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Solvability for a Differential System of Duffing Type Via Caputo-Hadamard Approach

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Abstract: In this work, we investigate a new sequential coupled differential system of Duffing type. The considered system involves Caputo Hadamard derivatives. Based on both Banach contraction principle and Scheafer fixed point theorems, we establish two results on the existence and uniqueness of solutions for the introduced problem. Some examples are presented to show the validity of our results. To give more interpretation to the examples, we establish a new approximation of Caputo-Hadamard derivative for the case $1<\beta<2$. Then, we plot the dynamics of one of the examples in terms of time and space coordinates.

Keywords: Caputo-Hadamard derivative, Duffing system, existence of solution, fixed point.

1 Introduction

Mathematical models involve fractional order derivatives have been introduced significantly for studying several phenomena in engineering and scientific disciplines, such as physics, population dynamics, biology and health, complex system, decision and control, see, for example, [1–5]. In the present paper, in general we shall be concerned with an important type of differential problem that has many applications in real word phenomena. The equation is called Duffing equation. For more information and some applications of classical Duffing equation on electric circuits and propagation in electromagnetic, we refer the reader to the papers [6–15]. The standard form of such equation has been introduced by G. Duffing [16] as follows:

$$m\frac{d^2}{dt^2}x(t)+c\frac{d}{dt}x(t)+kx(t)+\lambda x^3(t)=A\sin(\omega t),$$

such that m,c,k,λ,A and ω are respectively, the mass, the damping coefficient, the linear stiffness, the nonlinear stiffness, the excitation amplitude and the excitation frequency.
For some other papers on Duffing equations of classical or fractional order, one can see the papers [3–5, 17–20]. For some other papers on applications of fractional derivatives in real word phenomena, the reader can consult the references [1, 16, 21, 22], and [25–41].
Before introducing our Duffing type system, we cite also the following nonlinear forced Duffing problem which can be seen as a fractional version of the above standard Duffing equation:

$$\begin{cases} D^{\beta}u(t)+\delta D^{\alpha}u(t)+\rho u(t)+\mu u^3(t)=\lambda\sin(\omega t), \\ t\in[0,1], a>0 \\ \\ u(0)=A^*\in\mathbb{R}, D^{\alpha}u(0)=B^*\in\mathbb{R}, 0<\alpha<1, \\ 1<\beta<2, t\in[0,1] \end{cases}$$

where D^{α},D^{β} are the Caputo fractional derivatives and $\delta,\rho,\mu,\lambda>0$.
The motivation of our present work is in the use of Caputo Hadamard approach and also in dealing with sequential derivatives, this is in one hand. On the other hand, the motivation of the present paper can be seen also in the fact that Caputo Hadamard approach has many advantages with respect to the usual Hadamard (and the other) derivatives; the reader can confirm that the Hadamard fractional derivatives cannot be used to

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generalize the fundamental theorem of fractional calculus (FTFC) whereas the Caputo-Hadamard derivative can be applied perfectly. The obtained Caputo Hadamard FTFC is then very used to formulate other results with applications in the study of Green and Stoke theorems, as well as in the study of anomalous diffusion in some other works. Many important properties like, semigroup and commutativity for the derivatives are studied in details for Caputo Hadamard approach.
So based on these advantages, we feel motivated in this work to present a new contribution in this study since, to the best of our knowledge, this is the first time in the literature where such problem is investigated.
So, we consider the following nonlinear sequential differential problem:

$$\begin{cases} {}_H^C D^{\beta_1}\left({}_H^C D^{\alpha_1}+L_1\right)u_1(t)+\theta_1 f_1\left(t,u_1(t),u_2(t),\right.\\ \left.{}_H^C D^{\sigma_1}u_1(t)\right)+g_1\left(t,u_1(t),u_2(t),I^{\rho_1}u_1(t)\right)\\ =h_1(t),t\in[1,T],T>1,\\ {}_H^C D^{\beta_2}\left({}_H^C D^{\alpha_2}+L_2\right)u_2(t)+\theta_2 f_2\left(t,u_1(t),u_2(t),\right.\\ \left.{}_H^C D^{\sigma_2}u_2(t)\right)+g_2\left(t,u_1(t),u_2(t),I^{\rho_2}u_2(t)\right)\\ =h_2(t),t\in[1,T],T>1,\\ \left({}_H^C D^{\alpha_i}+L_i\right)u_i(1)=0,u_i(1)=u_i(T)=I^{\delta_i}u_i(\eta_i),\\ 0<\sigma_i<\alpha_i<1,1<\beta_i<2,1<\eta_i<T,\rho_i,\delta_i>0,\\ i=1,2, \end{cases} \tag{1}$$

where ${}_H^C D^{\beta_i}, {}_H^C D^{\alpha_i}$ and ${}_H^C D^{\sigma_i}$ are the derivatives in the sense of Caputo-Hadamard, I^{ρ_i} denotes the Hadamard integral of order ρ_i, with: $L_i,\theta_i>0, J=[1,T]$, the functions $f_i,g_i\in C\left(J\times\mathbb{R}^3,\mathbb{R}\right)$ and h_i are defined over $J, i=1,2$.

More precisely, in Section 2, we will recall some preliminary related to fractional calculus concepts for our problem. In Section 3, by proving two main theorems, we apply the fractional integral inequality theory combined with the fixed point theory to study the questions of existence and uniqueness of solutions for the considered system. In Section 4, some examples are studied; we establish a new approximation of the Caputo-Hadamard derivative for the case $1<\beta<2$, we present some comparative graphs for one of our examples . At the end, a conclusion follows.

2 Preliminaries

In this section, we need to work with the references [10, 11, 18, 23, 24].

Definition 2.1. Let $\mu>0$. The Hadamard fractional integral of order μ for a continuous function f is defined by:

$$I^{\mu}f(t)=\frac{1}{\Gamma(\mu)}\int_1^t\left(\log\frac{t}{s}\right)^{\mu-1}f(s)\frac{ds}{s}, n-1<\mu<n,$$

$$n=[\mu]+1,$$

where $[\mu]$ denotes the integer part of a real number $\mu, \log(\cdot)=\log_e(\cdot)$ and

$$\Gamma(\mu):=\int_0^{+\infty}e^{-s}s^{\mu-1}ds.$$

Definition 2.2. Let

$$AC_\delta^n([a,b]):=\left\{f:[a,b]\to\mathbb{R},\delta^{n-1}f\in AC[a,b],\delta=t\frac{d}{dt}\right\}.$$

The Caputo-Hadamard fractional derivative of order μ for a function $f\in AC_\delta^n([a,b],\mathbb{R})$ is defined by:

$${}_H^C D_{a+}^{\mu}f(t)=\frac{1}{\Gamma(n-\mu)}\int_a^t\left(\log\frac{t}{s}\right)^{n-\mu-1}\left(t\frac{dt}{t}\right)^n f(s)\frac{ds}{s},$$

whenever the right-hand side integral exists. **Lemma 2.1.** Let $\alpha,\beta>0$ and $f\in L^1([a,b],\mathbb{R})$.Then $I^{\alpha}I^{\beta}f(t)=I^{\alpha+\beta}f(t)$ and $D^{\alpha}I^{\alpha}f(t)=f(t)$.

Lemma 2.2. Let $\beta>\alpha>0$ and $f\in L^1([a,b],\mathbb{R})$.Then $D^{\alpha}I^{\beta}f(t)=I^{\beta-\alpha}f(t)$.

