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On Grüss Type Inequality Involving a Fractional Integral Operator with a Multi-Index Mittag-Leffler Function as a Kernel

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Abstract: Many disciplines of pure and applied mathematics have found fractional integral inequalities to be one of the most significant and powerful instruments for their progress. These inequalities get a variety of applications in numerical quadrature, transform theory, probability, and statistical problems, however the most relevant one is determining the uniqueness of fractional boundary value problem solutions. They also offer upper and lower limits for the solutions to the equations above. Among this article, we define an integral inequality of Grüss type linked to the bounded integrable function associated with the fractional integral operator, which involves the generalized multi-index Mittag-Leffler function as a kernel. Our key finding is of a general nature and may give rise, as a special case, to integral inequalities of the type Grüss representing different fractional integral operators described in the literature.

Keywords: Grüss integral inequality and Grüss type integral inequalities, Mittag-Leffler function, Multi-index Mittag-Leffler functions, Fractional integral operators.

1 Introduction and Mathematical Preliminaries

In 1935, Grüss [1] has defined and shown a valuable inequality that defines the relationship between the integral product of the two functions and the product of the integrals of the individual functions, as:

Suppose p and q are two integrable functions on $[c,d]$ and satisfy the inequalities: $l \leq p(t) \leq L$ and $m \leq q(t) \leq M$ for all $t \in [c,d]$ and $l, L, m, M \in \mathbb{R}$. The preceding inequalities therefore hold true:

$$\left| \frac{1}{(d-c)} \int_c^d p(\xi) q(\xi) d\xi - \frac{1}{(d-c)^2} \int_c^d p(\xi) d\xi \int_c^d q(\xi) d\xi \right| \leq \frac{1}{4}(L-l)(M-m). \tag{1}$$

The Grüss-inequality (1) has many applications in diverse research subjects such as coding theory, difference equations, integral arithmetic mean, numerical analysis, spaces with inner product and statistics. Therefore, many researchers have given ample attention to this inequality (see, e.g., [2,3,4,5,6,7]). In addition, many researchers have learned a great deal of fractional integral inequalities and related applications through the usage of fractional integral operators (see, for example, [8,9,10,11,12,13, 14]).

Recently, Srivastava et al. [15] defined a new fractional integral operator containing generalized multi-index Mittag-Leffler function (GMIMLF) as a kernel. Using this fractional integral operator we propose a new generalization of (1). First, we recall some basic definitions available in the literature to establish the main results.

In 1903, Gosta Mittag-Leffler [16] analyzed and defined the Mittag-Leffler (M-L) function $\mathrm{E}_\mu(u)$ as

$$\mathrm{E}_\mu(u) = \sum_{r=0}^{\infty} \frac{u^r}{\Gamma(\mu r+1)}, \quad (\mu \in \mathbb{C}, \Re(\mu) > 0). \tag{2}$$

Since then numerous notable generalizations of popularly known M-L function (2) have been presented in literature.

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In 1905, Wiman [17] generalized the M-L function and defined as

$$\mathrm{E}_{\mu,\nu}(u)=\sum_{r=0}^{\infty}\frac{u^r}{\Gamma(\mu r+\nu)},\quad (\mu,\nu\in\mathbb{C},\ \Re(\mu)>0).\quad (3)$$

In 1971, Prabhakar [18] established a new generalization of the M-L and Wiman's functions as

$$\mathrm{E}^{\gamma}_{\mu,\nu}(u)=\sum_{r=0}^{\infty}\frac{(\gamma)_r}{\Gamma(\mu r+\nu)}\frac{u^r}{r!},\ (\mu,\nu,\gamma\in\mathbb{C},\ \Re(\mu)>0),\quad (4)$$

where, $(\gamma)_r$ is a symbol of the Pochhammer and is described as:

$$(\gamma)_r=\frac{\Gamma(\gamma+r)}{\Gamma(\gamma)},\qquad (\gamma,r\in\mathbb{C}).\quad (5)$$

Further, Srivastava and Tomovski [19] generalized the function $\mathrm{E}^{\gamma}_{\mu,\nu}(u)$, and expressed as

$$\mathrm{E}^{\gamma,\rho}_{\mu,\nu}(u)=\sum_{r=0}^{\infty}\frac{(\gamma)_{\rho r}}{\Gamma(\mu r+\nu)}\frac{u^r}{r!},\quad (6)$$

$$(u,\nu,\gamma\in\mathbb{C},\ \Re(\mu)>\max\left(0,\Re(\rho)-1\right),\Re(\rho)>0).$$

Also, Shukla and Prajapati [20] gave the generalization of the function $\mathrm{E}^{\gamma}_{\mu,\nu}(u)$, which may be obtained by setting $\rho=q$ in (6).

