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On Additive ρ-Functional Equations Arising from Cauchy-Jensen Functional Equations and their Stability

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Abstract: In this paper, three new ρ–functional equations arising from Cauchy-Jensen functional equations are presented. Their relations with the additive function are obtained, and these ρ–functional equations are solved without any regularity assumptions. Their stability is also carried out in complex Banach spaces.

Keywords: Functional equation, functional inequality, stability

1 Introduction

The stability theory of functional equations originated from a question of Ulam in his famous lecture in 1940 to the Mathematics Club of the University of Wisconsin. Such question is related to the stability of group homomorphisms as follows: *Let* $(G,*_1)$ *be a group,* $(H_d,*_2)$ *a metric group, and* $f: G\to H_d$ *a mapping. For any* $\varepsilon>0$*, does there exist a* $\delta>0$ *such that*

$$d(f(x*_1 y), f(x)*_2 f(y))<\delta$$

for all $x,y\in G$ *implies there is a homomorphism* $A: G\to H_d$ *such that*

$$d(f(x),A(x))<\varepsilon \qquad \text{for all } x\in G?$$

If the answer is affirmative, then we say that the functional equation

$$f(x*_1 y)=f(x)*_2 f(y) \qquad \text{for all } x,y\in G$$

is *stable*. This kind of such question forms the basic of stability theory. In 1941, Hyers [1] obtained the first important result in this field as follows:

Theorem 1.1 ([1]). Let G and H be Banach spaces and let f be a δ–linear transformation of G into H, i.e.,

$$\|f(x+y)-f(x)-f(y)\|<\delta$$

for all $x,y\in G$, where $\delta>0$. Then the limit

$$\ell(x):=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

exists for each $x\in G$, ℓ is a linear transformation of G into H, and the inequality

$$\|f(x)-\ell(x)\|\le\delta$$

is true for all $x\in G$. Moreover, ℓ is the only linear transformation satisfying this inequality.

This theorem was generalized by Aoki [2] for additive mappings, and by Rassias [3] for linear mappings by considering an unbounded Cauchy difference. Latter, the Rassias theorem was generalized by replacing the unbounded Cauchy difference by a general control function, in the spirit of Rassias' approach, by Găvruta [4].

Recently, a number of papers [5,6,7,8] have been published dealing with the following ρ–functional inequalities:

$$\begin{aligned}&\|f(x+y)-f(x)-f(y)\|\\ &\le\left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\|,\end{aligned} \tag{1}$$

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\|\\ &\le\|\rho\left(f(x+y)-f(x)-f(y)\right)\|,\end{aligned} \tag{2}$$

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$$\left\|f\left(\sum_{j=1}^{k} x_j\right)-\sum_{j=1}^{k} f(x_j)\right\| \leq \left\|\rho\left(kf\left(\frac{1}{k}\sum_{j=1}^{k} x_j\right)-\sum_{j=1}^{k} f(x_j)\right)\right\|, \quad (3)$$

$$\left\|kf\left(\frac{1}{k}\sum_{j=1}^{k} x_j\right)-\sum_{j=1}^{k} f(x_j)\right\| \leq \left\|\rho\left(f\left(\sum_{j=1}^{k} x_j\right)-\sum_{j=1}^{k} f(x_j)\right)\right\|, \quad (4)$$

$$\|f(x+y)-f(x)-f(y)\| \leq \|\rho(f(x-y)-f(x)-f(-y))\|, \quad (5)$$

$$\|f(x+y+z)-f(x)-f(y)-f(z)\| \leq \left\|\hat{\rho}\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right)\right\|, \quad (6)$$

$$\left\|2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right\| \leq \left\|\rho\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z)\right)\right\|, \quad (7)$$

$$\|f(x+y+z)-f(x)-f(y)-f(z)\| \leq \left\|\rho\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z)\right)\right\|, \quad (8)$$

where ρ and $\hat{\rho}$ are nonzero complex numbers with $|\rho|<1$ and $|\hat{\rho}|<1/2$, as well as their stability. It is easily see that the additive mapping satisfies the inequalities (1) and (2), while the inequalities (3) and (4) are the generalized forms of (1) and (2), respectively. These first two functional inequalities are in the forms of Cauchy and Jensen functional equations. The inequalities (5), (6), (7) and (8) are related to the additive mapping. Closely involved to the inequalities (6), (7) and (8) are following functional equations:

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)=f(x)+2f(z), \quad (9)$$

$$f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)=f(y), \quad (10)$$

$$2f\left(\frac{x+y}{2}+z\right)=f(x)+f(y)+2f(z), \quad (11)$$

which were introduced and solved in 2006 by Baak [9]; their stability was also proved in complex Banach spaces. These equations are called the *Cauchy–Jensen functional equations*. Notice that the functional equation (11) becomes to the Cauchy additive functional equation when $y=x$, while when $z=0$ it becomes to the Jensen functional equation.

