Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/160218

Fekete-Szegö Inequality and Application of Poisson Distribution Series for some Subclasses of Analytic Functions related with Bessel Functions

Gangadharan Murugusundaramoorthy[1,*] *and Hemen Dutta*[2]

[1]Department of Mathematics, SAS, Vellore Institute of Technology, Vellore - 632014, India
[2]Department of Mathematics, Gauhati University, Guwahati-781014, India

Received: 8 Jun. 2020, Revised: 30 Oct. 2021, Accepted: 25 Dec. 2021
Published online: 1 Mar. 2022

Abstract: The determination of this current paper is to find certain coefficient estimates, Fekete-Szegö inequality results for a normalized analytic function defined in the open unit disk $\mathbb{D}=\{z: z\in\mathbb{C} \quad \text{and} \quad |z|<1\}$ by convolution operator with Bessel function. In particular, we derived Fekete-Szegö inequality for a class of functions defined through Poisson distribution.The results presented in this paper would generalize some related works of several earlier authors.

Keywords: Analytic functions, Starlike functions, Convex functions, Subordination, Fekete-Szegö inequality, Poisson distribution, Bessel function, Hadamard product.

1 Introduction, Definitions and Preliminary results

Let $\mathscr{A}$ be the set of all analytic functions,comprising of the functions

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1}$$

in the open unit disc

$$\mathbb{D}:=\{z: z\in\mathbb{C} \quad \text{and} \quad |z|<1\}$$

and $\mathscr{S}$ be the subclass of $\mathscr{A}$ comprising of univalent functions. Let the functions f and g be analytic in $\mathbb{D}$. We say that the function f is subordinate to g, if there exists a Schwarz function ω, which is analytic in $\mathbb{D}$ with

$$\omega(0)=0 \qquad \text{and} \qquad |\omega(z)|<1 \quad (z\in\mathbb{D}),$$

such that

$$f(z)=g\big(\omega(z)\big).$$

This subordination is denoted by

$$f\prec g \qquad \text{or} \qquad f(z)\prec g(z) \quad (z\in\mathbb{D}).$$

It is well known that (see [1]), if the function g is univalent in $\mathbb{D}$, then

$$f\prec g \quad (z\in\mathbb{D}) \iff f(0)=g(0) \quad \text{and} \quad f(\mathbb{D})\subseteq g(\mathbb{D}).$$

We recall here a Bessel function of the first kind of order υ, denoted by $J_\upsilon(z)$, is defined by the infinite series:

$$J_\upsilon(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+\upsilon+1)}\left(\frac{z}{2}\right)^{2n+\upsilon} \quad (z\in\mathbb{C},\upsilon\in\mathbb{R}) \tag{2}$$

which is the particular solution of the second order linear homogeneous differential equation

$$z^2\omega''(z)+z\omega'(z)+[z^2-\upsilon^2]\omega(z)=0, \tag{3}$$

where $\upsilon\in\mathbb{C}$, which is the natural Bessel's equation. Solutions of (3) are referred to as Bessel function of order υ. Although the series defined in (2) is convergent every where, in general J_υ is not univalent in $\mathbb{U}$. Latterly, Szász and Kupán[2] inspected the univalence of the normalized Bessel function of the first kind

$$u_\upsilon:\mathbb{D}\to\mathbb{C}$$

given by the transformation (see also[3,4])

$$u_\upsilon(z)=2^\upsilon\Gamma(\upsilon+1)\;z^{1\frac{-\upsilon}{2}}\;J_\upsilon(\sqrt{z}),\;\;\sqrt{1}=1.$$

* Corresponding author e-mail: gmsmoorthy@yahoo.com

We can express $u_\upsilon(z)$ as

$$u_\upsilon(z) = z + \sum_{n=1}^{\infty} \frac{(-1/4)^{n-1}\Gamma(\upsilon+1)}{(n-1)!\Gamma(n+\upsilon)} z^n, \qquad (4)$$

is analytic on $\mathbb{C}$ and satisfies the differential equation

$$4z^2u''(z) + 4(\upsilon+1)zu'(z) + cu(z) = 0.$$

Quantum calculus (*q-calculus* and *h-calculus*)simply the study of classical calculus without the notion of limits. Here, h represents Planckís constant, while q represents quantum. Due to its application in a variety of branches such as physics, mathematics, the area of *q-calculus* has added excessive prominence for researchers. The first study on *q-calculus* was systematically established by Jackson [5]. Recently, Kanas and Raducanu [6] defined a $q-$ analogue of the Ruscheweyh differential operator[7] by using the concept of convolution and then studied some of its properties by Aldweby and Darus [8]. Now, we give some notational details of *q-calculus* which are used in the paper.

For $f \in \mathscr{A}$ the Jackson's *q-derivative* $(0 < q < 1)$ is expressed by

$$\mathfrak{D}_q f(z) = \begin{cases} \dfrac{f(z)-f(qz)}{(1-q)z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases} \qquad (5)$$

and $\mathscr{D}_q^2 f(z) = \mathfrak{D}_q(\mathfrak{D}_q f(z))$. Thus, from (5), we presume that

$$\mathfrak{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

If $q \to 1^-$, we get $[n]_q \to n$. For the function $h(z) = z^n$, we get $\mathfrak{D}_q h(z) = \mathfrak{D}_q z^n = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1}$ and $\lim_{q\to 1^-} \mathfrak{D}_q h(z) = \lim_{q\to 1^-} \left([n]_q z^{n-1}\right) = nz^{n-1} = h'(z)$, where h' is the usual derivative.