Additionally, Salim and Faraj [21] introduced and studied the subsequent two generalizations of above mentioned Mittag-Leffler functions

$$\mathrm{E}^{\eta,\delta,q}_{\mu,\nu,p}(u)=\sum_{r=0}^{\infty}\frac{(\eta)_{qr}}{\Gamma(\mu r+\nu)}\frac{u^r}{(\delta)_{pr}},\quad (7)$$

$$(p,q\in\mathbb{R}^+;\ \mu,\nu,\eta,\delta\in\mathbb{C};\ \Re(\mu)>0)$$

and

$$\mathrm{E}^{\mu,\vartheta,\eta,q}_{\mu,\nu,\nu,\sigma,\delta,p}(u)=\sum_{r=0}^{\infty}\frac{(\mu)_{\vartheta r}(\eta)_{qr}}{(\nu)_{\sigma r}(\delta)_{pr}}\frac{u^r}{\Gamma(\mu r+\nu)},\quad (8)$$

$$p,q\in\mathbb{R}^+;\ q\leq\Re(\mu)+p;\ \mu,\nu,\eta,\delta,\mu,\nu,\vartheta,\sigma\in\mathbb{C};\ \min\{\Re(\mu),\Re(\vartheta),\Re(\sigma)\}>0.$$

In 2010, Saxena and Nishimoto [22,23] introduced and defined a new GMIMLF, $\mathrm{E}^{\gamma,\rho}_{(\mu_j,\nu_j)_m}(u)$, which is considered in this paper, and expressed in the following manner

$$\mathrm{E}^{\gamma,\rho}_{(\mu_j,\nu_j)_m}(u)=\mathrm{E}_{\gamma,\rho}[(\mu_j,\nu_j)_{j=1}^m;u]=\sum_{r=0}^{\infty}\frac{(\gamma)_{\rho r}}{\prod_{j=1}^{m}\Gamma(\mu_j r+\nu_j)}\frac{u^r}{r!},\quad (9)$$

$$\mu_j,\nu_j,\gamma,\rho,u\in\mathbb{C},\ \Re(\mu_j)>0\ (j=1,....,m),\ \Re(\sum_{j=1}^{m}\mu_j)>\max\{0,\Re(\rho)-1\}.\quad (10)$$

Finally, we recall the familiar Fox-Wright function ${}_p\Psi_q$, which is given by the following series (see [24]):

$${}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\cdots,(a_p,A_p);\\(b_1,B_1),\cdots,(b_p,B_p);\end{matrix}\ u\right]=\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+A_j r)}{\prod_{j=1}^{q}\Gamma(b_j+B_j r)}\frac{u^r}{r!},\quad (11)$$

where, $A_j,B_j\in\mathbb{R}^+$, $a_j,b_j\in\mathbb{C}$ and series converges absolutely for all $z\in\mathbb{C}$ when $\Delta=1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j>0$.

*Remark.*GMIMLF and Fox-Wright function are related as follows:

$$\mathrm{E}^{\gamma,\rho}_{(\mu_j,\nu_j)_m}(u)=\frac{1}{\Gamma(\gamma)}\,{}_1\Psi_m\left[\begin{matrix}(\gamma,\rho);\\(\nu_1,\mu_1),\cdots,(\nu_m,\mu_m);\end{matrix}\ u\right],\quad (12)$$

may be obtained from (9) and (11).