In the present work, the following ρ–functional equations:

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)-f(x)-2f(z) = \rho_1\left(f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)-f(y)\right), \quad (12)$$

$$g\left(\frac{x+y}{2}+z\right)-g\left(\frac{x-y}{2}+z\right)-g(y) = \rho_2\left(2g\left(\frac{x+y}{2}+z\right)-g(x)-g(y)-2g(z)\right), \quad (13)$$

$$h\left(\frac{x+y}{2}+z\right)+h\left(\frac{x-y}{2}+z\right)-h(x)-2h(z) = \rho_3\left(2h\left(\frac{x+y}{2}+z\right)-h(x)-h(y)-2h(z)\right), \quad (14)$$

are derived via their inequality forms (in Section 2), without any regularity assumptions, where ρ_1,ρ_2,ρ_3 are fixed nonzero complex numbers with $|\rho_1|<1$ and $|\rho_2|,|\rho_3|<1/2$. An analysis of their stability is investigated in complex Banach spaces (in Section 3). These equations arised from the regarding Cauchy-Jensen functional equations. In Section 4, the isomorphisms between unital Banach algebras are here also carried out. As this section is related to the multiplicative equation, the readers can be read the following works [10,11,12, 13] for more details involving multiplicative inverse functional equations which were published recently. For the last section, we give some remarks related to the duality of some theorems.

2 Additive ρ–Functional Inequalities and Equations

Throughout this paper, denote generalically, unless otherwise specified, by X a complex normed space, and denote by Y a complex Banach space. Let ρ_1,ρ_2,ρ_3 be fixed nonzero complex numbers with $|\rho_1|<1$ and $|\rho_2|,|\rho_3|<1/2$.

The solutions of our results involve the use of *Cauchy additive* and *Jensen functional equations* ([14, Chapter 1]), which are the functional equations of the following forms, respectively,

$$f(x+y)=f(x)+f(y), \quad f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}.$$

The function solutions of these equations are called *Cauchy additive mapping* and *Jensen mapping*, respectively. This section shows the relation between three focussed ρ-functional equations with the additive functional equation. We begin with the following lemma.

Lemma 2.1. Let X and Y be real or complex vector spaces. Then the mappings $f,g,h : X\to Y$ satisfy the following functional inequalities

$$\left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)-f(x)-2f(z)\right\| \leq \left\|\rho_1\left(f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)-f(y)\right)\right\|, \tag{15}$$

$$\left\|g\left(\frac{x+y}{2}+z\right)-g\left(\frac{x-y}{2}+z\right)-g(y)\right\| \leq \left\|\rho_2\left(2g\left(\frac{x+y}{2}+z\right)-g(x)-g(y)-2g(z)\right)\right\|, \tag{16}$$

$$\left\|h\left(\frac{x+y}{2}+z\right)+h\left(\frac{x-y}{2}+z\right)-h(x)-2h(z)\right\| \leq \left\|\rho_3\left(2h\left(\frac{x+y}{2}+z\right)-h(x)-h(y)-2h(z)\right)\right\| \tag{17}$$

for all $x,y,z \in X$ if and only if $f,g,h : X \to Y$ are additive.

Proof. As the proofs of (15), (16), and (17) are similar, we here show only that of (17). Taking $y = x$ in (17), we have

$$\|h(x+z)-h(x)-h(z)\| \leq \|2\rho_3\,(h(x+z)-h(x)-h(z))\|$$

for all $x,y,z \in X$. Since $|\rho_3| < \frac{1}{2}$, the result follows. The converse is obviously holds.

Immediate from Lemma 2.1 is:

Corollary 2.1. Let X and Y be real or complex vector spaces. Then the mappings $f,g,h : X \to Y$ satisfy the following functional equations

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)-f(x)-2f(z) = \rho_1\left(f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)-f(y)\right),$$

$$g\left(\frac{x+y}{2}+z\right)-g\left(\frac{x-y}{2}+z\right)-g(y) = \rho_2\left(2g\left(\frac{x+y}{2}+z\right)-g(x)-g(y)-2g(z)\right),$$

$$h\left(\frac{x+y}{2}+z\right)+h\left(\frac{x-y}{2}+z\right)-h(x)-2h(z) = \rho_3\left(2h\left(\frac{x+y}{2}+z\right)-h(x)-h(y)-2h(z)\right),$$

for all $x,y,z \in X$ if and only if $f,g,h : X \to Y$ are additive.

3 Stability Results

In this section, we investigate the stability of the functional equations (12), (13), and (14). The results so obtained can be applied to the next section. We now first prove the stability of the ρ–functional equation (12) as follows:

Theorem 3.1. Let $\phi : X^3 \to [0,\infty)$ be a fixed function satisfying

$$\Phi(x,y,z) := \sum_{j=1}^{\infty} 2^j \phi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^j}\right) < \infty \tag{18}$$

for all $x,y,z \in X$. Assume that $f : X \to Y$ is a function satisfying the following inequality

$$\left\|f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)-f(x)-2f(z) -\rho_1\left(f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)-f(y)\right)\right\| \leq \phi(x,y,z) \tag{19}$$

for all $x,y,z \in X$. Then there exists the unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \leq \left(\frac{1}{2\,|1-\rho_1|}\right)\Phi(x,x,x) \tag{20}$$

for all $x \in X$.