Lemma 2.3. Let $u\in AC_\delta^n([a,b],\mathbb{R}), n-1<\mu<n$. Then, the general solution of

$${}_H^C D^{\mu}u(t)=0$$

is given by:

$$u(t)=\sum_{j=0}^{n-1}c_j\left(\log\frac{t}{a}\right)^j, t>a>0,$$

and we have:

$$I^{\mu}\,{}_H^C D^{\mu}u(t)=u(t)+\sum_{j=0}^{n-1}c_j\left(\log\frac{t}{a}\right)^j,$$

such that $c_j\in\mathbb{R}, j=0,1,2,...,n-1$. We need also the following result:

Lemma 2.4. Let $\psi_i\in C(J,\mathbb{R}), i=1,2$. Then, the problem:

$$\begin{cases} {}_H^C D^{\beta_i}\left({}_H^C D^{\alpha_i}+L_i\right)u_i(t)=\psi_i(t), t\in J,\\ \left({}_H^C D^{\alpha_i}+L_i\right)u_i(1)=0,u_i(1)=u_i(T)=I^{\delta_i}u_i(\eta_i),\\ 0<\alpha_i<1,1<\beta_i<2,1<\eta_i<T,\delta_i>0,i=1,2, \end{cases} \tag{2}$$

admits the following solution

$$\begin{aligned} & u_i(t) := \\ & \frac{1}{\Gamma(\alpha_i)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha_i-1} \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta_i-1} \right. \\ & \left. \times \psi_i(\tau) \frac{d\tau}{\tau} - L_i u_i(s)\right) \frac{ds}{s} - \frac{(\log t)^{\alpha_i+1}}{(\log T)^{\alpha_i+1} \Gamma(\alpha_i)} \\ & \times \int_1^T \left(\log \frac{T}{s}\right)^{\alpha_i-1} \left(\frac{1}{\Gamma(\beta_i)} \right. \\ & \left. \times \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta_i-1} \psi_i(\tau) \frac{d\tau}{\tau} - L_i u_i(s)\right) \frac{ds}{s} \\ & - \frac{\Gamma(\delta_i+1)}{(\log \eta_i)^{\delta_i} - \Gamma(\delta_i+1)} \left[\frac{1}{\Gamma(\delta_i+\alpha_i)} \right. \\ & \times \int_0^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\delta_i+\alpha_{ii}-1} \\ & \times \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta_i-1} \psi_i(\tau) \frac{d\tau}{\tau} - L_i u_i(s)\right) \frac{ds}{s} \\ & - \frac{\Gamma^2(\alpha_i+2)}{\Gamma(\delta_i+\alpha_i+2)} \frac{(\log \eta_i)^{\delta_i+\alpha_i+1}}{(\log T)^{\alpha_i+1}} \frac{1}{\Gamma(\alpha_i)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha_i-1} \\ & \left. \times \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta_i-1} \psi_i(\tau) \frac{d\tau}{\tau} - L_i u_i(s)\right) \frac{ds}{s}\right], \end{aligned} \tag{3}$$

where $(\log \eta_i)^{\delta_i} \neq \Gamma(\delta_i+1)$.

Proof. Let $u_i, i = 1,2$ be a solution for the Caputo-Hadamard problem (2). By using Lemma 2.3, we have

$$\begin{aligned} & u_i(t) := \\ & \frac{1}{\Gamma(\alpha_i)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha_i-1} \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta_i-1} \right. \\ & \left. \times \psi_i(\tau) \frac{d\tau}{\tau} - L_i u_i(s)\right) \frac{ds}{s} \\ & - \frac{(\log t)^{\alpha_i}}{\Gamma(\alpha_i+1)} a_i - \frac{(\log t)^{\alpha_i+1}}{\Gamma(\alpha_i+2)} b_i - c_i, \end{aligned} \tag{4}$$

for some real constants $a_i, b_i, c_i, i = 1,2$.
Since $\left({}_H^C D^{\alpha_i} + k_i\right) u_i(1) = 0$, we get $a_i = 0$
and by using the condition $u_i(1) = u_i(T)$, we obtain

$$\begin{aligned} b_i = & \frac{\Gamma(\alpha_i+2)}{(\log T)^{\alpha_i+1} \Gamma(\alpha_i)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha_i-1} \\ & \times \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta_i-1} \right. \\ & \left. \times \psi_i(\tau) \frac{d\tau}{\tau} - L_i u_i(s)\right) \frac{ds}{s}. \end{aligned}$$

By using the fact that $u_i(1) = I^{\delta_i} u_i(\eta_i)$, we have

$$\begin{aligned} c_i = & \frac{\Gamma(\delta_i+1)}{(\log \eta_i)^{\delta_i} - \Gamma(\delta_i+1)} \left[\frac{1}{\Gamma(\delta_i+\alpha_i)} \right. \\ & \times \int_0^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\delta_i+\alpha_{ii}-1} \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta_i-1} \right. \\ & \left. \times \psi_i(\tau) \frac{d\tau}{\tau} - L_i u_i(s)\right) \frac{ds}{s} \\ & \left. - b_i \frac{\Gamma(\alpha_i+2)}{\Gamma(\delta_i+\alpha_i+2)} (\log \eta_i)^{\delta_i+\alpha_i+1}\right]. \end{aligned}$$

Substituting (5) , in (5) , we get

$$\begin{aligned} c_i = & \frac{\Gamma(\delta_i+1)}{(\log \eta_i)^{\delta_i} - \Gamma(\delta_i+1)} \left[\frac{1}{\Gamma(\delta_i+\alpha_i)} \right. \\ & \times \int_0^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\delta_i+\alpha_{ii}-1} \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta_i-1} \right. \\ & \left. \times \psi_i(\tau) \frac{d\tau}{\tau} - L_i u_i(s)\right) \frac{ds}{s} \\ & - \frac{\Gamma^2(\alpha_i+2)}{\Gamma(\delta_i+\alpha_i+2)} \frac{(\log \eta_i)^{\delta_i+\alpha_i+1}}{(\log T)^{\alpha_i+1}} \frac{1}{\Gamma(\alpha_i)} \\ & \times \int_1^T \left(\log \frac{T}{s}\right)^{\alpha_i-1} \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta_i-1} \right. \\ & \left.\left. \times \psi_i(\tau) \frac{d\tau}{\tau} - L_i u_i(s)\right) \frac{ds}{s}\right]. \end{aligned}$$

Then, replacing a_i, b_i, c_i in (4), we obtain (3).

3 Main Results

For computational convenience, we introduce the notions:
Let

$$X = \left\{u_i \mid u_i \in C(J,\mathbb{R}),\ {}_H^C D^{\sigma_i} u_i(t) \in C(J,\mathbb{R}); i = 1,2\right\}.$$

Obviously $\left(X \times X, \|(u_1,u_2)\|_{X\times X}\right)$ is a Banach space, endowed with the norm

$$\begin{aligned} & \|(u_1,u_2)\|_{X\times X} = \\ & \max\left\{\|u_1\|_\infty, \left\|{}_H^C D^{\sigma_1} u_1\right\|_\infty, \|u_2\|_\infty, \left\|{}_H^C D^{\sigma_2} u_2\right\|_\infty\right\}, \end{aligned}$$

such that

$$\|u_i\|_\infty = \sup_{t\in J} |u_i(t)|, \quad \left\|{}_H^C D^{\sigma_i} u_i\right\|_\infty = \sup_{t\in J} \left|{}_H^C D^{\sigma_i} u_i(t)\right|$$

. We need also the bounded closed ball:

$$C_R = \left\{(u_1,u_2) \in X \times X : \left\|(u_1,u_2)_{X\times X} \leq R\right\|\right\}.$$

In view of Lemma 2.4, we can define
$\Lambda : X \times X \to X \times X$, by

$$\Lambda(u_1,u_2)(t) = (\Lambda_1(u_1,u_2)(t), \Lambda_2(u_1,u_2)(t)),$$

where, for any $t \in J$, we have

$$\begin{aligned} & \Lambda_i(u_1,u_2)(t) := \frac{1}{\Gamma(\alpha_i)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha_i-1} \\ & \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta_i-1} [h_i(\tau) - \theta_i \right. \\ & f_i\left(\tau, u_1(\tau), u_2(\tau), {}_H^C D^{\sigma_i} u_i(\tau)\right), \\ & -g_i(\tau, u_1(\tau), u_2(\tau)\ I^{\rho_i} u_i(\tau))] \frac{d\tau}{\tau} \\ & \left. -L_i u_i(s)\right) \frac{ds}{s} - \frac{(\log t)^{\alpha_i+1}}{(\log T)^{\alpha_i+1} \Gamma(\alpha_i)} \\ & \times \int_1^T \left(\log \frac{T}{s}\right)^{\alpha_i-1} \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta_i-1} \right. \\ & [h_i(\tau) - \theta_i f_i\left(\tau, u_1(\tau), u_2(\tau), {}_H^C D^{\sigma_i} u_i(\tau)\right) \\ & -g_i \tau, u_1(\tau), u_2(\tau)\ I^{\rho_i} u_i(\tau))] \frac{d\tau}{\tau} \end{aligned}$$

$$
\begin{aligned}
&-L_i u_i(s)) \frac{ds}{s} - \frac{\Gamma(\delta_i+1)}{(\log\eta_i)^{\delta_i} - \Gamma(\delta_i+1)} \\
&\times \left[\frac{1}{\Gamma(\delta_i+\alpha_i)} \int_0^{\eta_i} \left(\log\frac{\eta_i}{s}\right)^{\delta_i+\alpha_{ii}-1} \right. \\
&\times \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta_i-1} [h_i(\tau) \right. \\
&-\theta_i f_i\left(\tau, u_1(\tau), u_2(\tau), {}_H^C D^{\sigma_i} u_i(\tau)\right) - g_i \tau, u_1(\tau), \\
&u_2(\tau), I^{\rho_i} u_i(\tau))] \frac{d\tau}{\tau} - L_i u_i(s)) \frac{ds}{s} \\
&-\frac{\Gamma^2(\alpha_i+2)}{\Gamma(\delta_i+\alpha_i+2)} \frac{(\log\eta_i)^{\delta_i+\alpha_i+1}}{(\log T)^{\alpha_i+1}\Gamma(\alpha_i)} \int_1^T \left(\log\frac{T}{s}\right)^{\alpha_i-1} \\
&\times \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta_i-1} [h_i(\tau) \right. \\
&-\theta_i f_i\left(\tau, u_1(\tau), u_2(\tau), {}_H^C D^{\sigma_i} u_i(\tau)\right) - g_i \tau, u_1(\tau), \\
&\left. u_2(\tau), I^{\rho_i} u_i(\tau))] \frac{d\tau}{\tau} \; - L_i u_i(s)) \frac{ds}{s} \right].
\end{aligned}
$$