Various integral operators involving distinct generalizations of M-L function have been studied by several mathematicians (see for details, [15,18,19,21,25]). Among them, we recall the following integral operator introduced by [15]:

$$\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{a+;(\mu_j,\nu_j)_m;\nu}\psi\right)(x)=\int_a^x(x-t)^{\nu-1}\mathrm{E}^{\gamma,\rho}_{(\mu_j,\nu_j)_m}(\omega(x-t)^{\mu})\psi(t)dt,\ (x>a),\quad (13)$$

where, $\mu,\nu,\mu_j,\nu_j,\gamma,\rho,\omega\in\mathbb{C};\Re(\mu_j)>0;\min\{\Re(\nu),\Re(\rho)\}>0;\Re\left(\sum_{j=1}^{m}\mu_j\right)>\max\{0,\Re(\rho)-1\}$.

Motivated by the above cited work, we propose to investigate Grüss type inequalities involving integral operator (13) containing GMIMLF in the kernel.

Before going to prove the main results, first we prove important result of fractional integral operator (13) containing GMIMLF in the kernel. This result will be used to prove the subsequent important results.

Lemma 1.*If $\zeta>0,\ \mu,\ \nu,\ \mu_j,\ \nu_j,\ \gamma,\ \rho,\ \omega\in\mathbb{C},\ \Re(\mu_j)>0$, $\min\{\Re(\nu),\Re(\rho)\}>0$ and $\Re\left(\sum_{j=1}^{m}\mu_j\right)>\max\{0,\Re(\rho)-1\}$, then,*

$$\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}t^{\zeta-1}\right)(x)=x^{\nu+\zeta-1}\frac{\Gamma(\zeta)}{\Gamma(\gamma)}\times{}_2\Psi_{m+1}\left[\begin{matrix}(\nu,\mu),(\gamma,\rho);\\(\zeta+\nu,\mu),(\nu_1,\mu_1),\cdots,(\nu_m,\mu_m);\end{matrix}\ \omega x^{\mu}\right].\quad (14)$$

*Proof.*Let us consider the L.H.S of (14) as

$$I=\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}t^{\zeta-1}\right)(x). \tag{15}$$

Using the definitions (9) and (13) in above equation, interchanging the order of integration and summation, and then after little simplification, we arrive at

$$I=x^{\nu+\zeta-1}\frac{\Gamma(\zeta)}{\Gamma(\gamma)}\times \sum_{r=0}^{\infty}\frac{\Gamma(\nu+\mu r)\Gamma(\gamma+\rho r)}{\Gamma(\zeta+\nu+\mu r)\prod_{j=1}^{m}\Gamma(\nu_j+\mu_j r)}\frac{(\omega x^{\mu})^r}{r!}, \tag{16}$$

finally, using (12), we obtain the desire result (14).

Throughout this paper we consider the existence conditions given by (10) of GMIMLF (9) and fractional integral operator (13) containing GMIMLF in the kernel unless otherwise stated.

2 Main Results

Below we create a modified integral inequality concerning the (13) fractional operators that provides an approximation of the fractional integral of the product of two functions as regards the product of the individual fractional integrals of both the functions. For this purpose, we first get a functional relationship for fractional operators, which includes GMIMLF in the kernel associated with the bounded integrable function described by the subsequent lemma.

Lemma 2.*Assume that* f, ψ_1 *and* ψ_2 *are integrable functions described in* $[0,\infty)$ *so that if*

$$\psi_1(t)\le \mathrm{f}(t)\le \psi_2(t), \qquad t\in[0,\infty). \tag{17}$$

Then the preceding relation holds true:

$$\begin{aligned}
&\mathscr{A}^2_{m+1}(\omega x^{\mu})\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}^2(t)\right)(x)-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)^2(x)\\
&=\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_2(t)-\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)(x)\times\\
&\quad\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)-\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\right)(x)\\
&-\mathscr{A}^2_{m+1}(\omega x^{\mu})\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}(\psi_2(t)-\mathrm{f}(t))(\mathrm{f}(t)-\psi_1(t))\right)(x)\\
&+\mathscr{A}^2_{m+1}(\omega x^{\mu})\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\mathrm{f}(t)\right)(x)\\
&\quad-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)(x)\\
&+\mathscr{A}^2_{m+1}(\omega x^{\mu})\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_2(t)\mathrm{f}(t)\right)(x)\\
&\quad-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_2(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)(x)\\
&+\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_2(t)\right)(x)\\
&\quad-\mathscr{A}^2_{m+1}(\omega x^{\mu})\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\psi_2(t)\right)(x),
\end{aligned}\tag{18}$$

where

$$\mathscr{A}^2_{m+1}(\omega x^{\mu})=\frac{x^{\nu}}{\Gamma(\gamma)}\times {}_2\Psi_{m+1}\left[\begin{matrix}(\nu,\mu),(\gamma,\rho);\\(1+\nu,\mu),(\nu_1,\mu_1),\cdots,(\nu_m,\mu_m);\end{matrix}\ \omega x^{\mu}\right]. \tag{19}$$