Proof. Letting $x = y = z$ in (19), we get

$$\|f(2x)-2f(x)\| \leq \left(\frac{1}{|1-\rho_1|}\right)\phi(x,x,x) \tag{21}$$

and so

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| \leq \left(\frac{1}{|1-\rho_1|}\right)\phi\left(\frac{x}{2},\frac{x}{2},\frac{x}{2}\right) \tag{22}$$

for all $x \in X$. Triangle inequality and (22) yield

$$\begin{aligned}
&\left\|2^m f\left(\frac{x}{2^m}\right)-2^n f\left(\frac{x}{2^n}\right)\right\| \\
&\leq \sum_{i=n}^{m-1}\left\|2^i f\left(\frac{x}{2^i}\right)-2^{i+1} f\left(\frac{x}{2^{i+1}}\right)\right\| \\
&\leq \sum_{i=n}^{m-1}\left(\frac{2^i}{|1-\rho_1|}\right)\phi\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}\right) \\
&= \left(\frac{1}{2\,|1-\rho_1|}\right)\sum_{i=n}^{m-1} 2^{i+1}\phi\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}\right) \qquad (23)\\
&= S_{m-1}-S_{n-1}, \qquad (24)
\end{aligned}$$

where

$$S_k := \left(\frac{1}{2\,|1-\rho_1|}\right)\sum_{i=1}^{k} 2^{i+1}\phi\left(\frac{x}{2^{i+1}},\frac{x}{2^{i+1}},\frac{x}{2^{i+1}}\right) < \infty,$$

and so there exists $s \geq 0$ such that $S_k \to s$ as $n \to \infty$. Taking limit as $n,m \to \infty$ in (24), we have

$$\left\|2^m f\left(\frac{x}{2^m}\right)-2^n f\left(\frac{x}{2^n}\right)\right\| \to 0 \quad (n,m \to \infty),$$

showing that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence in Y. By the completeness of Y, one can define a mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{25}$$

for all $x \in X$. Putting $n = 0$ and also taking limit as $m \to \infty$ in (23), and using (25) we get

$$\|f(x) - A(x)\| \leq \left(\frac{1}{2|1-\rho_1|}\right) \Phi(x,x,x) \quad (26)$$

for all $x \in X$, as desired. To show that A is an additive, consider

$$\begin{aligned}
&\left\| A\left(\frac{x+y}{2}+z\right) + A\left(\frac{x-y}{2}+z\right) - A(x) - 2A(z) \right.\\
&\left. -\rho_1\left(A\left(\frac{x+y}{2}+z\right) - A\left(\frac{x-y}{2}+z\right) - A(y)\right)\right\|\\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y}{2^n\cdot 2}+\frac{z}{2^n}\right) + f\left(\frac{x-y}{2^n\cdot 2}+\frac{z}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right.\\
&\quad -2f\left(\frac{z}{2^n}\right) - \rho_1\left(f\left(\frac{x+y}{2^n\cdot 2}+\frac{z}{2^n}\right) - f\left(\frac{x-y}{2^n\cdot 2}+\frac{z}{2^n}\right)\right.\\
&\quad \left.\left. -f\left(\frac{y}{2^n}\right)\right)\right\|\\
&\leq \lim_{n\to\infty} 2^n \phi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) = 0,
\end{aligned}$$

and the result follows from Corollary 2.1.

It now remains to verify the uniqueness of a mapping A. Indeed, assume that there exists another additive mapping $\hat{A} : X \to Y$ satisfying the inequality (20). Since the mappings A and $\hat{A}$ are additive, we see that

$$\begin{aligned}
\|A(x) - \hat{A}(x)\| &= \left\|2^t A\left(\frac{x}{2^t}\right) - 2^t \hat{A}\left(\frac{x}{2^t}\right)\right\|\\
&\leq \left\|2^t A\left(\frac{x}{2^t}\right) - 2^t f\left(\frac{x}{2^t}\right)\right\|\\
&\quad + \left\|2^t f\left(\frac{x}{2^t}\right) - 2^t \hat{A}\left(\frac{x}{2^t}\right)\right\|\\
&\leq \left(\frac{2^t}{|1-\rho_1|}\right) \Phi\left(\frac{x}{2^t},\frac{x}{2^t},\frac{x}{2^t}\right)\\
&\to 0 \quad (t \to \infty).
\end{aligned}$$

This proves the uniqueness of A, and the proof of this theorem is completed.

Immediate from Theorem 3.1 are the following, which illustrate some examples of functions ϕ satisfying (18).

Corollary 3.1. Let $p > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying the following inequality

$$\begin{aligned}
&\left\| f\left(\frac{x+y}{2}+z\right) + f\left(\frac{x-y}{2}+z\right) - f(x) - 2f(z) \right.\\
&\left. -\rho_1\left(f\left(\frac{x+y}{2}+z\right) - f\left(\frac{x-y}{2}+z\right) - f(y)\right)\right\|\\
&\leq \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \qquad (27)
\end{aligned}$$

for all $x,y,z \in X$. Then there exists the unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{3\theta}{(2^p-2)|1-\rho_1|}\right) \|x\|^p \quad (28)$$

for all $x \in X$.