We begin by taking into account the hypotheses:
(H_1) : The functions f_i, g_i are continuous over $J\times\mathbb{R}^3$ and h_i is continuous over J, $i=1,2$.
(H_2) :There exist nonnegative constants P_{ij}, Q_{ij}, $i=1,2$, $j=1,2,3$,
such that for all $t\in J, u_j, v_j\in\mathbb{R}$:

$$
|f_i(t,v_1,v_2,v_3) - f_i(t,u_1,u_2,u_3)| \le \sum_{j=1}^{3} P_{ij}\left|v_j-u_j\right|,
$$

$$
|g_i(t,v_1,v_2,v_3) - g_i(t,u_1,u_2,u_3)| \le \sum_{j=1}^{3} Q_{ij}\left|v_j-u_j\right|,
$$

we put for $i=1,2$,

$$
\Omega_{f_i} = \max_{j=1,2}\{P_{ij}\} \quad \text{and} \quad \Omega_{g_i} = \max_{j=1,2}\{Q_{ij}\}.
$$

(H_3) : There exist nonnegative constants K_{1i}, K_{2i}, K_{3i}, $i=1,2$, such that for all $t\in J, u_j\in\mathbb{R}, j=1,2,3$:

$$
|f_i(t,u_1,u_2,u_3)| \le K_{1i}, \quad |g_i(t,u_1,u_2,u_3)| \le K_{2i}, \quad |h_i(t)| \le K_{3i}.
$$

Then, for all $i=1,2$, we set the following quantities:

$$
\begin{aligned}
M_i =\ & \frac{2(\log T)^{\alpha_i+\beta_i}}{\Gamma(\alpha_i+\beta_i+1)} + \frac{\Gamma(\delta_i+1)}{\left|(\log\eta_i)^{\delta_i} - \Gamma(\delta_i+1)\right|} \\
&\times \left(\frac{(\log\eta_i)^{\alpha_i+\beta_i+\delta_i}}{\Gamma(\alpha_i+\beta_i+\delta_i+1)} + \frac{\Gamma^2(\alpha_i+2)}{\Gamma(\delta_i+\alpha_i+2)} \right. \\
&\left. \times \frac{(\log\eta_i)^{\delta_i+\alpha_i+1}}{(\log T)^{\alpha_i+1}} \frac{(\log T)^{\alpha_i+\beta_i}}{\Gamma(\alpha_i+\beta_i+1)} \right),
\end{aligned}
$$

$$
\overline{M}_i = \frac{(\log T)^{\alpha_i+\beta_i-\sigma_i}}{\Gamma(\alpha_i+\beta_i-\sigma_i+1)} + \frac{\Gamma(\alpha_i+2)}{\Gamma(\alpha_i-\sigma_i+2)} \times \frac{(\log T)^{\alpha_i+\beta_i-\sigma_i}}{\Gamma(\alpha_i+\beta_i+1)},
$$

$$
\begin{aligned}
N_i =\ & \frac{2(\log T)^{\alpha_i}}{\Gamma(\alpha_i+1)} + \frac{\Gamma(\delta_i+1)}{\left|(\log\eta_i)^{\delta_i} - \Gamma(\delta_i+1)\right|} \\
&\times \left(\frac{(\log\eta_i)^{\alpha_i+\delta_i}}{\Gamma(\delta_i+\alpha_i+1)} + \frac{(\alpha_i+1)^2\Gamma(\alpha_i+1)(\log\eta_i)^{\delta_i+\alpha_i+1}}{\Gamma(\delta_i+\alpha_i+2)\log T} \right),
\end{aligned}
$$

$$
\overline{N}_i = \frac{(\log T)^{\alpha_{ii}-\sigma_i}}{\Gamma(\alpha_i-\sigma_i+1)} + \frac{(\alpha_i+1)(\log T)^{\alpha_i-\sigma_i}}{\Gamma(\alpha_i-\sigma_i+2)}.
$$

Theorem 3.1. If (H_1) and (H_2) are satisfied and suppose also that

$$
\max\left\{ \left(\theta_i\Omega_{f_i}+\Omega_{g_i}\right)M_i + L_iN_i \,,\, \left(\theta_i\Omega_{f_i}+\Omega_{g_i}\right)\overline{M}_i + L_i\overline{N}_i \right\} < 1.
$$

Then, (1) has a unique solution on J.
Proof. We have to prove that Λ is a contraction mapping.

For $(u_1,u_2),(v_1,v_2)\in X\times X$, we can write

$$
\begin{aligned}
&\|\Lambda_i(v_1,v_2) - \Lambda_i(u_1,u_2)\|_\infty \\
&:\le \sup_{t\in J} \left\{ \frac{1}{\Gamma(\alpha_i)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha_i-1} \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta_i-1} \right.\right. \\
&\times \left[\theta_i \left| f_i\left(\tau, v_1(\tau), v_2(\tau), {}_H^C D^{\sigma_i} v_i(\tau)\right) - f_i(\tau, u_1(\tau), \right.\right. \\
&\left. u_2(\tau), {}_H^C D^{\sigma_i} u_i(\tau)) \right| + |g_i(\tau, v_1(\tau), v_2(\tau), I^{\rho_i} v_i(\tau)) \\
&- g_i(\tau, u_1(\tau), u_2(\tau), I^{\rho_i} u_i(\tau))|] \frac{d\tau}{\tau} \\
&+ L_i |v_i(s) - u_i(s)|) \frac{ds}{s} + \frac{(\log t)^{\alpha_i+1}}{(\log T)^{\alpha_i+1}\Gamma(\alpha_i)} \\
&\times \int_1^T \left(\log\frac{T}{s}\right)^{\alpha_i-1} \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta_i-1} \right. \\
&\times \left[\theta_i \left| f_i\left(\tau, v_1(\tau), v_2(\tau), {}_H^C D^{\sigma_i} v_i(\tau)\right) - f_i(\tau, u_1(\tau) \right.\right. , \\
&\left. u_2(\tau), {}_H^C D^{\sigma_i} u_i(\tau)) \right| + [|g_i(\tau, v_1(\tau), v_2(\tau), I^{\rho_i} v_i(\tau)) \\
&- g_i(\tau, u_1(\tau), u_2(\tau), I^{\rho_i} u_i(\tau))|] \frac{d\tau}{\tau} \\
&+ L_i |v_i(s) - u_i(s)|) \frac{ds}{s} + \frac{\Gamma(\delta_i+1)}{\left|(\log\eta_i)^{\delta_i} - \Gamma(\delta_i+1)\right|} \\
&\times \left[\frac{1}{\Gamma(\delta_i+\alpha_i)} \int_0^{\eta_i} \left(\log\frac{\eta_i}{s}\right)^{\delta_i+\alpha_{ii}-1} \right. \\
&\times \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta_i-1} \right. \\
&\times \left[\theta_i \left| f_i\left(\tau, v_1(\tau), v_2(\tau), {}_H^C D^{\sigma_i} v_i(\tau)\right) - f_i(\tau, u_1(\tau), \right.\right. \\
&\left. u_2(\tau), {}_H^C D^{\sigma_i} u_i(\tau)) \right| + |g_i(\tau, v_1(\tau), v_2(\tau), I^{\rho_i} v_i(\tau)) \\
&- g_i(\tau, u_1(\tau), u_2(\tau), I^{\rho_i} u_i(\tau))] \frac{d\tau}{\tau}
\end{aligned}
$$

