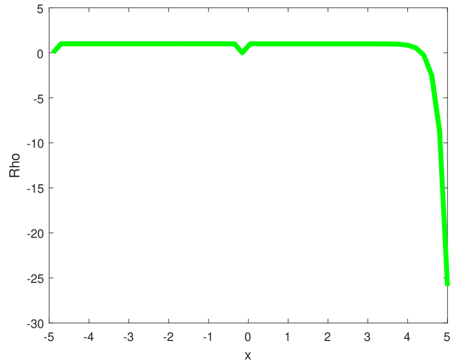


**Fig. 18:** Numerical solution of the system in (1.2) with delay $\tau = 5$ and at time (t) = 5:(d)Behaviour of the concentration of Extracellular Matrix (ECM)

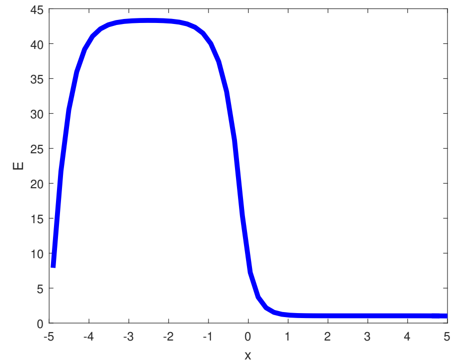


**Fig. 19:** Numerical solution of the system in (1.2) with delay $\tau = 5$ and at time (t) = 5: (a) Behaviour of the concentration of Epidermal Growth Factor molecules (EGF)

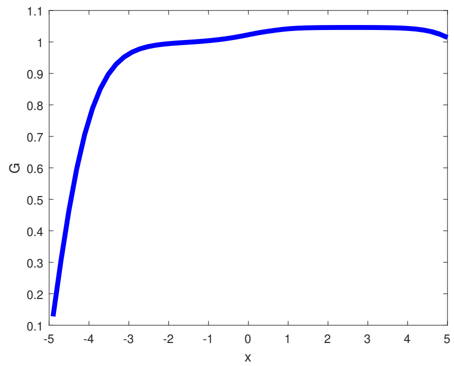


**Fig. 20:** Numerical solution of the system in (1.2) with delay $\tau = 5$ and at time (t) = 5: (b) Behaviour of the concentration of Transformed Growth Factor molecules (TGF-$\beta$)

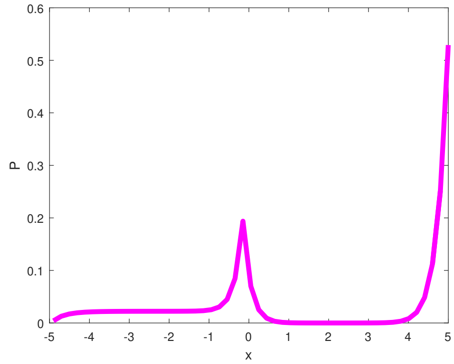


**Fig. 21:** Numerical solution of the system in (1.2) with delay $\tau = 5$ and at time (t) = 5: (c) Behaviour of the concentration of Matrix MetalloProteinase (MMP)

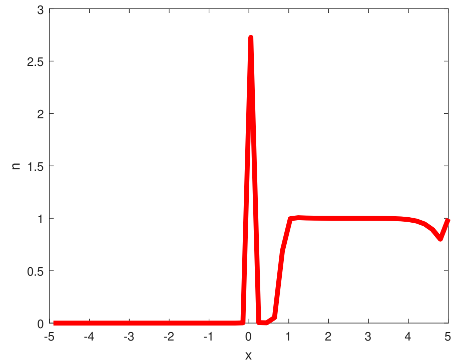


**Fig. 22:** Numerical solution of the system in (1.2) with delay $\tau = 15$ and at time (t) = 20:(a)Behaviour of Transformed Epithelial cells (TECs)

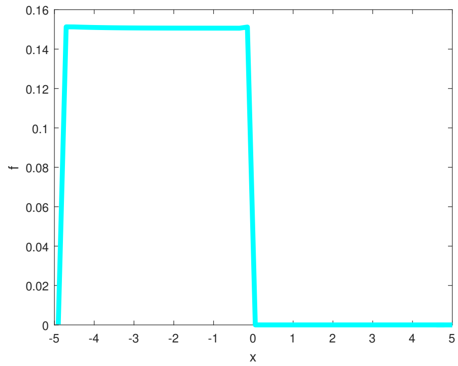


**Fig. 23:** Numerical solution of the system in (1.2) with delay $\tau = 15$ and at time (t) = 20: (b) Behaviour of Fibroblasts cells