Proof. Substituting $\phi(x,y,z) := \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right)$ into (19), the result follows.

Corollary 3.2. Let p_1, p_2, p_3 and θ be positive real numbers with $p_1 + p_2 + p_3 < 1$, and let $f : X \to Y$ be a mapping satisfying the following inequality

$$\begin{aligned}
&\left\| f\left(\frac{x+y}{2}+z\right) + f\left(\frac{x-y}{2}+z\right) - f(x) - 2f(z) \right.\\
&\left. -\rho_1\left(f\left(\frac{x+y}{2}+z\right) - f\left(\frac{x-y}{2}+z\right) - f(y)\right)\right\|\\
&\leq \theta \cdot \left(\|x\|^{p_1} \cdot \|y\|^{p_2} \cdot \|z\|^{p_3}\right) \qquad (29)
\end{aligned}$$

for all $x,y,z \in X$. Then there exists the unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{\theta}{(2^{p_1+p_2+p_3}-2)|1-\rho_1|}\right) \|x\|^{p_1+p_2+p_3} \quad (30)$$

for all $x \in X$.

Proof. Substituting $\phi(x,y,z) := \theta \cdot \left(\|x\|^{p_1} \cdot \|y\|^{p_2} \cdot \|z\|^{p_3}\right)$ into (19), the result follows.

The same conclusion as Therem 3.1 remains holds when the function ϕ satisfies the condition that similar to (18).

Theorem 3.2. Let $\phi : X^3 \to [0,\infty)$ be a fixed function satisfying

$$\Phi(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \phi\left(2^j x, 2^j y, 2^j z\right) < \infty \quad (31)$$

for all $x,y,z \in X$. Assume that $f : X \to Y$ be a function satisfying the following inequality

$$\begin{aligned}
&\left\| f\left(\frac{x+y}{2}+z\right) + f\left(\frac{x-y}{2}+z\right) - f(x) - 2f(z) \right.\\
&\left. -\rho_1\left(f\left(\frac{x+y}{2}+z\right) - f\left(\frac{x-y}{2}+z\right) - f(y)\right)\right\| \leq \phi(x,y,z) \qquad (32)
\end{aligned}$$

for all $x,y,z \in X$. Then there exists the unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{1}{2|1-\rho_1|}\right) \Phi(x,x,x) \quad (33)$$

for all $x \in X$.

Proof. Letting $x = y = z$ in (32), we get

$$\|f(2x) - 2f(x)\| \leq \left(\frac{1}{|1-\rho_1|}\right) \phi(x,x,x) \quad (34)$$

and so

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \left(\frac{1}{2|1-\rho_1|}\right) \phi(x,x,x) \quad (35)$$

for all $x \in X$. By using the same arguments as in the proof of Theorem 3.1, one can shows that

$$\left\| \frac{1}{2^m} f(2^m) - \frac{1}{2^n} f(2^n) \right\| \leq \left(\frac{1}{2|1-\rho_1|} \right) \sum_{i=n}^{m-1} \frac{1}{2^i} \phi\left(2^i x, 2^i x, 2^i x\right), \quad (36)$$

yielding that the sequence $\left\{\frac{1}{2^n} f(2^n x)\right\}$ is a Cauchy sequence in Y. By the completeness of Y, one can define a mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n) \quad (37)$$

for all $x \in X$. Using (36) and (37), we get

$$\|f(x) - A(x)\| \leq \left(\frac{1}{2|1-\rho_1|} \right) \Phi(x,x,x) \quad (38)$$

for all $x \in X$, which is the desired assertion. The rest of the proof is similar to that of Theorem 3.1.

Similar to Corollaries 3.1 and 3.2 are the following results whose analogous proofs are omitted.

Corollary 3.3. Let $p > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying the following inequality

$$\Big\| f\left(\frac{x+y}{2}+z\right) + f\left(\frac{x-y}{2}+z\right) - f(x) - 2f(z) - \rho_1\left(f\left(\frac{x+y}{2}+z\right) - f\left(\frac{x-y}{2}+z\right) - f(y)\right)\Big\| \leq \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \quad (39)$$

for all $x,y,z \in X$. Then there exists the unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{3\theta}{(2-2^p)|1-\rho_1|} \right) \|x\|^p \quad (40)$$

for all $x \in X$.

Corollary 3.4. Let p_1, p_2, p_3 and θ be positive real numbers with $p_1 + p_2 + p_3 < 1$, and let $f : X \to Y$ be a mapping satisfying the following inequality

$$\Big\| f\left(\frac{x+y}{2}+z\right) + f\left(\frac{x-y}{2}+z\right) - f(x) - 2f(z) - \rho_1\left(f\left(\frac{x+y}{2}+z\right) - f\left(\frac{x-y}{2}+z\right) - f(y)\right)\Big\| \leq \theta\cdot\left(\|x\|^{p_1} \cdot \|y\|^{p_2} \cdot \|z\|^{p_3}\right) \quad (41)$$

for all $x,y,z \in X$. Then there exists the unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{\theta}{(2-2^{p_1+p_2+p_3})|1-\rho_1|} \right) \|x\|^{p_1+p_2+p_3} \quad (42)$$

for all $x \in X$.