*Proof.*For the functions given in (17), and any ξ, $\vartheta>0$, we consider

$$\begin{aligned}
&(\psi_2(\vartheta)-\mathrm{f}(\vartheta))(\mathrm{f}(\xi)-\psi_1(\xi))+(\psi_2(\xi)-\mathrm{f}(\xi))(\mathrm{f}(\vartheta)-\psi_1(\vartheta))\\
&-(\psi_2(\xi)-\mathrm{f}(\xi))(\mathrm{f}(\xi)-\psi_1(\xi))-(\psi_2(\vartheta)-\mathrm{f}(\vartheta))(\mathrm{f}(\vartheta)-\psi_1(\vartheta))\\
&=\mathrm{f}^2(\xi)+\mathrm{f}^2(\vartheta)-2\mathrm{f}(\xi)\mathrm{f}(\vartheta)+\psi_2(\vartheta)\mathrm{f}(\xi)+\psi_1(\xi)\mathrm{f}(\vartheta)\\
&-\psi_1(\xi)\psi_2(\vartheta)+\psi_2(\xi)\mathrm{f}(\vartheta)+\psi_1(\vartheta)\mathrm{f}(\xi)-\psi_1(\vartheta)\psi_2(\xi)\\
&-\psi_2(\xi)\mathrm{f}(\xi)+\psi_1(\xi)\psi_2(\xi)-\psi_1(\xi)\mathrm{f}(\xi)-\psi_2(\vartheta)\mathrm{f}(\vartheta)\\
&+\psi_1(\vartheta)\psi_2(\vartheta)-\psi_1(\vartheta)\mathrm{f}(\vartheta).
\end{aligned}\tag{20}$$

For $x>0$ and $\xi\in(0,x)$, assume that

$$F(x,\xi)=(x-\xi)^{\nu-1}\mathrm{E}^{\gamma,\rho}_{(\mu_j,\nu_j)_m}(\omega(x-\xi)^{\mu}). \tag{21}$$

Multiplying the two sides of (20) by $F(x,\xi)$ and integrating the subsequent relation with respect to ξ from 0 to x, and employing (13), we get

$$\begin{aligned}
&(\psi_2(\vartheta)-\mathrm{f}(\vartheta))\left(\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)(x)-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\right)(x)\right)\\
&+\left(\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_2(t)\right)(x)-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)(x)\right)(\mathrm{f}(\vartheta)-\psi_1(\vartheta))\\
&-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}(\psi_2(t)-\mathrm{f}(t))(\mathrm{f}(t)-\psi_1(t))\right)(x)\\
&-(\psi_2(\vartheta)-\mathrm{f}(\vartheta))(\mathrm{f}(\vartheta)-\psi_1(\vartheta))\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\{1\}\right)(x)\\
&=\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}^2(t)\right)(x)+\mathrm{f}^2(\vartheta)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\{1\}\right)(x)\\
&-2\mathrm{f}(\vartheta)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)(x)+\psi_2(\vartheta)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)(x)\\
&+\mathrm{f}(\vartheta)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\right)(x)-\psi_2(\vartheta)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\right)(x)\\
&+\mathrm{f}(\vartheta)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_2(t)\right)(x)+\psi_1(\vartheta)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)(x)\\
&-\psi_1(\vartheta)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_2(t)\right)(x)-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_2(t)\mathrm{f}(t)\right)(x)\\
&+\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\psi_2(t)\right)(x)-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\mathrm{f}(t)\right)(x)\\
&-\psi_2(\vartheta)\mathrm{f}(\vartheta)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\{1\}\right)(x)\\
&+\psi_1(\vartheta)\psi_2(\vartheta)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\{1\}\right)(x)\\
&-\psi_1(\vartheta)\mathrm{f}(\vartheta)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\{1\}\right)(x).
\end{aligned}\tag{22}$$