$$+L_i\left|v_i(s)-u_i(s)\right|)\frac{ds}{s}+\frac{\Gamma^2(\alpha_i+2)}{\Gamma(\delta_i+\alpha_i+2)}$$
$$\times\frac{(\log\eta_i)^{\delta_i+\alpha_i+1}}{(\log T)^{\alpha_i+1}}\frac{1}{\Gamma(\alpha_i)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha_i-1}$$
$$\times\left(\frac{1}{\Gamma(\beta_i)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta_i-1}\right.$$
$$\times\left[\theta_i\left|f_i\left(\tau,v_1(\tau),v_2(\tau),{}_H^CD^{\sigma_i}v_i(\tau)\right)-f_i\left(\tau,u_1(\tau),\right.\right.\right.$$
$$\left.u_2(\tau),{}_H^CD^{\sigma_i}u_i(\tau)\right)\big|$$
$$+\left|g_i(\tau,v_1(\tau),v_2(\tau),I^{\rho_i}v_i(\tau))-g_i(\tau,u_1(\tau)\right.,$$
$$\left.\left.u_2(\tau),I^{\rho_i}u_i(\tau))\right|\right]\frac{d\tau}{\tau}+L_i\left|v_i(s)-u_i(s)\right|)\frac{ds}{s}\Bigg]\Bigg\},$$

which implies that

$$\|\Lambda_i(v_1,v_2)-\Lambda_i(u_1,u_2)\|_\infty$$
$$\leq\left(\theta_i\Omega_{f_i}+\Omega_{g_i}\right)\left[\frac{2(\log T)^{\alpha_i+\beta_i}}{\Gamma(\alpha_i+\beta_i+1)}\right.$$
$$+\frac{\Gamma(\delta_i+1)}{\left|(\log\eta_i)^{\delta_i}-\Gamma(\delta_i+1)\right|}\left(\frac{(\log\eta_i)^{\alpha_i+\beta_i+\delta_i}}{\Gamma(\alpha_i+\beta_i+\delta_i+1)}\right.$$
$$+\frac{\Gamma^2(\alpha_i+2)}{\Gamma(\delta_i+\alpha_i+2)}\frac{(\log\eta_i)^{\delta_i+\alpha_i+1}}{(\log T)^{\alpha_i+1}}$$
$$\left.\left.\times\frac{(\log T)^{\alpha_i+\beta_i}}{\Gamma(\alpha_i+\beta_i+1)}\right)\right]\|(v_1,v_2)-(u_1,u_2)\|_{X\times X}$$
$$+L_i\left[\frac{2(\log T)^{\alpha_i}}{\Gamma(\alpha_i+1)}+\frac{\Gamma(\delta_i+1)}{\left|(\log\eta_i)^{\delta_i}-\Gamma(\delta_i+1)\right|}\right.$$
$$\times\left(\frac{(\log\eta_i)^{\alpha_i+\delta_i}}{\Gamma(\delta_i+\alpha_i+1)}\right.$$
$$\left.\left.+\frac{(\alpha_i+1)^2\Gamma(\alpha_i+1)}{\Gamma(\delta_i+\alpha_i+2)}\frac{(\log\eta_i)^{\delta_i+\alpha_i+1}}{\log T}\right)\right]$$
$$\times\|(v_1,v_2)-(u_1,u_2)\|_{X\times X}$$
$$\leq\left[\left(\theta_i\Omega_{f_i}+\Omega_{g_i}\right)M_i+L_iN_i\right]\|(v_1,v_2)-(u_1,u_2)\|_{X\times X}.$$

Also, the reader can observe that

$$D^{\sigma_i}\Lambda_i(u_1,u_2)(t)=\frac{1}{\Gamma(\alpha_i-\sigma_i)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha_i-\sigma_i-1}$$
$$\times\left(\frac{1}{\Gamma(\beta_i)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta_i-1}[h_i(\tau)\right.$$
$$-\theta_if_i\left(\tau,u_1(\tau),u_2(\tau),{}_H^CD^{\sigma_i}u_i(\tau)\right)$$
$$\left.-g_i(\tau,u_1(\tau),u_2(\tau),I^{\rho_i}u_i(\tau))]\frac{d\tau}{\tau}-L_iu_i(s)\right)\frac{ds}{s}$$
$$-\frac{\Gamma(\alpha_i+2)(\log t)^{\alpha_i-\sigma_i+1}}{\Gamma(\alpha_i-\sigma_i+2)(\log T)^{\alpha_i+1}}\frac{1}{\Gamma(\alpha_i)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha_i-1}$$
$$\times\left(\frac{1}{\Gamma(\beta_i)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta_i-1}[h_i(\tau)\right.,$$
$$-\theta_if_i\left(\tau,u_1(\tau),u_2(\tau),{}_H^CD^{\sigma_i}u_i(\tau)\right)$$
$$\left.-g_i(\tau,u_1(\tau),u_2(\tau),I^{\rho_i}u_i(\tau))\frac{d\tau}{\tau}-L_iu_i(s)\right)\frac{ds}{s}$$

We have also

$$\|D^{\sigma_i}\Lambda_i(v_1,v_2)-D^{\sigma_i}\Lambda_i(u_1,u_2)\|_\infty$$
$$:\leq\sup_{t\in J}\left\{\frac{1}{\Gamma(\alpha_i-\sigma_i)}\int_1^t\left(\left(\log\frac{t}{s}\right)^{\alpha_i-\sigma_i-1}\right.\right.$$
$$\times\frac{1}{\Gamma(\beta_i)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta_i-1}$$
$$\times\left[\theta_i\left|f_i\left(\tau,v_1(\tau),v_2(\tau),{}_H^CD^{\sigma_i}v_i(\tau)\right)\right.\right.$$
$$\left.-f_i\left(\tau,u_1(\tau),u_2(\tau),{}_H^CD^{\sigma_i}u_i(\tau)\right)\right|$$
$$+\left|g_i(\tau,v_1(\tau),v_2(\tau),I^{\rho_i}v_i(\tau))\right.$$
$$\left.\left.-g_i(\tau,u_1(\tau),u_2(\tau),I^{\rho_i}u_i(\tau))\right|\right]\frac{d\tau}{\tau}$$
$$+L_i\left|v_i(s)-u_i(s)\right|)\frac{ds}{s}$$
$$+\frac{\Gamma(\alpha_i+2)(\log t)^{\alpha_i-\sigma_i+1}}{\Gamma(\alpha_i-\sigma_i+2)(\log T)^{\alpha_i+1}}\frac{1}{\Gamma(\alpha_i)}$$
$$\times\int_1^T\left(\log\frac{T}{s}\right)^{\alpha_i-1}\left(\frac{1}{\Gamma(\beta_i)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta_i-1}\right.$$
$$\times\left[\theta_i\left|f_i\left(\tau,v_1(\tau),v_2(\tau),{}_H^CD^{\sigma_i}v_i(\tau)\right)\right.\right.$$
$$\left.-f_i\left(\tau,u_1(\tau),u_2(\tau),{}_H^CD^{\sigma_i}u_i(\tau)\right)\right|$$
$$+\left|g_i(\tau,v_1(\tau),v_2(\tau),I^{\rho_i}v_i(\tau))\right.$$
$$\left.\left.-g_i(\tau,u_1(\tau),u_2(\tau),I^{\rho_i}u_i(\tau))\right|\right]\frac{d\tau}{\tau}$$
$$\left.+L_i\left|v_i(s)-u_i(s)\right|)\frac{ds}{s}\right\}$$
$$\leq\left(\theta_i\Omega_{f_i}+\Omega_{g_i}\right)\left[\frac{(\log T)^{\alpha_i+\beta_i-\sigma_i}}{\Gamma(\alpha_i+\beta_i-\sigma_i+1)}\right.$$
$$\left.+\frac{\Gamma(\alpha_i+2)}{\Gamma(\alpha_i-\sigma_i+2)}\frac{(\log T)^{\alpha_i+\beta_i-\sigma_i}}{\Gamma(\alpha_i+\beta_i+1)}\right]$$
$$\times\|(v_1,v_2)-(u_1,u_2)\|_{X\times X}$$
$$+L_i\left[\frac{(\log T)^{\alpha_{ii}-\sigma_i}}{\Gamma(\alpha_{ii}-\sigma_i+1)}+\frac{(\alpha_i+1)(\log T)^{\alpha_i-\sigma_i}}{\Gamma(\alpha_i-\sigma_i+2)}\right]$$
$$\times\|(v_1,v_2)-(u_1,u_2)\|_{X\times X}.$$
$$\leq\left[\left(\theta_i\Omega_{f_i}+\Omega_{g_i}\right)\overline{M}_i+L_i\overline{N}_i\right]$$
$$\times\|(v_1,v_2)-(u_1,u_2)\|_{X\times X}.$$

Thus,

$$\|\Lambda_i(v_1,v_2)-\Lambda_i(u_1,u_2)\|_\infty$$
$$\leq\max\left\{\left(\theta_i\Omega_{f_i}+\Omega_{g_i}\right)M_i\right.$$
$$\left.+L_iN_i,\left(\theta_i\Omega_{f_i}+\Omega_{g_i}\right)\overline{M}_i+L_i\overline{N}_i\right\}$$
$$\times\|(v_1,v_2)-(u_1,u_2)\|_{X\times X}.$$

We conclude that $\Lambda_i, i=1,2$ is contractive, so Λ is contractive. As a consequence of Banach contraction principle, we deduce that Λ has a unique fixed point which is the unique solution of (1). Now, we prove the existence solution by applying following Lemma of

Schaefer fixed point.