We now move on to the proof of stability of the ρ–functional equation (13).

Theorem 3.3. Let $\phi : X^3 \to [0,\infty)$ be a fixed function satisfying

$$\Phi(x,y,z) := \sum_{j=1}^{\infty} 2^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty \quad (43)$$

for all $x,y,z \in X$. Assume that $g : X \to Y$ be a function satisfying the following inequality

$$\Big\| g\left(\frac{x+y}{2}+z\right) - g\left(\frac{x-y}{2}+z\right) - g(y) - \rho_2\left(2g\left(\frac{x+y}{2}+z\right) - g(x) - g(y) - 2g(z)\right)\Big\| \leq \phi(x,y,z) \quad (44)$$

for all $x,y,z \in X$. Then there exists the unique additive mapping $A : X \to Y$ such that

$$\|g(x) - A(x)\| \leq \left(\frac{1}{2|1-2\rho_2|} \right) \Phi(x,x,x) \quad (45)$$

for all $x \in X$.

Proof. Letting $x = y = z$ in (44), we get

$$\|g(2x) - 2g(x)\| \leq \left(\frac{1}{|1-2\rho_2|} \right) \phi(x,x,x) \quad (46)$$

since $|s_2| < \frac{1}{2}$, and so

$$\left\| g(x) - 2g\left(\frac{x}{2}\right) \right\| \leq \left(\frac{1}{|1-2\rho_2|} \right) \phi\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) \quad (47)$$

for all $x \in X$. The rest of the proof is similar to that of the Theorem 3.1.

Immediate from Theorem 3.3 are the following:

Corollary 3.5. Let $p > 1$ and θ be positive real numbers, and let $g : X \to Y$ be a mapping satisfying the following inequality

$$\Big\| g\left(\frac{x+y}{2}+z\right) - g\left(\frac{x-y}{2}+z\right) - g(y) - \rho_2\left(2g\left(\frac{x+y}{2}+z\right) - g(x) - g(y) - 2g(z)\right)\Big\| \leq \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \quad (48)$$

for all $x,y,z \in X$. Then there exists the unique additive mapping $A : X \to Y$ such that

$$\|g(x) - A(x)\| \leq \left(\frac{3\theta}{(2^p-2)|1-2\rho_2|} \right) \|x\|^p \quad (49)$$

for all $x \in X$.

Corollary 3.6. Let p_1, p_2, p_3 and θ be positive real numbers with $p_1 + p_2 + p_3 < 1$, and let $g : X \to Y$ be a mapping satisfying the following inequality

$$\left\| g\left(\frac{x+y}{2}+z\right) - g\left(\frac{x-y}{2}+z\right) - g(y) -\rho_2\left(2g\left(\frac{x+y}{2}+z\right) - g(x) - g(y) - 2g(z)\right)\right\| \leq \theta\cdot(\|x\|^{p_1}\cdot\|y\|^{p_2}\cdot\|z\|^{p_3}) \tag{50}$$

for all $x,y,z \in X$. Then there exists the unique additive mapping $A : X \to Y$ such that

$$\|g(x) - A(x)\| \leq \left(\frac{\theta}{(2^{p_1+p_2+p_3} - 2)\,|1-2\rho_2|}\right)\|x\|^{p_1+p_2+p_3} \tag{51}$$

for all $x \in X$.

If the function ϕ satisfies the condition that similar to (43), the same conclusion as Therem 3.1 then still holds.

Theorem 3.4. Let $\phi : X^3 \to [0,\infty)$ be a fixed function satisfying

$$\Phi(x,y,z) := \sum_{j=0}^{\infty}\frac{1}{2^j}\phi\left(2^jx, 2^jy, 2^jz\right) < \infty \tag{52}$$

for all $x,y,z \in X$. Assume that $g : X \to Y$ be a function satisfying the following inequality

$$\left\| g\left(\frac{x+y}{2}+z\right) - g\left(\frac{x-y}{2}+z\right) - g(y) -\rho_2\left(2g\left(\frac{x+y}{2}+z\right) - g(x) - g(y) - 2g(z)\right)\right\| \leq \phi(x,y,z) \tag{53}$$

for all $x,y,z \in X$. Then there exists the unique additive mapping $A : X \to Y$ such that

$$\|g(x) - A(x)\| \leq \left(\frac{1}{2\,|1-2\rho_2|}\right)\Phi(x,x,x) \tag{54}$$

for all $x \in X$.

Proof. Letting $x = y = z$ in (53), we get

$$\|g(2x) - 2g(x)\| \leq \left(\frac{1}{|1-2\rho_2|}\right)\phi(x,x,x) \tag{55}$$

and so

$$\left\| g(x) - \frac{1}{2}g(2x)\right\| \leq \left(\frac{1}{2\,|1-2\rho_2|}\right)\phi(x,x,x) \tag{56}$$

for all $x \in X$. The rest of the proof is similar to that of the former.