For $0 < q < 1$, and $f \in \mathscr{A}$ of the form (1),the $q-$ derivative of u_υ is defined by:

$$\begin{aligned} \mathfrak{D}_q u_\upsilon(z) &= \mathfrak{D}_q \left[z + \sum_{n=1}^{\infty} \frac{(-1/4)^{n-1}\Gamma(\upsilon+1)}{(n-1)!\Gamma(n+\upsilon)} z^n \right] \\ &= \frac{u_\upsilon(z) - u_\upsilon(qz)}{(1-q)z} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1/4)^{n-1}\Gamma(\upsilon+1)}{(n-1)!\Gamma(n+\upsilon)} [n]_q z^{n-1}. \qquad (6) \end{aligned}$$

Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. The *q-generalized Pochhammer symbol* is defined by

$$[a;n]_q = [a]_q[a+1]_q[a+2]_q...[a+n-1]_q \qquad (7)$$

and for $a > 0$ the *q-gamma function* is defined by

$$\Gamma_a(a+1) = [a]_q\Gamma_q(a) \qquad \text{and} \qquad \Gamma_q(1) = 1. \qquad (8)$$

For $f \in \mathscr{A}$, Kanas and Raducanu [6] defined the Ruscheweyh *q-differential operator* as below:

$$\mathscr{R}_q^\delta f(z) = f(z) * F_{q,\delta+1}(z) \qquad (\delta > -1, z \in \mathbb{U}) \qquad (9)$$

where

$$\begin{aligned} F_{q,\delta+1}(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\delta)}{[n-1]_q!\Gamma_q(1+\delta)} z^n \\ &= z + \sum_{n=2}^{\infty} \frac{[\delta+1;n]_q}{[n-1]_q!} z^n. \qquad (10) \end{aligned}$$

using, (9) and (10), Aldweby and Darus[8] defined the $q-$analogue of Ruscheweyh operator $\mathscr{R}_q^\delta : \mathscr{A} \to \mathscr{A}$ as follows:

$$\mathscr{R}_q^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\delta)}{[n-1]_q!\Gamma_q(1+\delta)} a_n z^n \qquad (z \in \mathbb{U}).. \qquad (11)$$

As $q \to 1^-$, we note that

$\mathscr{R}_q^0 f(z) = f(z), \qquad \mathscr{R}_q^1 f(z) = zf'(z),$

It is easy to check that

$$z\mathfrak{D}_q(F_{q,\delta+1}(z)) = \left(1 + \frac{[\delta]}{q^\delta}\right) F_{q,\delta+2}(z) - \frac{[\delta]}{q^\delta} F_{q,\delta+1}(z), \qquad (12)$$

$z \in \mathbb{U}$. From (9), (12) and by the concept of Hadamard product, we have

$$z(\mathscr{R}_q^\delta f(z))' = (1+\delta)\mathscr{R}_q^{1+\delta} f(z) - \delta\mathscr{R}_q^\delta f(z), \quad (z \in \mathbb{U}). \qquad (13)$$

From (11),as $q \to 1^-$ we note that

$$\lim_{q\to 1^-} F_{q,\delta+1}(z) = \frac{z}{(1-z)^{\delta+1}},$$

$$\lim_{q\to 1^-} \mathscr{R}_q^\delta f(z) = f(z) * \frac{z}{(1-z)^{\delta+1}}$$

the usual Ruscheweyh derivative [7].

By the description of $q-derivative$ and the perception of Hadamard product ,we describe the linear operator

$$\Psi_{\upsilon,q}^\delta : \mathscr{A} \to \mathscr{A}$$

defined by

$$\Psi_{\upsilon,q}^\delta f(z) = u_\upsilon(z) * (\mathscr{R}_q^\delta f(z)) \qquad (14)$$

$$= z + \sum_{n=2}^{\infty} \frac{(-1/4)^{n-1}\Gamma(\upsilon+1)}{(n-1)!\Gamma(n+\upsilon)} \frac{\Gamma_q(n+\delta)}{[n-1]_q!\Gamma_q(1+\delta)} a_n z^n \qquad (15)$$

$$= z + \sum_{n=2}^{\infty} \Theta_n a_n z^n \qquad (z \in \mathbb{U}) \qquad (16)$$

where

$$\Theta_n = \frac{(-1/4)^{n-1}\Gamma(\upsilon+1)}{(n-1)!\Gamma(n+\upsilon)} \frac{\Gamma_q(n+\delta)}{[n-1]_q!\Gamma_q(1+\delta)}. \quad (17)$$

Ma and Minda[9], unified various subclasses of starlike and convex functions for which either of the functions

$$\frac{zf'(z)}{f(z)} \quad \text{and} \quad 1+\frac{zf''(z)}{f'(z)} \qquad (z\in\mathbb{D}).$$

is subordinate to a more general superordinate function and denoted such function classes by $\mathscr{S}^*(\phi)$ and $\mathscr{C}(\Phi)$, respectively. For this purpose,they considered an analytic function Φ as below ,

Definition 1.[9] *Suppose Φ is an analytic function such that*

1. $\Re\ (\Phi) > 0$ in $\mathbb{U}$
2. $\Phi(0) = 1, \quad \Phi'(0) > 0$
3. *Φ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis.*

Further they gave $\Phi(z)$ in series by

$$\Phi(z) = 1+B_1z+B_2z^2+B_3z^3+\cdots, \quad (18)$$

where $B_n's$ are real with $B_1 > 0; B_2 \geqq 0$.

Fixing $\Phi(z) = \dfrac{1+(1-2\beta)z}{1-z}, \quad (z\in\mathbb{D};\ 0\leqq\beta<1)$ we get the well-known classes $\mathscr{S}^*(\beta)$ (and $\mathscr{C}(\beta)$) of starlike functions (and the class of convex functions) of order $\beta(0\leqq\beta<1)$ respectively. In [10],Guo and Liu defined a subclass $M(\mu,\lambda,\rho)$ as below which unifies certain subclasses of analytic functions.

Let $\mu\geqq 0$, $\lambda\geqq 0$ and $0\leqq\rho<1$ and $f\in\mathscr{A}$. We say that $f\in M(\mu,\lambda,\rho)$ if it hold the analytic criterion

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\mu} + \lambda\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}+\mu\left(\frac{zf'(z)}{f(z)}-1\right)\right]\right\} > \rho.$$

In [9], the authors have obtained the Fekete-Szegö inequality f in $\mathscr{S}^*(\Phi)$ and $\mathscr{C}(\Phi)$.For a brief history of Fekete-Szegö problem various subclasses of analytic functions, one may refer to [11] and the references cited there in. Motivated essentially by the aforementioned works on Fekete-Szegö inequality and the definition of hadamard product we define a more general class of analytic functions which unifies the class $\mathscr{S}^*(\Phi)$ and $\mathscr{C}(\Phi)$, $\mathscr{M}_\lambda(\Phi)$ based on Bessel function. Also, we give applications of our results to certain functions defined through of Poisson distribution series.