Next, multiplying both sides of (22) by $F(x,\vartheta)$ given in (21), and integrating with respect to ϑ from 0 to x, we obtain

$$2\left(\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\psi_2(t)\right)(x)-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)\right)$$
$$\times\left(\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\psi_1(t)\right)(x)\right)$$
$$-2\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,(\psi_2(t)-\mathrm{f}(t))(\mathrm{f}(t)-\psi_1(t))\right)(x)\times\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\{1\}\right)(x)$$
$$=2\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\{1\}\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}^2(t)\right)(x)$$
$$-2\left(\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)\right)^2$$
$$+2\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\psi_1(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)$$
$$-2\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\{1\}\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\psi_1(t)\mathrm{f}(t)\right)(x)$$
$$+2\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\psi_2(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)$$
$$-2\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\{1\}\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\psi_2(t)\mathrm{f}(t)\right)(x)$$
$$-2\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\psi_1(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\psi_2(t)\right)(x)$$
$$+2\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\{1\}\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\psi_1(t)\psi_2(t)\right)(x), \tag{23}$$

which, by using the result (14), gives rise to the intended result (18).

Now, we are providing our key finding of satisfying the inequality of the Cauchy-Schwarz form as stated by the upcoming theorem.

Theorem 1. *Suppose* f, g, ψ_1, ψ_2, φ_1 *and* φ_2 *are integrable functions on* $[0,\infty)$ *such that*

$$\psi_1(t)\le \mathrm{f}(t)\le\psi_2(t),\quad \varphi_1(t)\le \mathrm{g}(t)\le\varphi_2(t),\; t\in[0,\infty). \tag{24}$$

Thereupon the preceeding inequality holds true:

$$\left|\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\mathrm{g}(t)\right)(x)-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{g}(t)\right)(x)\right|$$
$$\le\sqrt{\mathscr{T}(\mathrm{f},\psi_1,\psi_2)\;\mathscr{T}(\mathrm{g},\varphi_1,\varphi_2)}, \tag{25}$$

where,

$$\mathscr{T}(u,v,w)=$$
$$\left(\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,w(t)\right)(x)-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,u(t)\right)(x)\right)\times$$
$$\left(\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,u(t)\right)(x)-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,v(t)\right)(x)\right)$$
$$+\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,v(t)u(t)\right)(x)$$
$$-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,v(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,u(t)\right)(x)$$
$$+\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,w(t)u(t)\right)(x)$$
$$-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,w(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,u(t)\right)(x)$$
$$+\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,v(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,w(t)\right)(x)$$
$$-\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,v(t)w(t)\right)(x), \tag{26}$$

and $\mathscr{A}^2_{m+1}(\omega x^\mu)$ *is given in* (19).

Proof. Let us define a function $\mathscr{H}(\cdot,\cdot)$ by

$$\mathscr{H}(\xi,\vartheta)=(\mathrm{f}(\xi)-\mathrm{f}(\vartheta))\,(\mathrm{g}(\xi)-\mathrm{g}(\vartheta)),\;(x>0;\;0<\xi,\,\vartheta<x). \tag{27}$$

Multiplying both sides of (27) by $F(x,\xi)F(x,\vartheta)$, where $F(x,\xi)$ and $F(x,\vartheta)$ are given by (21), and integrating the subsequent relation with respect to ξ and ϑ, respectively, from 0 to x, we get

$$\int_0^x\int_0^x(x-\xi)^{\nu-1}\,(x-\vartheta)^{\nu-1}\mathrm{E}^{\gamma,\rho}_{(\mu_j,\nu_j)_m}(\omega(x-\xi)^\mu)\times\mathrm{E}^{\gamma,\rho}_{(\mu_j,\nu_j)_m}(\omega(x-\vartheta)^\mu)\mathscr{H}(\xi,\vartheta)\,d\xi\,d\vartheta$$
$$=\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\mathrm{g}(t)\right)(x)$$
$$-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{g}(t)\right)(x). \tag{28}$$

Now, implement the Cauchy-Schwarz inequality to the right-hand side of the above equation, we obtain

$$\left\{\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\mathrm{g}(t)\right)(x)-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{g}(t)\right)(x)\right\}^2$$
$$\le\left(\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}^2(t)\right)(x)-\left(\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)\right)^2\right)\times$$
$$\left(\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{g}^2(t)\right)(x)-\left(\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{g}(t)\right)(x)\right)^2\right). \tag{29}$$

The function $F(x,\xi)$ given by (21) stands positive for all $\xi\in(0,x)$ $(x>0)$. Thus, under the assumption of Lemma 2, it is noticeable that either if a function f is integrable and non-negative on $[0,\infty)$, then $\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)(x)\geq 0$; or if a function f is integrable and non-positive on $[0,\infty)$, then $\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)(x)\leq 0$.