Corollary 3.7. Let $p > 1$ and θ be positive real numbers, and let $g : X \to Y$ be a mapping satisfying the following inequality

$$\left\| g\left(\frac{x+y}{2}+z\right) - g\left(\frac{x-y}{2}+z\right) - g(y) -\rho_2\left(2g\left(\frac{x+y}{2}+z\right) - g(x) - g(y) - 2g(z)\right)\right\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{57}$$

for all $x,y,z \in X$. Then there exists the unique additive mapping $A : X \to Y$ such that

$$\|g(x) - A(x)\| \leq \left(\frac{3\theta}{(2-2^p)\,|1-2\rho_2|}\right)\|x\|^p \tag{58}$$

for all $x \in X$.

Corollary 3.8. Let p_1, p_2, p_3 and θ be positive real numbers with $p_1 + p_2 + p_3 < 1$, and let $g : X \to Y$ be a mapping satisfying the following inequality

$$\left\| g\left(\frac{x+y}{2}+z\right) - g\left(\frac{x-y}{2}+z\right) - g(y) -\rho_2\left(2g\left(\frac{x+y}{2}+z\right) - g(x) - g(y) - 2g(z)\right)\right\| \leq \theta\cdot(\|x\|^{p_1}\cdot\|y\|^{p_2}\cdot\|z\|^{p_3}) \tag{59}$$

for all $x,y,z \in X$. Then there exists the unique additive mapping $A : X \to Y$ such that

$$\|g(x) - A(x)\| \leq \left(\frac{\theta}{(2 - 2^{p_1+p_2+p_3})\,|1-2\rho_2|}\right)\|x\|^{p_1+p_2+p_3} \tag{60}$$

for all $x \in X$.

We now arrive at the last results whose analogous proofs are omitted.

Theorem 3.5. Let $\phi : X^3 \to [0,\infty)$ be a fixed function satisfying

$$\Phi(x,y,z) := \sum_{j=1}^{\infty}2^j\phi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty \tag{61}$$

for all $x,y,z \in X$. Assume that $h : X \to Y$ be a function satisfying the following inequality

$$\left\| h\left(\frac{x+y}{2}+z\right) + h\left(\frac{x-y}{2}+z\right) - h(x) - 2h(z) -\rho_3\left(2h\left(\frac{x+y}{2}+z\right) - h(x) - h(y) - 2h(z)\right)\right\| \leq \phi(x,y,z) \tag{62}$$

for all $x,y,z \in X$. Then there exists the unique additive mapping $A : X \to Y$ such that

$$\|h(x) - A(x)\| \leq \left(\frac{1}{2\,|1-2\rho_3|}\right)\Phi(x,x,x) \tag{63}$$

for all $x \in X$.

Proof. The proof is similar to that of Theorem 3.3.

Corollary 3.9. Let $p>1$ and θ be positive real numbers, and let $h: X\to Y$ be a mapping satisfying the following inequality

$$\begin{aligned}&\left\|h\left(\frac{x+y}{2}+z\right)+h\left(\frac{x-y}{2}+z\right)-h(x)-2h(z)\right.\\&\left.-\rho_3\left(2h\left(\frac{x+y}{2}+z\right)-h(x)-h(y)-2h(z)\right)\right\|\\&\leq\ \theta\left(\|x\|^p+\|y\|^p+\|z\|^p\right)\end{aligned}\tag{64}$$

for all $x,y,z\in X$. Then there exists the unique additive mapping $A: X\to Y$ such that

$$\|h(x)-A(x)\|\leq\left(\frac{3\theta}{(2^p-2)\,|1-2\rho_3|}\right)\|x\|^p\tag{65}$$

for all $x\in X$.

Corollary 3.10. Let p_1,p_2,p_3 and θ be positive real numbers with $p_1+p_2+p_3<1$, and let $h: X\to Y$ be a mapping satisfying the following inequality

$$\begin{aligned}&\left\|h\left(\frac{x+y}{2}+z\right)+h\left(\frac{x-y}{2}+z\right)-h(x)-2h(z)\right.\\&\left.-\rho_3\left(2h\left(\frac{x+y}{2}+z\right)-h(x)-h(y)-2h(z)\right)\right\|\\&\leq\ \theta\cdot\left(\|x\|^{p_1}\cdot\|y\|^{p_2}\cdot\|z\|^{p_3}\right)\end{aligned}\tag{66}$$

for all $x,y,z\in X$. Then there exists the unique additive mapping $A: X\to Y$ such that

$$\|h(x)-A(x)\|\leq\left(\frac{\theta}{(2^{p_1+p_2+p_3}-2)\,|1-2\rho_3|}\right)\|x\|^{p_1+p_2+p_3}\tag{67}$$

for all $x\in X$.

Similar to Theorems 3.2 and 3.4 are the following results.