Now, we define the following new function class $\mathscr{G}^{\upsilon,\delta,q}_{\mu,\lambda}(\Phi)$:

Definition 2.*For $\mu\geqq 0$, $\lambda\geqq 0$, let $\Phi(z)$ be in Definition1 and $f\in\mathscr{A}$ is in the class $\mathscr{G}^{\upsilon,\delta,q}_{\mu,\lambda}(\Phi)$ if*

$$\frac{z(\Psi^{\delta}_{\upsilon,q}f(z))'}{\Psi^{\delta}_{\upsilon,q}f(z)}\left(\frac{\Psi^{\delta}_{\upsilon,q}f(z)}{z}\right)^{\mu} + \lambda\left[1+\frac{z(\Psi^{\delta}_{\upsilon,q}f(z))''}{(\Psi^{\delta}_{\upsilon,q}f(z))'}-\frac{z(\Psi^{\delta}_{\upsilon,q}f(z))'}{\Psi^{\delta}_{\upsilon,q}f(z)} + \mu\left(\frac{z(\Psi^{\delta}_{\upsilon,q}f(z))'}{\Psi^{\delta}_{\upsilon,q}f(z)}-1\right)\right] \prec \Phi(z), \qquad z\in\mathbb{D}.$$

By specializing the parameters, suitably we deduce the following new subclasses based on Bessel functions , which are not yet been studied.

Definition 3.*For $\mu=0$, $\lambda\geqq 0$ and let $\Phi(z)$ be given in Definition1 and $f\in\mathscr{A}$ is in the class $\mathscr{G}^{\upsilon,\delta,q}_{0\lambda}(\Phi) = \mathscr{M}^{\upsilon,\delta,q}_{\lambda}(\Phi)$ if*

$$\lambda\left(1+\frac{z(\Psi^{\delta}_{\upsilon,q}f(z))''}{(\Psi^{\delta}_{\upsilon,q}f(z))'}\right)+(1-\lambda)\frac{z(\Psi^{\delta}_{\upsilon,q}f(z))'}{\Psi^{\delta}_{\upsilon,q}f(z)} \prec \Phi(z), z\in\mathbb{D}.$$

Note that

$$\mathscr{M}^{\upsilon,\delta,q}_{0}(\Phi)\equiv S^{\upsilon,\delta,q}(\Phi) \quad \text{and} \quad \mathscr{M}^{\upsilon,\delta,q}_{1}(\Phi)\equiv\mathscr{C}^{\upsilon,\delta,q}(\Phi).$$

Definition 4.*For $\mu\geqq 0$, $\lambda=0$ and let $\Phi(z)$ be in Definition1 and $f\in\mathscr{A}$ is in the class $\mathscr{G}^{\upsilon,\delta,q}_{\mu,0}(\Phi) = \mathscr{B}^{\upsilon,\delta,q}_{\mu}(\Phi)$ if*

$$(\Psi^{\delta}_{\upsilon,q}f(z))'\left(\frac{\Psi^{\delta}_{\upsilon,q}f(z)}{z}\right)^{\mu-1} \prec \Phi(z), z\in\mathbb{D}.$$

Definition 5.*For $\mu=1$, $\lambda=0$ and let $\Phi(z)$ be in Definition1 and $f\in\mathscr{A}$ is in the class $\mathscr{G}^{\upsilon,\delta,q}_{1,0}(\Phi) = \mathscr{R}^{\upsilon,\delta,q}(\Phi)$ if*

$$(\Psi^{\delta}_{\upsilon,q}f(z))' \prec \Phi(z), z\in\mathbb{D}.$$

To prove our main result, we need the following lemmas:

Lemma 1.[12] *If $\varpi\in\mathscr{P}$ and given by*

$$\varpi(z) = 1+c_1z+c_2z^2+\cdots \quad (19)$$

then $|c_j|\leq 2$ for all $j\geq 1$, and the result is best possible for $\phi_1(z)=\dfrac{1+\eta z}{1-\eta z}$, $|\eta|=1$.

Lemma 2.[9] *If $\varpi(z)\in\mathscr{P}$ and given by (19) then*

$$|c_2-\vartheta c_1^2| \leqq \begin{cases} -4\vartheta+2, & \text{if } \vartheta\leqq 0,\\ 2, & \text{if } 0\leqq\vartheta\leqq 1,\\ 4\vartheta-2, & \text{if } \vartheta\geqq 1.\end{cases}$$

When $\vartheta < 0$ or $\vartheta > 1$, the equality holds if and only if $p_1(z) = \frac{1+z}{1-z}$ or one of its rotations. If $0 < \vartheta < 1$, then the above upper bound is sharp,for $p_2(z) = \frac{1+z^2}{1-z^2}$ or one of its rotations. If $\vartheta = 0$,the above upper bound is sharp, for

$$p_3(z) = \left(\frac{1}{2}+\frac{\eta}{2}\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{\eta}{2}\right)\frac{1-z}{1+z} \quad (0 \leqq \eta \leqq 1)$$

or one of its rotations. If $\vartheta = 1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $\vartheta = 0$.

Although the above upper bound is sharp, when $0 < \vartheta < 1$, it can be improved as follows:

$$|c_2 - \vartheta c_1^2| + \vartheta |c_1|^2 \leqq 2 \quad (0 < \vartheta \leqq 1/2)$$

and

$$|c_2 - \vartheta c_1^2| + (1-\vartheta)|c_1|^2 \leqq 2 \quad (1/2 < \vartheta \leqq 1).$$

We also need the following:

Lemma 3.*[13] If $\varpi(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a member of $\mathscr{P}$, then*

$$|c_2 - \vartheta c_1^2| \leqq 2\max(1, |2\vartheta - 1|).$$

The result is sharp for the functions

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{D}).$$

2 Coefficient Estimates and Fekete-Szegö inequality

To start with in this section we determine the initial Coefficient estimates a_2 anda_3. Unless otherwise stated, we let the following in our study:

$$\Theta_2 = \frac{(-1/4)\Gamma(\upsilon+1)}{\Gamma(2+\upsilon)} \frac{\Gamma_q(2+\delta)}{[1]_q!\Gamma_q(1+\delta)}$$

and

$$\Theta_3 = \frac{(-1/4)^2\Gamma(\upsilon+1)}{2\Gamma(3+\upsilon)} \frac{\Gamma_q(3+\delta)}{[2]_q!\Gamma_q(1+\delta)}. \quad (20)$$

Theorem 1.*Let $\mu \geqq 0$ and $\lambda \geqq 0$ and μ a real number and $\Phi(z)$ be given by (18). If $f(z)$ given by (1) belongs to $\mathscr{G}_{\mu,\lambda}^{\upsilon,\delta,q}(\Phi)$, then*

$$|a_2| \leq \left|\frac{B_1}{\tau\Theta_2}\right|,$$

$$|a_3| \leq \frac{B_1}{2\xi\Theta_3}\left|-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}\right|,$$

where $\Lambda = \frac{1}{2}(\mu^2+\mu) - (\mu+3)\lambda - 1$;

$$\tau := (1+\mu)(1+\lambda), \quad \text{and} \quad \xi := (\mu+2)(1+2\lambda). \quad (21)$$

These results are sharp.