Lemma 3.1. Let X be a Banach space and $S : X \to X$

be a completely continuous operator. If the set

$$G = \{x \in X : x = \mu Sx,\ 0 < \mu < 1\}$$

is bounded, then S has a fixed point in X.

Theorem 3.2. Assume that the hypotheses (H_1),(H_3) are satisfied. Then, (1) has at least one solution defined over J.

Proof. We proceed as follows:

Step 1: Since f_i, g_i and h_i $i = 1,2$ are continuous, then Λ is continuous on $X \times X$.

Step 2: For all $(u_1,u_2) \in C_R$ and by (H_3) we have

$$\begin{aligned}
&\|\Lambda_i(u_1,u_2)\|_\infty \\
\le\ & [\theta_i K_{1i} + K_{2i} + K_{3i}] \left[\frac{2(\log T)^{\alpha_i+\beta_i}}{\Gamma(\alpha_i+\beta_i+1)} + \frac{\Gamma(\delta_i+1)}{\left|(\log\eta_i)^{\delta_i} - \Gamma(\delta_i+1)\right|} \right. \\
&\times \left(\frac{(\log\eta_i)^{\alpha_i+\beta_i+\delta_i}}{\Gamma(\alpha_i+\beta_i+\delta_i+1)} \left. + \frac{\Gamma^2(\alpha_i+2)(\log T)^{\beta_i-1}(\log\eta_i)^{\delta_i+\alpha_i+1}}{\Gamma(\delta_i+\alpha_i+2)\Gamma(\alpha_i+\beta_i+1)} \right) \right] \\
&+ L_i R \left[\frac{2(\log T)^{\alpha_i}}{\Gamma(\alpha_i+1)} + \frac{\Gamma(\delta_i+1)}{\left|(\log\eta_i)^{\delta_i} - \Gamma(\delta_i+1)\right|} \right. \\
&\times \left(\frac{(\log\eta_i)^{\alpha_i+\delta_i}}{\Gamma(\delta_i+\alpha_i+1)} \left. + \frac{(\alpha_i+1)\Gamma(\alpha_i+2)(\log\eta_i)^{\delta_i+\alpha_i+1}}{\Gamma(\delta_i+\alpha_i+2)\log T} \right) \right] \\
\le\ & [\theta_i K_{1i} + K_{2i} + K_{3i}] M_i + L_i R N_i < +\infty, i = 1,2.
\end{aligned}$$

and

$$\begin{aligned}
&\|D^{\sigma_i}\Lambda_i(u_1,u_2)\|_\infty \le \\
&[\theta_i K_{1i} + K_{2i} + K_{3i}] \left[\frac{(\log T)^{\alpha_i+\beta_i-\sigma_i}}{\Gamma(\alpha_i+\beta_i-\sigma_i+1)} + \frac{\Gamma(\alpha_i+2)(\log T)^{\alpha_i+\beta_i-\sigma_i}}{\Gamma(\alpha_i-\sigma_i+2)\Gamma(\alpha_i+\beta_i+1)} \right] \\
&+ L_i R \left[\frac{(\log T)^{\alpha_i-\sigma_i}}{\Gamma(\alpha_i-\sigma_i+1)} + \frac{\Gamma(\alpha_i+2)(\log T)^{\alpha_i-\sigma_i}}{\Gamma(\alpha_i-\sigma_i+2)\Gamma(\alpha_i+1)} \right] \\
\le\ & [\theta_i K_{1i} + K_{2i} + K_{3i}] \overline{M}_i + L_i R \overline{N}_i < +\infty, i = 1,2.
\end{aligned}$$

Hence, for any $(u_1,u_2) \in C_R$, we obtain $\|\Lambda_i(u_1,u_2)\|_{X\times X} < +\infty, i = 1,2$, which implies that the operator Λ is bounded on C_R.

Step 3: Λ can map bounded sets into some equicontinuous ones of $X \times X$: Let $t_1,t_2 \in J$ with $t_1 < t_2$ and let C_R be the above bounded set of $X \times X$.

For all $(u_1,u_2) \in C_R$, we have

$$\begin{aligned}
&|\Lambda_i(u_1,u_2)(t_2) - \Lambda_i(u_1,u_2)(t_1)| \\
\le\ & \left| \frac{1}{\Gamma(\alpha_i)} \int_1^{t_1} \left[\left(\log\frac{t_2}{s}\right)^{\alpha_i-1} - \left(\log\frac{t_1}{s}\right)^{\alpha_i-1} \right] \right. \\
&\times \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta_i-1} [h_i(\tau) \right. \\
&-\theta_i f_i\left(\tau,u_1(\tau),u_2(\tau),{}_H^C D^{\sigma_i}u_i(\tau)\right) \\
&-g_i\left(\tau,u_1(\tau),u_2(\tau),I^{\rho_i}u_i(\tau)\right)] \frac{d\tau}{\tau} \left. -L_i u_i(s) \right) \frac{ds}{s} \\
&+\frac{1}{\Gamma(\alpha_i)} \int_{t_1}^{t_2} \left(\log\frac{t_2}{s}\right)^{\alpha_i-1} \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta_i-1} \right. \\
&\times [h_i(\tau) - \theta_i f_i\left(\tau,u_1(\tau),u_2(\tau),{}_H^C D^{\sigma_i}u_i(\tau)\right) \\
&- g_i\left(\tau,u_1(\tau),u_2(\tau),I^{\rho_i}u_i(\tau)\right)] \frac{d\tau}{\tau} \\
&\left. -L_i u_i(s) \right) \frac{ds}{s} - \frac{(\log t_2)^{\alpha_i+1} - (\log t_1)^{\alpha_i+1}}{(\log T)^{\alpha_i+1}} \\
&\times \frac{1}{\Gamma(\alpha_i)} \int_1^T \left(\log\frac{T}{s}\right)^{\alpha_i-1} \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta_i-1} \right. \\
&\times [h_i(\tau) - \theta_i f_i\left(\tau,u_1(\tau),u_2(\tau),{}_H^C D^{\sigma_i}u_i(\tau)\right) \\
&- g_i\left(\tau,u_1(\tau),u_2(\tau),I^{\rho_i}u_i(\tau)\right)] \frac{d\tau}{\tau} \\
&\left. \left. -L_i u_i(s) \right) \frac{ds}{s} \right| \\
\le\ & \frac{\theta_i K_{1i} + K_{2i} + K_{3i}}{\Gamma(\alpha_i+\beta_i+1)} \left[\left| (\log t_2)^{\alpha_i+\beta_i} - (\log t_1)^{\alpha_i+\beta_i} \right| \right. \\
&\left. + \left| (\log t_2)^{\alpha_i+1} - (\log t_1)^{\alpha_i+1} \right| (\log T)^{\beta_i-1} \right] \\
&+ \frac{L_i R}{\Gamma(\alpha_i+1)} \left[\left| (\log t_2)^{\alpha_i} - (\log t_1)^{\alpha_i} \right| \right. \\
&\left. + \frac{\left| (\log t_2)^{\alpha_i+1} - (\log t_1)^{\alpha_i+1} \right|}{\log T} \right].
\end{aligned}$$