Now, by noting the relation that for all $t\in(0,\infty)$, $(\psi_2(t)-\mathrm{f}(t))(\mathrm{f}(t)-\psi_1(t))\geq 0$ and $(\varphi_2(t)-\mathrm{g}(t))(\mathrm{g}(t)-\varphi_1(t))\geq 0$, we get

$$\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}(\psi_2(t)-\mathrm{f}(t))(\mathrm{f}(t)-\psi_1(t))\right)(x)\geq 0,$$

and

$$\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}(\varphi_2(t)-\mathrm{g}(t))(\mathrm{g}(t)-\varphi_1(t))\right)(x)\geq 0.$$

Hence, on employing Lemma 2, we arrive at

$$\begin{aligned}&\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}^2(t)\right)(x)-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)^2(x)\\ &\leq\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_2(t)-\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)(x)\times\\ &\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)-\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\right)(x)\\ &+\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\mathrm{f}(t)\right)(x)\\ &-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)(x)\\ &+\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_2(t)\mathrm{f}(t)\right)(x)\\ &-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_2(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{f}(t)\right)(x)\\ &+\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_2(t)\right)(x)\\ &-\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\psi_1(t)\psi_2(t)\right)(x)\\ &=\mathscr{T}(\mathrm{f},\psi_1,\psi_2).\end{aligned}\tag{30}$$

A similar argument will give rise to the preceding inequality:

$$\left\{\mathscr{A}^2_{m+1}(\omega x^\mu)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{g}^2(t)\right)(x)-\left(\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\mathrm{g}(t)\right)(x)\right)^2\right\}\leq\mathscr{T}(g,\varphi_1,\varphi_2).\tag{31}$$

Subsequently, taking advantage of the inequalities (29), (30) and (31), we are immediately led to the desired inequality (25). This finishes the theorem assertion.

3 Consequent Results and Special Cases

The fractional integral operator involving GMIMLF as kernel defined in (13) contains, as particular cases, the integral operators characterized by Kilbas et al. [25], Shukla and Prajapati [20], Srivastava and Tomovski [19] and Kiryakova [12]. If we set:

(i) $\rho=m=1$, we obtain the fractional integration operator defined by Prabhakar [18] and Kilbas et al. [25].

$$\left(\varepsilon^{\gamma}_{\mu,\nu;\omega,a+}\psi\right)(x)=\int_a^x(x-t)^{\nu-1}\mathrm{E}^{\gamma}_{\mu,\nu}(\omega(x-t)^\mu)\psi(t)dt,\ (x>a),$$

where, $\mu,\nu,\gamma,\omega\in\mathbb{C}$; $\Re(\mu)>0$ and $\Re(\nu)>0$.

(ii) $\rho=m=1$ and $\omega\to 0$, then by virtue of the limit formula we obtain the familiar Riemann-Liouville fractional integral defined as

$$(I^\mu_{a+}\psi)(x)=\frac{1}{\Gamma(\mu)}\int_a^x(x-t)^{\mu-1}\psi(t)dt,\ (x>a;\ \Re(\mu)>0).$$

(iii) $m=1$, we obtain the integral operator defined by Srivastava and Tomovski [19]

$$\left(\varepsilon^{\omega;\gamma,\rho}_{a+;\mu,\nu}\psi\right)(x)=\int_a^x(x-t)^{\nu-1}\mathrm{E}^{\gamma,\rho}_{\mu,\nu}(\omega(x-t)^\mu)\psi(t)dt,\ (x>a),$$

where, $\mu,\nu,\gamma,\omega\in\mathbb{C}$; $\Re(\mu)>0,\Re(\nu)>0,\Re(\rho)>0$ and $\Re(\mu)=\Re(\rho)-1>0$, which for $\rho=q$ is reduced to that given by Shukla and Prajapati [20].