*Proof.*If $f \in \mathscr{G}_{\mu,\lambda}^{\upsilon,\delta,q}(\Phi)$, then there is a Schwarz function $\omega(z)$, analytic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{D}$ such that

$$\frac{z(\Psi_{\upsilon,q}^{\delta}f(z))'}{\Psi_{\upsilon,q}^{\delta}f(z)}\left(\frac{\Psi_{\upsilon,q}^{\delta}f(z)}{z}\right)^{\mu} + \lambda\left[1+\frac{z(\Psi_{\upsilon,q}^{\delta}f(z))''}{(\Psi_{\upsilon,q}^{\delta}f(z))'} - \frac{z(\Psi_{\upsilon,q}^{\delta}f(z))'}{\Psi_{\upsilon,q}^{\delta}f(z)} + \mu\left(\frac{z(\Psi_{\upsilon,q}^{\delta}f(z))'}{\Psi_{\upsilon,q}^{\delta}f(z)}-1\right)\right] \prec \Phi(z) = \Phi(\omega(z)). \quad (22)$$

Define the function $p_1(z)$ by

$$p_1(z) := \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \quad (23)$$

Since $\omega(z)$ is a Schwarz function, we see that $\Re(p_1(z)) > 0$ and $p_1(0) = 1$. Let us define the function $p(z)$ by

$$\begin{aligned} p(z) \quad := \quad & \frac{z(\Psi_{\upsilon,q}^{\delta}f(z))'}{\Psi_{\upsilon,q}^{\delta}f(z)}\left(\frac{\Psi_{\upsilon,q}^{\delta}f(z)}{z}\right)^{\mu} \\ & + \lambda\left[1+\frac{z(\Psi_{\upsilon,q}^{\delta}f(z))''}{(\Psi_{\upsilon,q}^{\delta}f(z))'} - \frac{z(\Psi_{\upsilon,q}^{\delta}f(z))'}{\Psi_{\upsilon,q}^{\delta}f(z)}\right. \\ & + \left.\mu\left(\frac{z(\Psi_{\upsilon,q}^{\delta}f(z))'}{\Psi_{\upsilon,q}^{\delta}f(z)}-1\right)\right] \\ = \quad & 1 + b_1 z + b_2 z^2 + \cdots. \end{aligned} \quad (24)$$

In view of (22), (23), (24), we have

$$p(z) = \Phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right). \quad (25)$$

Using (23) in (25), we get,

$$b_1 = \frac{1}{2}B_1 c_1 \quad \text{and} \quad b_2 = \frac{1}{2}B_1\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}B_2 c_1^2.$$

A computation shows that

$$\begin{aligned} \frac{z(\Psi_{\upsilon,q}^{\delta}f(z))'}{\Psi_{\upsilon,q}^{\delta}f(z)} &= 1 + \Theta_2 a_2 z + (2\Theta_3 a_3 - \Theta_2^2 a_2^2)z^2 \\ &+ (3\Theta_4 a_4 + \Theta_2^3 a_2^3 - 3\Theta_3\Theta_2 a_3 a_2)z^3 \\ &+ \cdots. \end{aligned}$$

Similarly we have

$$1 + \frac{z(\Psi_{\upsilon,q}^{\delta}f(z))''}{(\Psi_{\upsilon,q}^{\delta}f(z))'} = 1 + 2\Theta_2 a_2 z + (6\Theta_3 a_3 - 4\Theta_2^2 a_2^2)z^2 + \cdots.$$

An easy computation shows that

$$\frac{z(\Psi^{\delta}_{\upsilon,q}f(z))'}{\Psi^{\delta}_{\upsilon,q}f(z)}\left(\frac{\Psi^{\delta}_{\upsilon,q}f(z)}{z}\right)^{\mu}$$
$$+\lambda\left[1+\frac{z(\Psi^{\delta}_{\upsilon,q}f(z))''}{(\Psi^{\delta}_{\upsilon,q}f(z))'}-\frac{z(\Psi^{\delta}_{\upsilon,q}f(z))'}{\Psi^{\delta}_{\upsilon,q}f(z)}\right.$$
$$\left.+\mu\left(\frac{z(\Psi^{\delta}_{\upsilon,q}f(z))'}{\Psi^{\delta}_{\upsilon,q}f(z)}-1\right)\right]$$
$$=1+(1+\mu)(1+\lambda)\Theta_2a_2z+(\mu+2)(1+2\lambda)\Theta_3a_3z^2$$
$$+\left(\frac{\mu^2+\mu}{2}-(\mu+3)\lambda-1\right)\Theta_2^2a_2^2z^2+\cdots.$$

In prospect of (24), we see that

$$\begin{aligned} b_1 &= (1+\mu)(1+\lambda)\Theta_2a_2\\ b_2 &= (\mu+2)(1+2\lambda)\Theta_3a_3\\ &\quad+\left(\frac{\mu^2+\mu}{2}-(\mu+3)\lambda-1\right)\Theta_2^2a_2^2\\ &= (\mu+2)(1+2\lambda)\Theta_3a_3+\Lambda\Theta_2^2a_2^2 \end{aligned}$$

where $\Lambda=\frac{1}{2}(\mu^2+\mu)-(\mu+3)\lambda-1$. Equivalently, we have

$$a_2 = \frac{B_1c_1}{2(1+\mu)(1+\lambda)\Theta_2}=\frac{B_1c_1}{2\tau\Theta_2}, \tag{26}$$

$$\begin{aligned} a_3 &= \frac{B_1}{2(\mu+2)(1+2\lambda)\Theta_3}\\ &\quad\times\left[c_2-\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{B_1\Lambda}{[(1+\mu)(1+\lambda)]^2}\right)c_1^2\right]\\ &= \frac{B_1}{2\xi\Theta_3}\left[c_2-\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}\right)c_1^2\right]. \end{aligned} \tag{27}$$

From(26) and applying Lemma 1,, we get

$$|a_2|\leq\left|\frac{B_1}{\tau\Theta_2}\right|.$$

From (27), by using the estimate

$$|c_2-\vartheta c_1^2|\leqq 2\max\{1,|2\vartheta-1|\}$$

where $\vartheta=\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}\right)$, given in Lemma 3 we have

$$\begin{aligned} |a_3| &\leqq \frac{B_1}{(\mu+2)(1+2\lambda)\Theta_3}\\ &\quad\times\max\{1,|2\times\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}\right)-1|\}\\ &= \frac{B_1}{\xi\Theta_3}\max\{1,|-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}|\} \end{aligned}$$

where τ,ξ are as assumed in (21) and $\Lambda=\frac{1}{2}(\mu^2+\mu)-(\mu+3)\lambda-1$.