Similarly as before, we have

$$\begin{aligned}
&|D^{\sigma_i}\Lambda_i(u_1,u_2)(t_2) - D^{\sigma_i}\Lambda_i(u_1,u_2)(t_1)| \le \\
&\left| \frac{1}{\Gamma(\alpha_i-\sigma_i)} \int_1^{t_1} \left[\left(\log\frac{t_2}{s}\right)^{\alpha_i-\sigma_i-1} - \left(\log\frac{t_1}{s}\right)^{\alpha_i-\sigma_i-1} \right] \right. \\
&\times \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta_i-1} [h_i(\tau) \right. \\
&-\theta_i f_i\left(\tau,u_1(\tau),u_2(\tau),{}_H^C D^{\sigma_i}u_i(\tau)\right) \\
&- g_i\left(\tau,u_1(\tau),u_2(\tau),I^{\rho_i}u_i(\tau)\right)] \frac{d\tau}{\tau} \left. -L_i u_i(s) \right) \frac{ds}{s} \\
&+\frac{1}{\Gamma(\alpha_i-\sigma_i)} \int_{t_1}^{t_2} \left(\log\frac{t_2}{s}\right)^{\alpha_i-\sigma_i-1} \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta_i-1} \right. \\
&\times [h_i(\tau) - \theta_i f_i\left(\tau,u_1(\tau),u_2(\tau),{}_H^C D^{\sigma_i}u_i(\tau)\right) \\
&- g_i\left(\tau,u_1(\tau),u_2(\tau),I^{\rho_i}u_i(\tau)\right)] \frac{d\tau}{\tau} \\
&\left. -L_i u_i(s) \right) \frac{ds}{s} - \frac{\Gamma(\alpha_i+2)}{\Gamma(\alpha_i-\sigma_i+2)} \\
&\times \frac{(\log t_2)^{\alpha_i-\sigma_i+1} - (\log t_1)^{\alpha_i-\sigma_i+1}}{(\log T)^{\alpha_i+1}} \\
&\times \frac{1}{\Gamma(\alpha_i)} \int_1^T \left(\log\frac{T}{s}\right)^{\alpha_i-1} \left(\frac{1}{\Gamma(\beta_i)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta_i-1} \right.
\end{aligned}$$

$$[h_i(\tau)-\theta_i f_i(\tau,u_1(\tau),u_2(\tau),{}^C_H D^{\sigma_i}u_i(\tau)) -g_i(\tau,u_1(\tau),u_2(\tau),I^{\rho_i}u_i(\tau))]\frac{d\tau}{\tau}-L_i u_i(s))\frac{ds}{s}\Big| \le [\theta_i K_{1i}+K_{2i}+K_{3i}] \Bigg[\frac{\left|(\log t_2)^{\alpha_i+\beta_i-\sigma_i}-(\log t_1)^{\alpha_i+\beta_i-\sigma_i}\right|}{\Gamma(\alpha_i+\beta_i-\sigma_i+1)} +\left|(\log t_2)^{\alpha_i-\sigma_i+1}-(\log t_1)^{\alpha_i-\sigma_i+1}\right| \times\frac{\Gamma(\alpha_i+2)(\log T)^{\beta_i-1}}{\Gamma(\alpha_i-\sigma_i+2)\Gamma(\alpha_i+\beta_i+1)}\Bigg] +L_iR\Bigg[\frac{\left|(\log t_2)^{\alpha_i-\sigma_i}-(\log t_1)^{\alpha_i-\sigma_i}\right|}{\Gamma(\alpha_i-\sigma_i+1)} +\frac{(\alpha_i+1)\left|(\log t_2)^{\alpha_i-\sigma_i+1}-(\log t_1)^{\alpha_i-\sigma_i+1}\right|}{\Gamma(\alpha_i-\sigma_i+2)\log T}\Bigg].$$

The right hand sides of (5) and (5) tend to zero and they do not depend on (u_1,u_2) as $t_1\to t_2$.

As an implication of Steps 1,2 and 3, and thanks to Ascoli-Arzela theorem, we can conclude that Λ is completely continuous.

Step 4: The set

$$G=\{(u_1,u_2)\in X\times X:(u_1,u_2)=\mu\Lambda(u_1,u_2),\ 0<\mu<1\},$$

can be bounded. Let $(u_1,u_2)\in G$, then, we have $(u_1,u_2)=\mu\Lambda(u_1,u_2)$ for some $0<\mu<1$, so that means $(u_1,u_2)=\mu\Lambda_i(u_1,u_2), i=1,2$.

Hence, for $i=1,2$ we can write

$$\|\Lambda_i\mu(u_1,u_2)\|_\infty\le \mu\max_{i=1,2}\Bigg\{[\theta_iK_{1i}+K_{2i}+K_{3i}]\Bigg[\frac{2(\log T)^{\alpha_i+\beta_i}}{\Gamma(\alpha_i+\beta_i+1)} +\frac{\Gamma(\delta_i+1)}{\left|(\log\eta_i)^{\delta_i}-\Gamma(\delta_i+1)\right|}\Bigg(\frac{(\log\eta_i)^{\alpha_i+\beta_i+\delta_i}}{\Gamma(\alpha_i+\beta_i+\delta_i+1)} +\frac{\Gamma^2(\alpha_i+2)(\log T)^{\beta_i-1}(\log\eta_i)^{\delta_i+\alpha_i+1}}{\Gamma(\delta_i+\alpha_i+2)\Gamma(\alpha_i+\beta_i+1)}\Bigg)\Bigg] +L_iR\Bigg[\frac{2(\log T)^{\alpha_i}}{\Gamma(\alpha_i+1)}+\frac{\Gamma(\delta_i+1)}{\left|(\log\eta_i)^{\delta_i}-\Gamma(\delta_i+1)\right|} \times\Bigg(\frac{(\log\eta_i)^{\alpha_i+\delta_i}}{\Gamma(\delta_i+\alpha_i+1)} +\frac{(\alpha_i+1)\Gamma(\alpha_i+2)(\log\eta_i)^{\delta_i+\alpha_i+1}}{\Gamma(\delta_i+\alpha_i+2)\log T}\Bigg)\Bigg]\Bigg\}\le \mu\max_{i=1,2}\{[\theta_iK_{1i}+K_{2i}+K_{3i}]M_i+L_iN_iR\}<+\infty,$$

and

$$\|D^{\sigma_i}\mu\Lambda_i(u_1,u_2)\|_\infty\le \mu\max_{i=1,2}\Bigg\{[\theta_iK_{1i}+K_{2i}+K_{3i}]\Bigg[\frac{(\log T)^{\alpha_i+\beta_i-\sigma_i}}{\Gamma(\alpha_i+\beta_i-\sigma_i+1)}+ \frac{\Gamma(\alpha_i+2)(\log T)^{\alpha_i+\beta_i-\sigma_i}}{\Gamma(\alpha_i-\sigma_i+2)\Gamma(\alpha_i+\beta_i+1)}\Bigg]+ L_iR\Bigg[\frac{(\log T)^{\alpha_i-\sigma_i}}{\Gamma(\alpha_i-\sigma_i+1)}+\frac{(\alpha_i+1)(\log T)^{\alpha_i-\sigma_i}}{\Gamma(\alpha_i-\sigma_i+2)}\Bigg]\Bigg\}\le \mu\max_{i=1,2}\{[\theta_iK_{1i}+K_{2i}+K_{3i}]\overline{M}_i+L_i\overline{N}_iR\}<+\infty.$$

From (5) and (5), we see that $\|(u_1,u_2)\|_{X\times X}<+\infty$. Consequently, G is bounded.

As a consequence of our above lemma, it can be concluded that Λ has a fixed point which is one of the solutions of the coupled system (1).

4 Examples

Example 1. Consider the following system:

$$\begin{cases} {}^C_HD^{1.69}\left({}^C_HD^{0.55}+10^{-2}\right)u_1(t)+\left|{}^C_HD^{0.5}u_2(t)\right| \dfrac{1}{50\pi^5e^{t+1}\left(1+\left|{}^C_HD^{0.5}u_2(t)\right|\right)} +\dfrac{\cos\left(u_1(t)+I^{0.4}u_2(t)\right)}{99e^t}=e^{\frac{1}{2}t}, t\in[1,e],\\ {}^C_HD^{1.71}\left({}^C_HD^{0.67}+\dfrac{1}{66\pi^3}\right)u_2(t)+\dfrac{10^{-3}}{e^7} \times\left(\dfrac{\cos u_1(t)}{370e^3\sqrt{t+1}}+\dfrac{\left|{}^C_HD^{0.49}u_2(t)\right|}{\pi^9e^t\left(1+\left|D^{0.49}x_1(t)\right|\right)}\right) +\dfrac{\sin\left(u_1(t)-I^{0.6}u_2(t)+t\right)}{87e^{t^2+1}}=t+5, t\in[1,e],\\ \left({}^C_HD^{0.55}+10^{-2}\right)u_1(1)=0,\\ \left({}^C_HD^{0.67}+\dfrac{1}{66\pi^3}\right)u_2(1)=0,\\ u_1(1)=u_1(e)=I^{0.75}u_1(1.33),\\ u_2(1)=u_2(e)=I^{0.72}u_2(1.6), \end{cases} \tag{5}$$