Theorem 3.6. Let $\phi: X^3\to[0,\infty)$ be a fixed function satisfying

$$\Phi(x,y,z):=\sum_{j=0}^{\infty}\frac{1}{2^j}\phi\left(2^jx,2^jy,2^jz\right)\ <\ \infty\tag{68}$$

for all $x,y,z\in X$. Assume that $h: X\to Y$ be a function satisfying the following inequality

$$\begin{aligned}&\left\|h\left(\frac{x+y}{2}+z\right)+h\left(\frac{x-y}{2}+z\right)-h(x)-2h(z)\right.\\&\left.-\rho_3\left(2h\left(\frac{x+y}{2}+z\right)-h(x)-h(y)-2h(z)\right)\right\|\ \leq\ \phi(x,y,z)\end{aligned}\tag{69}$$

for all $x,y,z\in X$. Then there exists the unique additive mapping $A: X\to Y$ such that

$$\|h(x)-A(x)\|\leq\left(\frac{1}{2\,|1-2\rho_3|}\right)\Phi(x,x,x)\tag{70}$$

for all $x\in X$.

Proof. The proof is similar to that of Theorem 3.4.

Corollary 3.11. Let $p>1$ and θ be positive real numbers, and let $h: X\to Y$ be a mapping satisfying the following inequality

$$\begin{aligned}&\left\|h\left(\frac{x+y}{2}+z\right)+h\left(\frac{x-y}{2}+z\right)-h(x)-2h(z)\right.\\&\left.-\rho_3\left(2h\left(\frac{x+y}{2}+z\right)-h(x)-h(y)-2h(z)\right)\right\|\\&\leq\ \theta\left(\|x\|^p+\|y\|^p+\|z\|^p\right)\end{aligned}\tag{71}$$

for all $x,y,z\in X$. Then there exists the unique additive mapping $A: X\to Y$ such that

$$\|h(x)-A(x)\|\leq\left(\frac{3\theta}{(2-2^p)\,|1-2\rho_3|}\right)\|x\|^p\tag{72}$$

for all $x\in X$.

Corollary 3.12. Let p_1,p_2,p_3 and θ be positive real numbers with $p_1+p_2+p_3<1$, and let $h: X\to Y$ be a mapping satisfying the following inequality

$$\begin{aligned}&\left\|h\left(\frac{x+y}{2}+z\right)+h\left(\frac{x-y}{2}+z\right)-h(x)-2h(z)\right.\\&\left.-\rho_3\left(2h\left(\frac{x+y}{2}+z\right)-h(x)-h(y)-2h(z)\right)\right\|\\&\leq\ \theta\cdot\left(\|x\|^{p_1}\cdot\|y\|^{p_2}\cdot\|z\|^{p_3}\right)\end{aligned}\tag{73}$$

for all $x,y,z\in X$. Then there exists the unique additive mapping $A: X\to Y$ such that

$$\|h(x)-A(x)\|\leq\left(\frac{\theta}{(2-2^{p_1+p_2+p_3})\,|1-2\rho_3|}\right)\|x\|^{p_1+p_2+p_3}\tag{74}$$

for all $x\in X$.

4 Applications

In this section, we show only the application of Theorem 3.1 as that of the rest are similar. We now investigate the isomorphisms between unital Banach algebras. Our results involve the use of *multiplicative function* ([14, Chapter 1]), which is the function satisfying the functional equation

$$f(xy)=f(x)f(y).$$

From now on, assume that B_1 is a unital Banach algebra over a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) with the unit e, and that B_2 is a unital Banach algebra over a field $\mathbb{F}$ with the unit e'.

Theorem 4.1. Let $\phi: B_1^3\to[0,\infty)$ be a fixed positive–valued function satisfying

$$\Phi(x,y,z):=\sum_{j=1}^{\infty}2^j\phi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^j}\right)\ <\ \infty\tag{75}$$

for all $x,y,z \in B_1$. Assume that $f : B_1 \to B_2$ is a bijective multiplicative mapping satisfying

$$\left\| f\left(\frac{\lambda x+\lambda y}{2}+\lambda z\right)+\lambda f\left(\frac{x-y}{2}+z\right)-\lambda f(x)-2\lambda f(z) -\rho_1\left(f\left(\frac{\lambda x+\lambda y}{2}+\lambda z\right)-\lambda f\left(\frac{x-y}{2}+z\right)-\lambda f(y)\right)\right\| \le \phi(x,y,z) \tag{76}$$

for all $x,y,z \in B_1$ and all $\lambda \in \mathbb{F}$ with the condition that

$$\lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right) = e'. \tag{77}$$

Then a mapping $f : B_1 \to B_2$ is an isomorphism between unital Banach algebras B_1 and B_2.