Using Lemma 2, we prove the following:

Theorem 2. *Let $\mu\geqq0$ and $\lambda\geqq0$ and ν a real number and $\Phi(z)$ be given by (18). If $f\in\mathscr{G}^{\upsilon,\delta,q}_{\mu,\lambda}(\Phi)(z)$, then*

$$|a_3-\nu a_2^2|\leqq\begin{cases}\frac{B_1}{\xi\Theta_3}\left(\frac{B_2}{B_1}-\frac{B_1\Lambda}{\tau^2}-\frac{\nu\xi B_1\Theta_3}{\tau^2\Theta_2^2}\right), & \text{if } \nu\leqq\sigma_1,\\ \frac{B_1}{\Theta_3\xi}, & \text{if } \sigma_1\leqq\nu\leqq\sigma_2,\\ \frac{B_1}{\xi\Theta_3}\left(-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}+\frac{\nu\xi B_1\Theta_3}{\tau^2\Theta_2^2}\right), & \text{if } \nu\geqq\sigma_2,\end{cases}$$

where, for convenience,

$$\sigma_1:=\frac{\tau^2\Theta_2^2}{\xi B_1\Theta_3}\left(-1+\frac{B_2}{B_1}-\frac{B_1\Lambda}{\tau^2}\right),$$

$$\sigma_2:=\frac{\tau^2\Theta_2^2}{\xi B_1\Theta_3}\left(1+\frac{B_2}{B_1}-\frac{B_1\Lambda}{\tau^2}\right),$$

$$\sigma_3:=\frac{\tau^2\Theta_2^2}{\xi B_1\Theta_3}\left(\frac{B_2}{B_1}-\frac{B_1\Lambda}{\tau^2}\right),$$

also τ,ξ are as defined in (21) and $\Lambda=\frac{1}{2}(\mu^2+\mu)-(\mu+3)\lambda-1$.
Further, if $\sigma_1\leqq\nu\leqq\sigma_3$, then

$$|a_3-\nu a_2^2|+\frac{\tau^2\Theta_2^2}{\xi B_1\Theta_3}\left(1-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}+\frac{\nu\xi B_1\Theta_3}{\tau^2\Theta_2^2}\right)|a_2|^2\leqq\frac{B_1}{\xi\Theta_3}.$$

If $\sigma_3\leqq\nu\leqq\sigma_2$, then

$$|a_3-\nu a_2^2|+\frac{\tau^2\Theta_2^2}{\xi B_1\Theta_3}\left(1+\frac{B_2}{B_1}-\frac{B_1\Lambda}{\tau^2}-\frac{\nu\xi B_1\Theta_3}{\tau^2\Theta_2^2}\right)|a_2|^2\leqq\frac{B_1}{\xi\Theta_3}.$$

These results are sharp.

Proof. From (26) and (27),we have

$$\begin{aligned} a_3-\nu a_2^2 &= \frac{B_1}{2\xi\Theta_3}\left[c_2-\frac{c_1^2}{2}\left(1-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}\right)\right]\\ &\quad-\frac{c_1^2}{4}\frac{\nu B_1^2}{(\tau\Theta_2)^2}\\ &= \frac{B_1}{2\xi\Theta_3}\left[c_2-\frac{c_1^2}{2}\left(1-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}+\frac{\nu\xi B_1\Theta_3}{(\tau\Theta_2)^2}\right)\right]. \end{aligned}$$

Therefore, we have

$$a_3-\nu a_2^2=\frac{B_1}{2\xi\Theta_3}(c_2-vc_1^2)$$

where

$$v:=\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}+\frac{\nu\xi B_1}{(\tau\Theta_2)^2}\right).$$

By applying Lemma 2 we get the desired result. To prove the bounds are sharp, we define the functions K_{Φ_n} $(n = 2,3,\ldots)$ with $K_{\Phi_n}(0) = 0 = [K_{\Phi_n}]'(0) - 1$, by

$$\frac{z(\Psi^{\delta}_{\upsilon,q}K_{\Phi_n}(z))'}{\Psi^{\delta}_{\upsilon,q}K_{\Phi_n}(z)}\left(\frac{\Psi^{\delta}_{\upsilon,q}K_{\Phi_n}(z)}{z}\right)^{\mu} + \lambda\left[1+\frac{z(\Psi^{\delta}_{\upsilon,q}K_{\Phi_n}(z))''}{(\Psi^{\delta}_{\upsilon,q}K_{\Phi_n}(z))'}-\frac{z(\Psi^{\delta}_{\upsilon,q}K_{\Phi_n}(z))'}{\Psi^{\delta}_{\upsilon,q}K_{\Phi_n}(z)} + \mu\left(\frac{z(\Psi^{\delta}_{\upsilon}K_{\Phi_n}(z))'}{\Psi^{\delta}_{\upsilon,q}K_{\Phi_n}(z)}-1\right)\right] = \Phi(z^{n-1})$$

and the functions F_ρ and G_ρ $(0 \leqq \rho \leqq 1)$, respectively, with $F_\rho(0) = 0 = F'_\rho(0) - 1$ and $G_\rho(0) = 0 = G'_\rho(0) - 1$ by