(iv) $\gamma=\rho=1$, we obtain integral operator associated with the multi-index Mittag-Leffler function defined by Kiryakova [12]

$$\left(\varepsilon^{\omega;\mu}_{a+;(\mu_j,\nu_j)_m;\nu}\psi\right)(x)=\int_a^x(x-t)^{\nu-1}\mathrm{E}_{(\mu_j,\nu_j)_m}(\omega(x-t)^\mu)\psi(t)dt,\ (x>a),$$

where, $\mu_j,\nu j,\omega\in\mathbb{C}$, $\Re(\mu_j)>0$, $\Re(\nu_j)>0$ $j=1,\cdots,m$.

Likewise, by appropriately specializing the values of the parameters, the inequality (25) in Theorem 1 would give rise to further Grüss type of integral inequalities containing the aforementioned integral operators.

If we specialize the integrable functions $\psi_1(t)$, $\psi_2(t)$, $\varphi_1(t)$ and $\varphi_2(t)$ in Theorem 1 as either constant functions or linear functions, then we obtain the fractional integral inequalities given by the preceding corollaries:

Corollary 1. *Suppose* f *and* g *be two integrable functions on* $[0,\infty)$ *satisfying the following inequalities*

$$m\leq\mathrm{f}(t)\leq M\quad and\quad p\leq\mathrm{g}(t)\leq P\quad(t\in[0,\infty)),\tag{32}$$

where, m, M, p and P are real constants. Then the following inequality holds true:

$$\Big|\mathscr{A}^2_{m+1}(\omega x^{\mu})\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\mathrm{g}(t)\right)(x) - \left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{g}(t)\right)(x)\Big| \leq \left(\mathscr{A}^2_{m+1}(\omega x^{\mu})\right)^2(M-m)(P-p). \quad (33)$$

Proof. If we set $\psi_1(t)=m$, $\psi_2(t)=M$, $\varphi_1(t)=p$ and $\varphi_2(t)=P$ into (24), then Theorem 1 asserted the corollary.

In addition, if we take $\psi_1(t)=t$, $\psi_2(t)=t+1$, $\varphi_1(t)=t-1$ and $\varphi_2(t)=t$ in Theorem 1, and use formula (14), we obtain the fractional integral inequality claimed by the following corollary.

Corollary 2. *Suppose* f *and* g *be two integrable functions on* $[0,\infty)$ *and satisfy the following inequalities*

$$t \leq \mathrm{f}(t) \leq t+1 \quad \textit{and} \quad t-1 \leq \mathrm{g}(t) \leq t,\ (t\in[0,\infty)). \quad (34)$$

Then the following inequality holds true:

$$\Big|\mathscr{A}^2_{m+1}(\omega x^{\mu})\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\mathrm{g}(t)\right)(x) - \left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{g}(t)\right)(x)\Big| \leq \sqrt{\mathscr{T}(\mathrm{f},t,t+1)\ \mathscr{T}(\mathrm{g},t-1,t)}, \quad (35)$$

here,

$$\begin{aligned}
&\mathscr{T}(\mathrm{f},t,t+1)=\\
&\left(\mathscr{B}^2_{m+1}(\omega x^{\mu})+\mathscr{A}^2_{m+1}(\omega x^{\mu})-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)\right)\\
&\qquad\times\left(\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)-\mathscr{B}^2_{m+1}(\omega x^{\mu})\right)\\
&+\mathscr{A}^2_{m+1}(\omega x^{\mu})\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,t\,\mathrm{f}(t)\right)(x)\\
&-\mathscr{B}^2_{m+1}(\omega x^{\mu})\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)\\
&+\mathscr{A}^2_{m+1}(\omega x^{\mu})\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,(t+1)\,\mathrm{f}(t)\right)(x)\\
&-\left(\mathscr{B}^2_{m+1}(\omega x^{\mu})+\mathscr{A}^2_{m+1}(\omega x^{\mu})\right)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{f}(t)\right)(x)\\
&+\mathscr{B}^2_{m+1}(\omega x^{\mu})\left(\mathscr{B}^2_{m+1}(\omega x^{\mu})+\mathscr{A}^2_{m+1}(\omega x^{\mu})\right)\\
&-\mathscr{A}^2_{m+1}(\omega x^{\mu})\left(\mathscr{C}^2_{m+1}(\omega x^{\mu})+\mathscr{B}^2_{m+1}(\omega x^{\mu})\right),
\end{aligned}$$