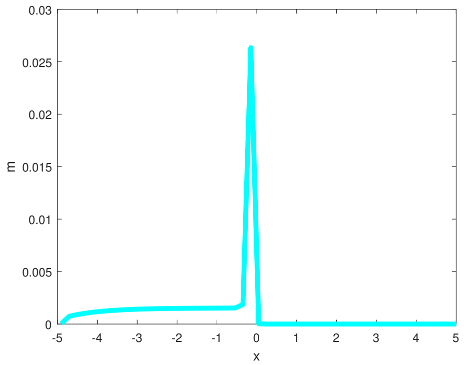


**Fig. 24:** Numerical solution of the system in (1.2) with delay $\tau = 15$ and at time (t) = 20: (c)Behaviour of Myfibroblasts cells

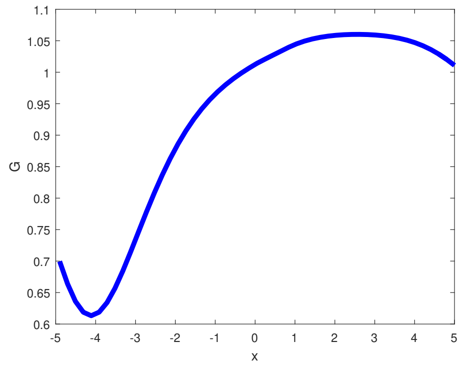


**Fig. 27:** Numerical solution of the system in (1.2) with delay $\tau = 15$ and at time (t) = 20: (b)Behaviour of the concentration of Transformed Growth Factor molecules (TGF-$\beta$)

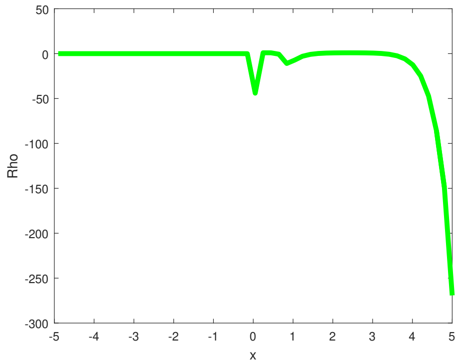


**Fig. 25:** Numerical solution of the system in (1.2) with delay $\tau = 15$ and at time (t) = 20: (d) Behaviour of the concentration of Extracellular Matrix (ECM)

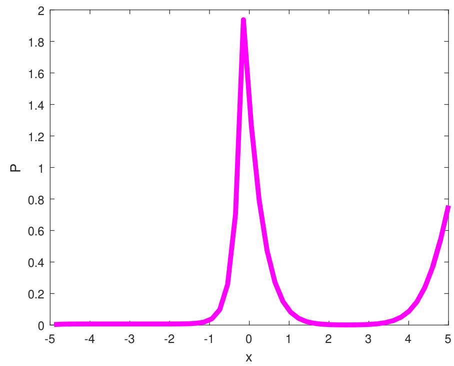


**Fig. 28:** Numerical solution of the system in (1.2) with delay $\tau = 15$ and at time (t) = 20: (c) Behaviour of the concentration of Matrix MetalloProteinase (MMP)

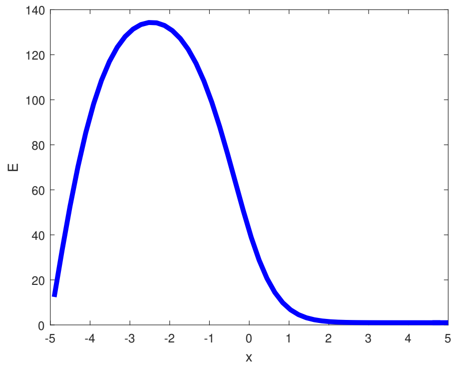


**Fig. 26:** Numerical solution of the system in (1.2) with delay $\tau = 15$ and at time (t) = 20: (a)Behaviour of the concentration of Epidermal Growth Factor molecules (EGF)

uniqueness of solution led us to the extension of Gronwall's inequality for linear delay differential equations. We have also reported on the analysis for the resulting system of non-linear delay parabolic partial differential equations, established the global asymptotically for the equilibrium point. Consequently, we were able to derive the a fitted operator finite difference method (FOFDM) for solving the modified system in equation (1.2). Our main findings are more vivid that the delay factor can be observed after some time of the delay term and enable us to see the sensitivity of the density of transformed epithelial cells. Thus, our finding are indeed essential for the design of the drug which can slow and/or confine tumor invasion, particularly when the analysis present that the Hopf bifurcation affects the entire experiment through the density of fibrolasts. Our future research is to extend our results to the higher dimensional space and make use of the recent developments reported in [26,27,28, 29,30,31].

## Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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