$$\frac{z(\Psi^{\delta}_{\upsilon,q}F_\rho(z))'}{\Psi^{\delta}_{\upsilon,q}F_\rho(z)}\left(\frac{\Psi^{\delta}_{\upsilon,q}F_\rho(z)}{z}\right)^{\mu} + \lambda\left[1+\frac{z(\Psi^{\delta}_{\upsilon,q}F_\rho(z))''}{(\Psi^{\delta}_{\upsilon,q}F_\rho(z))'}-\frac{z(\Psi^{\delta}_{\upsilon,q}F_\rho(z))'}{\Psi^{\delta}_{\upsilon,q}F_\rho(z)} + \mu\left(\frac{z(\Psi^{\delta}_{\upsilon,q}F_\rho(z))'}{\Psi^{\delta}_{\upsilon,q}F_\rho(z)}-1\right)\right] = \Phi\left(\frac{z(z+\rho)}{1+\rho z}\right),$$

and

$$\frac{z(\Psi^{\delta}_{\upsilon,q}G_\rho(z))'}{\Psi^{\delta}_{\upsilon,q}G_\rho(z)}\left(\frac{\Psi^{\delta}_{\upsilon,q}G_\rho(z)}{z}\right)^{\mu} + \lambda\left[1+\frac{z(\Psi^{\delta}_{\upsilon,q}G_\rho(z))''}{(\Psi^{\delta}_{\upsilon,q}G_\rho(z))'}-\frac{z(\Psi^{\delta}_{\upsilon}G_\rho(z))'}{\Psi^{\delta}_{\upsilon,q}F_\rho(z)} + \mu\left(\frac{z(\Psi^{\delta}_{\upsilon,q}G_\rho(z))'}{\Psi^{\delta}_{\upsilon,q}G_\rho(z)}-1\right)\right] = \Phi\left(-\frac{z(z+\rho)}{1+\rho z}\right),$$

respectively.

Clearly the functions $K_{\Phi_n}, F_\rho, G_\rho \in \mathscr{G}^{\upsilon,\delta,q}_{\mu,\lambda}(\Phi)$. Also we write $K_\Phi := K_{\Phi_2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the bounds are sharp if and only if f is K_Φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the bounds are sharp if and only if f is K_{Φ_3} or one of its rotations. If $\mu = \sigma_1$ then the bounds are sharp if and only if f is F_ρ or one of its rotations. If $\mu = \sigma_2$ then the bounds are sharp if and only if f is G_ρ or one of its rotations.

By making use of Lemma 3, we immediately obtain the following:

Theorem 3. *Let $\mu \geqq 0$ and $\lambda \geqq 0$ further, let $\Phi(z)$ be of the form (18). If $f \in \mathscr{G}^{\upsilon,\delta,q}_{\mu,\lambda}(\Phi)$, then for complex ν, we have*

$$|a_3 - \nu a_2^2| = \frac{B_1}{\xi\Theta_3}\max\left\{1, \left|-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}+\frac{\nu\xi\Theta_3}{\tau^2\Theta_2^2}B_1\right|\right\}$$

where ξ,τ are as assumed in (21) and $\Lambda = \frac{1}{2}(\mu^2+\mu) - (\mu+3)\lambda - 1$. The result is sharp.

3 Coefficient inequalities for the function $f^{-1} \in \mathscr{G}^{\upsilon,\delta,q}_{\mu,\lambda}(\Phi)$

Theorem 4. *If $f \in \mathscr{G}^{\upsilon,\delta,q}_{\mu,\lambda}(\Phi)$ and $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$ is the inverse function of f with $\left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$ the Koebe domain of the class $f \in \mathscr{G}^{\upsilon,\delta,q}_{\mu,\lambda}(\Phi)$, then for any complex number ν, we have*

$$|\, d_3 - \nu d_2^2 \,| \leq \frac{B_1}{\xi\Theta_3} \times \max\left\{1, \left| -\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}+\frac{(2-\nu)B_1\xi\Theta_3}{\tau^2\Theta_2^2} \right| \right\}. \tag{28}$$

where τ,ξ are as assumed in (21) and $\Lambda = \frac{1}{2}(\mu^2+\mu) - (\mu+3)\lambda - 1$.

Proof. As

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n \tag{29}$$

is the inverse function of f, it can be seen that

$$f^{-1}(f(z)) = f\{f^{-1}(z)\} = z. \tag{30}$$

From equations (1) and (30), we get

$$f^{-1}(z + \sum_{n=2}^{\infty} a_n z^n) = z. \tag{31}$$

From (30) and (31), one can obtain

$$z + (a_2+d_2)z^2 + (a_3+2a_2d_2+d_3)z^3 + \ldots\ldots\ldots = z. \tag{32}$$

By equating corresponding coefficients , of (32), we have

$$d_2 = -a_2 \tag{33}$$

$$d_3 = 2a_2^2 - a_3. \tag{34}$$

From relations (26),(27),(33) and (34)

$$d_2 = -\frac{B_1c_1}{2(1+\mu)(1+\lambda)\Theta_2} = -\frac{B_1c_1}{2\tau\Theta_2}; \tag{35}$$

$$\begin{aligned} d_3 &= \frac{B_1^2c_1^2}{2\tau^2\Theta_2^2} - \frac{B_1}{2\xi\Theta_3}\left(c_2 - \frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}\right)c_1^2\right); \\ &= \frac{B_1}{2\xi\Theta_3}\left[-c_2+\frac{c_1^2}{2}\left(1-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}+\frac{2B_1\xi\Theta_3}{\tau^2\Theta_2^2}\right)\right]; \\ &= \frac{-B_1}{2\xi\Theta_3}\left[c_2-\frac{c_1^2}{2}\left(1-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}+\frac{2B_1\xi\Theta_3}{\tau^2\Theta_2^2}\right)\right]; \end{aligned} \tag{36}$$

where τ,ξ are as assumed in (21) and $\Lambda = \frac{1}{2}(\mu^2+\mu) - (\mu+3)\lambda - 1$. For any complex number ν, consider

$$d_3 - \nu d_2^2 = -\frac{B_1}{2\xi\Theta_3} \times \left[c_2 - \frac{c_1^2}{2}\left(1-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}+\frac{(2-\nu)B_1\xi\Theta_3}{\tau^2\Theta_2^2}\right)\right]. \tag{37}$$

Taking modulus on both sides and by applying Lemma 3 on the right hand side of (37), one can obtain the result as in (28). Hence this completes the proof.