where
$\alpha_1=0.55, \alpha_2=0.67, \beta_1=1.69, \beta_2=1.71$,
$L_1=10^{-2}, L_2=\frac{1}{66\pi^3}, \theta_1=\frac{1}{50\pi^5}, \theta_2=\frac{10^{-3}}{e^7}$,
$\sigma_1=0.5, \sigma_2=0.49, \rho_1=0.4, \rho_2=0.6, \delta_1=0.75$,
$\delta_2=0.72, \eta_1=1.33, \eta_2=1.6$,

$$f_1\left(t,u_1(t),u_2(t),{}^C_HD^{0.5}u_1(t)\right)=\frac{\left|D^{0.5}u_2(t)\right|}{e^{t+1}\left(1+\left|D^{0.5}u_2(t)\right|\right)},$$

$$g_1\left(t,u_1(t),u_2(t),I^{0.4}u_1(t)\right)=\frac{\cos\left(u_1(t)+I^{0.4}u_2(t)\right)}{99e^t},$$

$$f_2\left(t,u_1(t),u_2(t),{}_H^CD^{0.49}u_2(t)\right)=\frac{\cos u_1(t)}{370e^3\sqrt{t}}+\frac{\left|D^{0.49}u_2(t)\right|}{\pi^9e^t\left(1+\left|D^{0.49}x_1(t)\right|\right)},$$

$$g_2\left(t,u_1(t),u_2(t),I^{0.6}u_2(t)\right)=\frac{\sin\left(u_1(t)-I^{0.6}u_2(t)+t\right)}{87e^{t^2+1}}.$$

We have:
$\Omega_{f_1}=\frac{1}{e^2},\Omega_{g_1}=\frac{1}{99e},\Omega_{f_2}=\frac{1}{370e^3},\Omega_{g_2}=\frac{1}{87e^2}.$
Since, it is found that

$$M_1=0.82663,N_1=2.6188,\overline{M}_1=1.1610,$$
$$\overline{N}_1=2.5436,M_2=0.84379,N_2=3.3805,$$
$$\overline{M}_2=1.0263,\overline{N}_2=2.6147,$$

and

$$\left(\theta_1\Omega_{f_1}+\Omega_{g_1}\right)M_1+L_1N_1=2.9267\times10^{-2},$$
$$\left(\theta_2\Omega_{f2}+\Omega_{g_2}\right)M_2+L_2N_2=2.9645\times10^{-3},$$
$$\left(\theta_1\Omega_{f_1}+\Omega_{g_1}\right)\overline{M}_1+L_1\overline{N}_1=0.02976,$$
$$\left(\theta_2\Omega_{f_2}+\Omega_{g_2}\right)\overline{M}_2+L_2\overline{N}_2=2.8742\times10^{-3}.$$

Note that

$$\max_{i=1,2}\left\{\left(\theta_i\Omega_{f_i}+\Omega_{g_i}\right)M_i+L_iN_i\ ,\ \left(\theta_i\Omega_{f_i}+\Omega_{g_i}\right)\overline{M}_i+L_i\overline{N}_i\right\}\le0.02976<1.$$

Thus,Theorem 3.1. implies that the system (5) has a unique solution on $[1,e]$.

*Example 2.*Consider the problem:

$$\begin{cases}{}_H^CD^{1.91}\left({}_H^CD^{0.63}+\frac{10^{-4}}{1+e}\right)u_1(t)+\frac{1}{(\pi e)^3}\\ \times\frac{\sin\left({}_H^CD^{0.51}u_2(t)+t^3+1\right)}{2e^t+3}\\ +\frac{\sin\left(u_1(t)-I^{0.6}u_2(t)+t\right)}{2\log(3t+5)}=\frac{1}{te^{1+t}},t\in[1,e],\\ {}_H^CD^{1.63}\left({}_H^CD^{0.91}+\frac{10^{-3}}{\log2}\right)u_2(t)\\ +\frac{(7+\log2)^{-5}\left|{}_H^CD^{0.7}u_2(t)\right|}{(t^2+1)\pi^2\left(1+\left|{}_H^CD^{0.7}u_2(t)\right|\right)}\\ +\frac{\left|I^{0.59}u_2(t)\right|}{3e^{t^2+1}\left(1+\left|I^{0.59}u_2(t)\right|\right)}=\frac{1}{t+30},t\in[1,e],\\ \left({}_H^CD^{0.63}+10^{-4}\right)u_1(1)=0,\\ \left({}_H^CD^{0.91}+10^{-3}\right)u_2(1)=0,\\ u_1(1)=u_1(e)=I^{0.55}u_1(2),\\ u_2(1)=u_2(e)=I^{0.82}u_2(2.5),\end{cases}\tag{6}$$

where

$$f_1\left(t,u_1(t),u_2(t),{}_H^CD^{0.51}u_1(t)\right)=\frac{\sin\left({}_H^CD^{0.51}u_2(t)+t^3+1\right)}{2e^t+3},$$

$$g_1\left(t,u_1(t),u_2(t),I^{0.6}u_1(t)\right)=\frac{\sin\left(u_1(t)-I^{0.6}u_2(t)+t\right)}{2\log(3t+5)},$$

$$f_2\left(t,u_1(t),u_2(t),{}_H^CD^{0.7}u_2(t)\right)=\frac{\left|{}_H^CD^{0.7}u_2(t)\right|}{(t^2+1)\pi^2\left(1+\left|{}_H^CD^{0.7}u_2(t)\right|\right)},$$

$$g_2\left(t,u_1(t),u_2(t),I^{0.59}u_2(t)\right)=\frac{\left|I^{0.59}u_2(t)\right|}{3e^{t^2+1}\left(1+\left|I^{0.59}u_2(t)\right|\right)},$$

$$h_1(t)=\frac{1}{te^{1+t}},\quad h_2(t)=\frac{1}{t+30},$$

and

$\beta_1=1.91,\beta_2=1.63,\alpha_1=0.63,\alpha_2=0.91,$
$L_1=\frac{10^{-3}}{\log2},L_2=\frac{10^{-4}}{1+e},\theta_1=\frac{1}{(\pi e)^3},\theta_2=(7+\log2)^{-5},$
$\sigma_1=0.51,\sigma_2=0.7,\ \rho_1=0.6,\ \rho_2=0.59\ ,\delta_1=0.55,$
$\delta_2=0.82,\eta_1=2,\eta_2=2.5,R=\frac{1}{10}.$

We have: $K_{11}=\frac{1}{2e+3},K_{12}=\frac{1}{2\pi^2},K_{21}=\frac{1}{2\log8},$
$K_{22}=\frac{1}{3e^2},K_{31}=\frac{1}{e^2},K_{32}=\frac{1}{31}$
and $\|\Lambda_1(x_1,x_2)\|_\infty\le0.98799,\|\Lambda_2(x_1,x_2)\|_\infty\le3.127,\left\|D^{0.51}\Lambda_1(x_1,x_2)\right\|_\infty\le0.43617,\left\|D^{0.7}\Lambda_2(x_1,x_2)\right\|_\infty\le8.1682\times10^{-2},\|\Lambda_i\mu(u_1,u_2)\|_\infty\le3.127\mu,$
$\|D^{\sigma_i}\mu\Lambda_i(u_1,u_2)\|_\infty\le0.43617\mu:0<\mu<1,i=1,2.$

Hence, Theorem 3.2. holds true, witch implies that (6) has at least one solution on $[1,e]$.

5 Numerical Approximations and Simulations

In this section, we present an approximation for Caputo Hadamard derivative for the case of $1<\beta<2$. To do that, we need the method developed in the work [10].