Proof. Putting $\lambda = 1$ into (76), Theorem 3.1 then implies that there exists a unique additive mapping $A : B_1 \to B_2$, which is defined as

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{78}$$

for all $x \in B_1$, satisfying

$$\|f(x)-A(x)\| \le \left(\frac{1}{2(1-|\rho_1|)}\right)\Phi(x,x,x) \tag{79}$$

for all $x \in B_1$. By (76) and (78), we see that

$$\begin{aligned} &|1-\rho_1|\,\|A(2\lambda x)-2\lambda A(x)\| \\ &= \lim_{n\to\infty} 2^n \left\| f\left(\frac{2\lambda x}{2^n}\right)-2\lambda f\left(\frac{x}{2^n}\right) -\rho_1\left(f\left(\frac{2\lambda x}{2^n}\right)-2\lambda f\left(\frac{x}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty} 2^n \phi\left(\frac{x}{2^n},\frac{x}{2^n},\frac{x}{2^n}\right) = 0 \end{aligned} \tag{80}$$

for all $x \in B_1$ and all $\lambda \in \mathbb{F}$. Since $|\rho_1| < 1$, we must have $A(2\lambda x) = 2\lambda A(x)$ yielding that

$$A(\lambda x) = \lambda A(x) \tag{81}$$

for all $x \in B_1$ and all $\lambda \in \mathbb{F}$. By (81) and the additivity of A, we conclude that $A : B_1 \to B_2$ is an $\mathbb{F}$–linear mapping.

To show that A is an isomorphism, by (78) and the multiplicity of f, we have

$$A(xy) = \lim_{n\to\infty} 2^n f\left(\frac{xy}{2^n}\right) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) f(y) = A(x)f(y) \tag{82}$$

for all $x,y \in B_1$. By (77) and (78), we have

$$A(e) = \lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right) = e'$$

and so

$$A(x) = A(ex) = A(e)f(x) = e'f(x) = f(x) \tag{83}$$

for all $x \in B_1$. Using (82) and (83), the result follows.

Immediate from Theorem 4.1 are the following:

Corollary 4.1. Let $p > 1$ and θ be positive real numbers, and let $f : B_1 \to B_2$ be a bijective multiplicative mapping satisfying the following inequality

$$\left\| f\left(\frac{\lambda x+\lambda y}{2}+\lambda z\right)+\lambda f\left(\frac{x-y}{2}+z\right)-\lambda f(x)-2\lambda f(z) -\rho_1\left(f\left(\frac{\lambda x+\lambda y}{2}+\lambda z\right)-\lambda f\left(\frac{x-y}{2}+z\right)-\lambda f(y)\right)\right\| \le \theta\left(\|x\|^p+\|x\|^p+\|x\|^p\right)$$

for all $x,y,z \in B_1$ and all $\lambda \in \mathbb{F}$ with the condition that

$$\lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right) = e'.$$

Then a mapping $f : B_1 \to B_2$ is an isomorphism between unital Banach algebras B_1 and B_2.

Proof. Substituting $\phi(x,y,z) := \theta\left(\|x\|^p+\|x\|^p+\|x\|^p\right)$ into (76) and applying Theorem 4.1, the result follows.

Corollary 4.2. Let p_1,p_2,p_3 and θ be positive real numbers with $p_1+p_2+p_3 < 1$, and let $f : B_1 \to B_2$ be a bijective multiplicative mapping satisfying the following inequality

$$\left\| f\left(\frac{\lambda x+\lambda y}{2}+\lambda z\right)+\lambda f\left(\frac{x-y}{2}+z\right)-\lambda f(x)-2\lambda f(z) -\rho_1\left(f\left(\frac{\lambda x+\lambda y}{2}+\lambda z\right)-\lambda f\left(\frac{x-y}{2}+z\right)-\lambda f(y)\right)\right\| \le \theta\cdot\left(\|x\|^{p_1}\cdot\|x\|^{p_2}\cdot\|x\|^{p_3}\right)$$

with the condition that

$$\lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right) = e'$$

for all $x,y,z \in B_1$ and all $\lambda \in \mathbb{F}$. Then a mapping $f : B_1 \to B_2$ is an isomorphism between unital Banach algebras B_1 and B_2.

Proof. Substituting $\phi(x,y,z) := \theta\cdot\left(\|x\|^{p_1}\cdot\|x\|^{p_2}\cdot\|x\|^{p_3}\right)$ into (20) and applying Theorem 4.1, the result follows.

5 Final Remarks

Note from the conditions (18) and (31) of Theorems 3.1 and 3.2, respectively, that they are dual of each other. Thus, we can say that Theorem 3.1 and Theorem 3.2 are dual of each other. Similarly, Theorems 3.3 and 3.4 as well as their corresponding corollaries are dual of each other. From this we can combine them, say (18) and (31), into single condition of the theorem as follows:

$$\Phi(x,y,z) := \sum_{j=1}^{\infty} 2^{\alpha j}\phi\left(2^{\alpha j}x,2^{\alpha j}y,2^{\alpha j}z\right) < \infty, \tag{84}$$

where $\alpha \in \{-1,1\}$ for a fixed function Φ, so that to prove the stabilities of these two theorems, for instance, it suffic to consider one only one condition, say (84). As the proof may not be straightforward, to obtain the stability result it requires some more steps which is worthy to present the future researches.

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Conflict of Interest The authors declare that they have no conflict of interest

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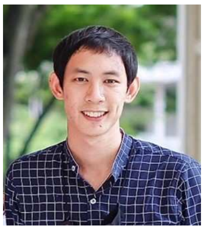

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