4 Application to Functions Defined by Poisson distribution

A variable χ is said to be Poisson distributed if it takes the values $0,1,2,3,\cdots$ with probabilities $e^{-\kappa}$, $m\frac{e^{-\kappa}}{1!}$, $\kappa^2\frac{e^{-\kappa}}{2!}$, $\kappa^3\frac{e^{-\kappa}}{3!},\ldots$ respectively, where κ is called the parameter. Thus

$$P(\chi=r)=\frac{\kappa^r e^{-\kappa}}{r!},\ r=0,1,2,3,\cdots.$$

In [14], Porwal introduced a power series whose coefficients are probabilities of Poisson distribution

$$\mathscr{I}(\kappa,z)=z+\sum_{n=2}^{\infty}\frac{\kappa^{n-1}}{(n-1)!}e^{-\kappa}z^n,\qquad z\in\mathbb{D},$$

where $\kappa>0$. We note by the familiar ratio test that the radius of convergence of the above series is infinity. Lately, using the Hadamard product, Porwal[14] (see also, [15,16,17]) introduced a new linear operator $\mathscr{I}^{\kappa}(z):\mathscr{A}\to\mathscr{A}$ defined by

$$\begin{aligned}\mathscr{I}^{\kappa}f=\mathscr{I}(\kappa,z)*f(z)&=z+\sum_{n=2}^{\infty}\frac{\kappa^{n-1}}{(n-1)!}e^{-\kappa}a_nz^n,\\ &=z+\sum_{n=2}^{\infty}\psi_n(\kappa)a_nz^n,\qquad z\in\mathbb{D},\end{aligned}$$

where

$$\psi_n=\psi_n(\kappa)=\frac{\kappa^{n-1}}{(n-1)!}e^{-\kappa}.$$

In particular

$$\psi_2=\kappa e^{-\kappa} \text{and } \psi_3=\frac{\kappa^2}{2}e^{-\kappa}\tag{38}$$

We describe the class $\mathscr{G}^{\upsilon,\delta,q}_{\kappa,\mu,\lambda}(\Phi)$ as below:

$$\mathscr{G}^{\upsilon,\delta,q}_{\kappa,\mu,\lambda}(\Phi):=\{f\in\mathscr{A}\quad\text{and}\quad\mathscr{I}^{\kappa}f\in\mathscr{G}^{\upsilon,\delta,q}_{\mu,\lambda}(\Phi)\}$$

where $\mathscr{G}^{\upsilon,\delta,q}_{\mu,\lambda}(\Phi)$ is given by Definition 2. We find the coefficient estimate for $f\in\mathscr{G}^{\upsilon,\delta,q}_{\kappa,\mu,\lambda}(\Phi)$, from the corresponding estimate for functions in the class $\mathscr{G}^{\upsilon,\delta,q}_{\mu,\lambda}(\Phi)$. Applying Theorem 2 and 3, for $\mathscr{I}^{\kappa}f=\mathscr{I}(\kappa,z)*f(z)=z+\psi_2a_2z^2+\psi_3a_3z^3+\cdots$, we get the following Theorems 5 and 6 after a noticeable modification of the parameter ν.

Theorem 5. *Let $\mu\geqq 0;\lambda\geqq 0$ and $\Phi(z)$ be given by (18). If $f\in\mathscr{G}^{\upsilon,\delta,q}_{\kappa,\mu,\lambda}(\Phi)$, then for complex μ, we have*

$$|a_3-\nu a_2^2|=\frac{2B_1}{\xi\Theta_3\kappa^2e^{-\kappa}}\times\max\left\{1,\left|-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}+\frac{\nu\xi B_1\Theta_3}{2\tau^2\Theta_2^2e^{-\kappa}}\right|\right\}$$

where τ,ξ are as assumed in (21) and $\Lambda=\frac{1}{2}(\mu^2+\mu)-(\mu+3)\lambda-1$. The result is sharp.

Theorem 6. *Let $\mu\geqq 0;\lambda\geqq 0$ and ν a real number. Further, let $\Phi(z)$ be given in (18). If $f\in\mathscr{G}^{\upsilon,\delta,q}_{\kappa,\mu,\lambda}(\Phi)$, then*

$$|a_3-\nu a_2^2|\leqq\begin{cases}\frac{2B_1}{\xi\Theta_3\kappa^2e^{-\kappa}}\left(\frac{B_2}{B_1}-\frac{B_1\Lambda}{\tau^2}-\frac{\nu\xi B_1\Theta_3}{2\tau^2\Theta_2^2e^{-\kappa}}\right),\ \text{if } \nu\leqq\sigma_1,\\ \frac{2B_1}{\xi\Theta_3\kappa^2e^{-\kappa}},\ \text{if } \sigma_1\leqq\nu\leqq\sigma_2,\\ \frac{2B_1}{\xi\Theta_3\kappa^2e^{-\kappa}}\left(-\frac{B_2}{B_1}+\frac{B_1\Lambda}{\tau^2}+\frac{\nu\xi B_1\Theta_3}{2\tau^2\Theta_2^2e^{-\kappa}}\right),\ \text{if } \nu\geqq\sigma_2,\end{cases}$$

where, for convenience,

$$\sigma_1=\frac{2\tau^2\Theta_2^2e^{-\kappa}}{\xi B_1\Theta_3}\left(-1+\frac{B_2}{B_1}-\frac{B_1\Lambda}{\tau^2}\right),$$
$$\sigma_2=\frac{2\tau^2\Theta_2^2e^{-\kappa}}{\xi B_1\Theta_3}\left(1+\frac{B_2}{B_1}-\frac{B_1\Lambda}{\tau^2}\right)$$

$\Lambda=\frac{1}{2}(\mu^2+\mu)-(\mu+3)\lambda-1$ and τ,ξ are as assumed in (21). These results are sharp.

Concluding Remarks

Suitably specializing the parameters μ and λ as stated in Definitions 3 to 5, in Theorems 2,3 and 4 one can easily state above result for the function classes defined in Definitions 3 to 5 related with Bessel Functions. Also , further fixing Φ as illustrated below:

1. For $(0<\alpha\leq 1)$ and $-1\leq B<A\leq 1$, taking the function Φ as

$$\begin{aligned}\Phi(z)&=\left(\frac{1+Az}{1+Bz}\right)^{\alpha}\\ &=1+\alpha(A-B)z\\ &\quad-\frac{\alpha}{2}[2B(A-B)+(1-\alpha)(A-B)^2]z^2+\cdots\end{aligned}$$

which gives $B_1=\alpha(A-B)$ and $B_2=-\frac{\alpha}{2}[2B(A-B)+(1-\alpha)(A-B)^2]$.