$$\begin{aligned}
&\mathscr{T}(\mathrm{g},t-1,t)=\\
&\left(\mathscr{B}^2_{m+1}(\omega x^{\mu})-\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{g}(t)\right)(x)\right)\times\\
&\left(\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{g}(t)\right)(x)-\mathscr{B}^2_{m+1}(\omega x^{\mu})+\mathscr{A}^2_{m+1}(\omega x^{\mu})\right)\\
&+\mathscr{A}^2_{m+1}(\omega x^{\mu})\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,(t-1)\,\mathrm{g}(t)\right)(x)\\
&-\left(\mathscr{B}^2_{m+1}(\omega x^{\mu})-\mathscr{A}^2_{m+1}(\omega x^{\mu})\right)\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{g}(t)\right)(x)\\
&+\mathscr{A}^2_{m+1}(\omega x^{\mu})\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,t\,\mathrm{g}(t)\right)(x)\\
&-\mathscr{B}^2_{m+1}(\omega x^{\mu})\left(\varepsilon^{\omega;\gamma,\rho;\mu}_{0+;(\mu_j,\nu_j)_m;\nu}\,\mathrm{g}(t)\right)(x)\\
&+\left(\mathscr{B}^2_{m+1}(\omega x^{\mu})-\mathscr{A}^2_{m+1}(\omega x^{\mu})\right)\mathscr{B}^2_{m+1}(\omega x^{\mu})\\
&-\mathscr{A}^2_{m+1}(\omega x^{\mu})\left(\mathscr{C}^2_{m+1}(\omega x^{\mu})-\mathscr{B}^2_{m+1}(\omega x^{\mu})\right),
\end{aligned}$$

$$\begin{aligned}
\mathscr{A}^2_{m+1}(\omega x^{\mu}) &= \frac{x^{\nu}}{\Gamma(\gamma)}\times\\
&{}_2\Psi_{m+1}\left[\begin{matrix}(\nu,\mu),(\gamma,\rho);\\ (1+\nu,\mu),(\nu_1,\mu_1),\cdots,(\nu_m,\mu_m);\end{matrix}\ \omega x^{\mu}\right],\\
\mathscr{B}^2_{m+1}(\omega x^{\mu}) &= \frac{x^{\nu+1}}{\Gamma(\gamma)}\times\\
&{}_2\Psi_{m+1}\left[\begin{matrix}(\nu,\mu),(\gamma,\rho);\\ (2+\nu,\mu),(\nu_1,\mu_1),\cdots,(\nu_m,\mu_m);\end{matrix}\ \omega x^{\mu}\right],
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{C}^2_{m+1}(\omega x^{\mu}) &= \frac{x^{\nu+2}\Gamma(3)}{\Gamma(\gamma)}\times\\
&{}_2\Psi_{m+1}\left[\begin{matrix}(\nu,\mu),(\gamma,\rho);\\ (3+\nu,\mu),(\nu_1,\mu_1),\cdots,(\nu_m,\mu_m);\end{matrix}\ \omega x^{\mu}\right].
\end{aligned}$$

4 Concluding Remark

Here, we have described Grüss type inequality involving fractional integrals, linked to the bounded integrable functions. The inequality is derived by taking into consideration the fractional integral operator that includes the generalized Mittag-Leffler multi-index function as a kernel. Some inequalities are also identified by the specialization of the values of the parameters, as particular instances.

We draw the conclusion by once again emphasizing that our main outcome here seems to be of a general type, may be skilled in producing lots of interesting fractional integral inequalities, along with some known consequences. By suitably specializing the arbitrary integrable functions $\psi_1(t)$, $\psi_2(t)$, $\varphi_1(t)$ and $\varphi_2(t)$ one can further easily obtain additional integral inequalities from our main results in in Theorem 1.

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Competing interests:

The authors declare that they have no competing interests.

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