Theorem 5.1. Let $y\in\mathscr{C}(J,\mathbb{R})$, and $0<\alpha<1$, then the Caputo-Hadamard derivative presented as the following convolutional form:

$${}_H^CD^{\alpha}y(t_k)=\sum_{j=0}^{k}b_{j,k}u(t_j)+R^k$$

where

$$b_{j,k}=\begin{cases}-a_{1,k}, & j=0\\ a_{j,k}-a_{j+1,k}, & j=1,2,\dots,k-1\\ a_{k,k}, & j=k\end{cases}$$

$$a_{j,k}=\frac{\left(\left(\log\frac{t_k}{t_{j-1}}\right)^{1-\alpha}-\left(\log\frac{t_k}{t_j}\right)^{1-\alpha}\right)}{\Gamma(2-\alpha)\log\frac{t_j}{t_{j-1}}},$$

$$R^k=\frac{\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\left(\log\frac{t_k}{s}\right)^{-\alpha}}{\Gamma(1-\alpha)}\times\left(\delta u(s)-\frac{y(t_j)-u(t_{j-1})}{\log\frac{t_j}{t_{j-1}}}\right)\frac{ds}{s}$$

Proof. Now, let us consider our case $1<\beta<2$. For $T>1$, we divide $[1,T]$ into N subintervals with $1=t_0<t_1<\cdots<t_{k-1}<t_k<\cdots<t_N=T$, with stepwise $h=t_k-t_{k-1}, 1\leq k\leq N$. By a linear interpolation at $t=t_k, 1\leq k\leq N$, we have:

$$\begin{aligned}{}^{C}_{H}D^{\beta}u(t_k)&=\frac{1}{\Gamma(2-\beta)}\int_a^{t_k}\left(\log\frac{t_k}{s}\right)^{1-\beta}\delta^2u(s)\frac{ds}{s}\\ &=\frac{1}{\Gamma(2-\beta)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\left(\log\frac{t_k}{s}\right)^{1-\beta}\delta^2u(s)\frac{ds}{s}\\ &=\frac{1}{\Gamma(2-\beta)}\sum_{j=2}^{k-2}\int_{t_{j-1}}^{t_j}\left(\log\frac{t_k}{s}\right)^{1-\beta}\frac{u(t_{j+1})-2u(t_j)+u(t_{j-1})}{\left(\log\frac{t_j}{t_{j-1}}\right)^2}\frac{ds}{s}+R\\ &\quad+\frac{1}{\Gamma(2-\beta)}\int_{t_0}^{t_1}\left(\log\frac{t_k}{s}\right)^{1-\beta}\times\frac{u(t_2)-2u(t_1)+u(t_0)}{\left(\log\frac{t_1}{t_0}\right)^2}\frac{ds}{s}\\ &\quad+\frac{1}{\Gamma(2-\beta)}\int_{t_{k-2}}^{t_k}\left(\log\frac{t_k}{s}\right)^{1-\beta}\times\frac{u(t_k)-2u(t_{k-1})+u(t_{k-2})}{\left(\log\frac{t_k}{t_{k-2}}\right)^2}\frac{ds}{s}\\ &=\sum_{j=2}^{k-2}C_{j,k}\left(u(t_{j+1})-2u(t_j)+u(t_{j-1})\right)\\ &\quad+F_k\left(u(t_0)-2u(t_1)+u(t_2)\right)\\ &\quad+B_k\left(u(t_k)-2u(t_{k-1})+u(t_{k-2})\right)+R,\end{aligned}\tag{7}$$

where

$$\begin{cases}C_{j,k}=\dfrac{\left(\left(\log\frac{t_k}{t_{j-1}}\right)^{2-\beta}-\left(\log\frac{t_k}{t_j}\right)^{2-\beta}\right)}{\Gamma(3-\beta)\log\frac{t_j}{t_{j-1}}},\\ F_k=\dfrac{\left(\left(\log\frac{t_k}{t_1}\right)^{2-\beta}-\left(\log\frac{t_k}{t_0}\right)^{2-\beta}\right)}{\Gamma(3-\beta)\log\frac{t_1}{t_0}},\\ B_k=\dfrac{1}{\Gamma(3-\beta)}\dfrac{1}{\log\frac{t_k}{t_{k-2}}}\left(\left(\log\frac{t_k}{t_{k-2}}\right)^{2-\beta}\right)\end{cases}$$

We omit the truncation error R. Then (7) can be rewritten as follows:

$$\begin{aligned}{}^{C}_{H}D^{\beta}u(t_k)&=\sum_{j=3}^{k-3}T_{5,j}u(t_j)+F_ku(t_0)+B_ku(t_k)+T_1u(t_1)\\ &+T_2u(t_2)+T_3u(t_{k-2})T_4u(t_{k-1}),\end{aligned}$$

where

$$\begin{cases}T_{5,j}=C_{j-1,k}-2C_{j,k}+C_{j+1,k},\\ T_1=C_{2,k}-2F_k,\\ T_2=C_{3,k}-2C_{2,k}+F_k,\\ T_3=C_{k-3,k}-2C_{k-2,k}+B_k,\\ T_4=C_{k-2,k}-2B_k.\end{cases}$$

To solve numerically our problem, we implement the previous approximation on

$${}^{C}_{H}D^{\beta_i}\left({}^{C}_{H}D^{\alpha_i}+L_i\right)u(t)=\psi(t),$$

so, it yields

$$\begin{aligned}&B_k\Big(L+bk,k\Big)\\ &=\psi(t_k)-L\Big(\sum_{j=3}^{k-3}T_{5,j}u(t_j)+u(t_0)F_k+u(t_1)T1\\ &\quad+u(t_2)T2+u(t_{k-2})b_{k-2,k}T3+u(t_{k-1}T4\Big)\\ &\quad-\Big(\sum_{j=3}^{k-3}T_{5,j}\sum_{i=0}^{j}b_{i,k}u(t_i)+u(t_0)b_{0,k}(T_1+T_2+T_3+T_4+F_k)\\ &\quad+u(t_1)b_{1,k}(T_1+T_2+T_3+T_4)\\ &\quad+u(t_2)b_{2,k}(T_2+T_3+T_4)+u(t_{k-2})b_{k-2,k}(T_3+T_4)\\ &\quad+u(t_{k-1})b_{k-1,k}*T_4+(T3+T4)\sum_{i=3}^{k-3}b_{i,k}u(t_i)\Big).\end{aligned}$$

Now we show the behavior of the dynamics of the first example, adopting the developed method.
First of all, we attempt to display the solution of the corresponding ODE by MuPAD, then we tackle the given problem using Matlab. Behaviors are plotted, respectively, in figures

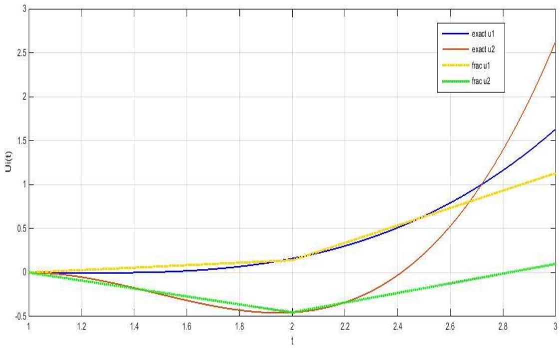

Fig. 1: A : Numerical simulation for examples (5) for $\alpha 1$ = 0.55, $\alpha 2$ = 0.67, $\beta 1$ = 1.69, $\beta 2$ =1.71 and exact solution.

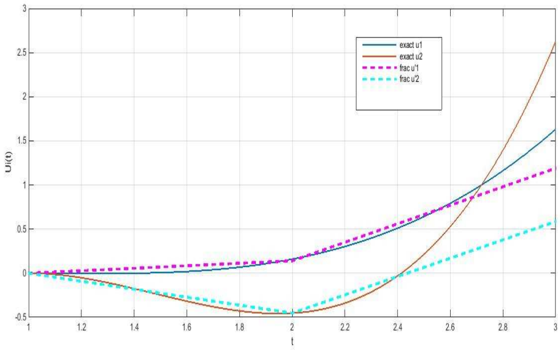

Fig. 2: B : numerical simulation for examples (5) for $\alpha 1$ = 0.85, $\beta 1$ = 1.9, $\beta 2$ = 1,85, $\alpha 2$ =0.75, and exact solution

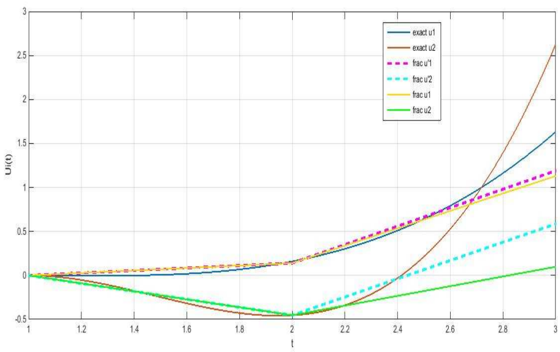

Fig. 3: comparative graph between A and B

6 conclusion

In this work, a sequential differential system of Duffing type, that involves Caputo Hadamard derivatives, has been investigated. For the above introduced system, we have proved two main results on the existence and uniqueness of solutions. Then, we have illustrated the results by some examples. Another important point that has been discussed in this work is the proposition of a new approximation for the Caputo Hadamard derivative in the case of $1 < \beta < 2$; such approximation has allowed us to present a numerical study with some graphs and simulations for one of our examples.

Conflict of Interest The authors declare that they have no conflict of interest.

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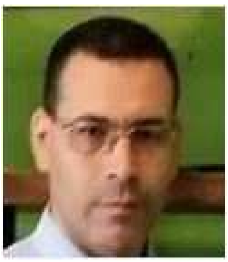

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