2. If we take $\alpha=1$ and $-1\leq B<A\leq 1$, then we have

$$\Phi(z)=\frac{1+Az}{1+Bz}=1+(A-B)z+B(A-B)z^2+\cdots\tag{39}$$

thus we have $B_1=A-B$ and $B_2=B(A-B)$.

3.By fixing $A = 1$ and $B = -1$ we have

$$\Phi(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots \quad (40)$$

thus we have $B_1 = 2$ and $B_2 = 2$

4.Further for some $c \in (0,1]$, taking

$$\Phi(z) = \sqrt{1+cz} = 1 + \frac{c}{2}z - \frac{c^2}{8}z^2 + \ldots \quad (41)$$

then the class is said to be associated with the right -loop of the Cassinian Ovals [18]. In particular if $c = 1$ then the class is associated with right-half of the lemniscate of Bernoulli [19] is given by

$$\Phi(z) = \sqrt{1+z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \ldots. \quad (42)$$

which gives $B_1 = \frac{1}{2}$ and $B_2 = -\frac{1}{8}$.

5.Taking

$$\Phi(z) = z + \sqrt{1+z^2} = 1 + z + \frac{1}{2}z^2 - \frac{1}{8}z^4 + \ldots. \quad (43)$$

which gives $B_1 = 1$ and $B_2 = \frac{1}{2}$,
then the class is said to be associated with the right crescent [20].

6.Again by taking

$$\Phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2 \quad (44)$$

which gives $B_1 = \frac{4}{3}$ and $B_2 = \frac{2}{3}$,
then the class is said to be associated with the cardioid [21].

7.By taking $\Phi(z) = 1 + \frac{8}{\pi^2}z + \frac{16}{3\pi^2}z^2$ where f is a parabolic starlike function (see [22]) in conic regions,

one can deduce the analogues results of above theorems, we left the proof as exercise to interested readers.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

Conflict of Interest The authors declare that they have no conflict of interest

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. **225**, Marcel Dekker Incorporated, New York and Basel, 2000.

[2] R.Szász and P.A.Kupan, About the univalence of the Bessel functions,*Studia Univ.Babes-Bolyai Math*. **54**(1),127–132 (2009)

[3] Á.Baricz, Geometric propertis of generalized Bessel functions,*Publ.Math.Debr*. **73** ,155–178(2008)

[4] K.A.Selvakumaran and R.Szász,Certain geometric properties of an integral operator involving Bessel functions, *Kyungpook Math.J*. **58**,507–517(2018).

[5] F. H. Jackson,On q-functions and a certain difference operator, *Trans. Royal Soc.of Edinburgh*, **46**, 253–281(1908).

[6] S. Kanas and D. Răducanu,Some subclass of analytic functions related to conic domains, *Math. Slovaca* **64**, no. 5, 1183–1196(2014).

[7] S.Ruscheweyh,New criteria for univalent functions, *Proc. Amer. Math. Soc*. **49** , 109–115(1975).

[8] H. Aldweby and M. Darus,Some subordination results on q -analogue of Ruscheweyh differential operator, *Abst. Appl. Anal*., 2014,6 pages, Article ID 958563,(2014).

[9] W. Ma and D. Minda, *A Unified treatment of some special classes of univalent functions*, in: Proceedings of the Conference on Complex Analysis, Z. Li, F. Ren, L. Yang, and S. Zhang(Eds.), Int. Press (1994), 157–169.

[10] D. Guo and M-S. Liu,On certain subclass of Bazilevič functions, *J.Inequal. Pure Appl. Math.* **8**(1), Article 12, 1–11(2007).

[11] H. M. Srivastava, A.K. Mishra and M.K. Das, The fekete-szegö-problem for a subclass of close-to-convex functions,*Complex Variables Theory Appl.,* **44** (2), 145–163 (2001).

[12] C. Carathéodory,Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen, *Rend. Circ. Mat. Palermo.*, **2***32*, 193–217(1911).

[13] R. J. Libera and E. J. Zlotkiewicz,Coefficient bounds for the inverse of a function with derivative in P , *Proc. Amer. Math. Soc*.,**87**(2): 251–257 (1983).

[14] S. Porwal,An application of a Poisson distribution series on certain analytic functions, *J. Complex Anal*., Art. ID 984135, 1–3 (2014).

[15] G. Murugusundaramoorthy, Subclasses of starlike and convex functions involving Poisson distribution series, *Afr. Mat*., **28**, 1357-1366 (2017).

[16] G. Murugusundaramoorthy, K. Vijaya and S. Porwal,Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series, *Hacettepe J. Math. Stat*. **45** (4), 1101-1107(2016).

[17] S. Porwal and M. Kumar, A unified study on starlike and convex functions associated with Poisson distribution series, *Afr. Mat*., **27** (5), 1021-1027(2016).

[18] M.K.Aouf,J.Dziok and J.Sokol, On a subclass of strongly starlike function, *Appl. Math. Lett*.,**24**,27–32(2011).

[19] J.Sokol and J.Stankiewicz,Radius of convexity of some subclasses of strongly starlike functions, *Fol.Sci.Univ.Tech.Res*., **147**,101–105(1996).

[20] R.K.Raina,J.Sokol,Some properties related to a certain class of starlike functions, *C.R.Acad.Sci.Paris,Ser.I* **353**,973-978(2015).

[21] V.Ravichandran, K.Sharma,Sufficient conditions for starlikeness, *J.Korean Math.Soc*., **52**(4),727–749(2015).

[22] H. M. Srivastava and A. K. Mishra, Applications of fractional calculus to parabolic starlike and uniformly convex functions, *Computer Math. Appln.* **39**, 57–69(2000).

Murugusundaramoorthy Gangadharan received the Ph.D., degree in Mathematics from University of Madras. His research interests are in the areas of pure mathematics especially in complex analysis (Geometric Function Theory). He has published research articles in the topics viz.,univalent functions, harmonic functions, special functions, integral operator, differential subordination and meromorphic functions in reputed international journals of mathematical sciences. Since 1995, he is working as a Professor in the Department of Mathematics, VIT University.

Hemen Dutta belongs to the Department of Mathematics at Gauhati University as a regular faculty member. He does research in the areas of functional analysis and mathematical modelling. He has to his credit several research papers in reputed journals and books.