Appl. Math. Inf. Sci. 16, No. 2, 259-268 (2022)

Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/160213

Approximation of Functions by Lupas-Kantorovich-Stancu Type Operators based on Polya Distribution

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Received: 14 Apr. 2020, Revised: 11 Oct. 2021, Accepted: 10 Nov. 2021
Published online: 1 Mar. 2022

Abstract: In this paper, we introduce a two parameter generalization of Lupas-Kantorovich operators based on Polya distribution. We obtain the moments of the operators by deriving a recurrence relation and then prove and study Voronovskaja-type asymptotic theorem, local approximation, weighted approximation, rate of convergence and pointwise estimates using the Lipschitz type maximal function.

Keywords: Asymptotic formula, modulus of continuity, K-functional, Polya distribution, weighted approximation.

1 Introduction and preliminaries

In the field of mathematical analysis, Karl Weierstrass established an elegant theorem, the first Weierstrass approximation theorem, in 1885. This theorem has specially a big role in polynomial interpolation corresponding to every continuous function f on interval $[a,b]$. The proof given by Weierstrass was rigorous and difficult to understand. In 1912, Bernstein [1] gave a simple proof of this theorem by introducing the Bernstein polynomials with the aid of the binomial distribution, hence for $f \in C[0,1]$, we have

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), n \in N,$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, x \in [0,1]$ is the Bernstein basis function. Many mathematicians researched in this direction and studied various modifications in several functional spaces using different error optimization techniques.

In 1930, Kantorovich [2] introduced the following integral modification of Bernstein polynomials for $f \in L_1[0,1]$(the class of Lebesgue integrable functions on $[0,1]$):

$$K_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_0^1 \phi_{n,k}(t) f(t)dt,$$

where $\phi_{n,k}(t)$ is the characteristic function of the interval $[k/(n+1),(k+1)/(n+1)]$. After that Kantorovich type modification of several sequences of linear positive operators has been made and studied for their approximation behaviour. Several researchers also defined different types of generalizations of these operators and studied their approximation properties, we refer the reader to e.g. [[3], [4], [5]] etc.

In [6], for $f \in C(I), I = [0,1]$, Miclaus studied some approximation properties of Bernsteini Stancu type operators based on Polya distribution given by

$$P_n^{(1/n)}(f;x) = \sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) f\left(\frac{k}{n}\right), \tag{1}$$

where

$$p_{n,k}^{(1/n)}(x) = \frac{2(n!)}{(2n)!}\binom{n}{k}(nx)_k(n-nx)_{n-k},$$

and $(n)_k = n(n+1)(n+2)...(x+k-1)$ is the rising factorial.

To approximate Lebesgue integrable functions, Agrawal et al. [7] introduced the following Kantorovich type modification of the operators defined by Lupas and Lupas [8] as follows:

$$D_n^{*(1/n)}(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt. \tag{2}$$

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In [7], Agrawal et al. studied the Voronovskaja type theorem, local approximation, pointwise estimates and global approximation results. Later, Ispir et al. [9] estimated the rate of convergence for absolutely continuous functions having a derivative coinciding a.e. with a function of bounded variation.

It is very well known that the polynomial approximation of continuous functions has an important role in numerical analysis. The Lagrange interpolating polynomials have a great practical interest in approximation theory of continuous functions, but they do not provide always uniform convergence of approximating sequences for any continuous function on a compact interval of the real axis, no matter how the nodes are chosen.

In 1905, Borel proposed a way to obtain an approximation polynomial of a function $f \in C[0,1]$ by using an interpolation polynomial having a similar form with the Lagrange ones and using the nodes $x_{n,k} = \frac{k}{n}, k = 0,1 \ldots n$ and with an appropriate selection of the basic polynomials $p_{n,k}(x)$.

In 1912, Bernstein had the wonderful idea to select $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, inspired by the binomial probability distribution. He considered the binomial probability distribution assuming that the discrete random variable has the value $f(\frac{k}{n})$ with probability $p_{n,k}(x)$ and then he calculate the mean value. In 1969, [10], Stancu wanted to choose the nodes in another different way, in order to obtain more flexibility. So, he considered the nodes such as, when $n \to \infty$ the distance between two consecutive nodes and the distance between 0 and first node and also between last node and 1 to tend all to zero. Thus, Stancu introduced the following linear positive operators which are known as Bernstein-Stancu polynomials in literature

$$P_n^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right),$$

acting from $C[0,1]$ into $C[0,1]$, the space of all real valued continuous functions defined on $[0,1]$, where $n \in N$, $f \in C[0,1], x \in [0,1]$ and α, β are any two real numbers which satisfy the condition that $0 \leq \alpha \leq \beta$.

In the recent years, Stancu type generalization of the certain operators introduced by several researchers and obtained different type of approximation properties of many operators, we refer some of the important papers in this direction as [11], [12], [13], [14], [15], [16], [17], [18] etc.

Inspired by the above work, for $f \in C[0,1]$ we introduce the Stancu type generalization of the operators (2):

$$D_{n,\alpha,\beta}^{*(1/n)}(f;x) = (n+\beta+1) \sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(t)dt. \quad (3)$$

The goal of the present paper is to study the basic convergence theorem, Voronovskaja type asymptotic result, local approximation theorem, rate of convergence, weighted approximation and pointwise estimation of the operators (3).

2 Moment and central moment estimates

In this section, we prove some basic results which are useful to prove several theorems and results.
Let $e_i(t) = t^i$, $i = 0,1,2,3,4$.

Lemma 1. *[7] For the operators $D_n^{*(1/n)}(f;x)$, we have*

1. $D_n^{*(1/n)}(1;x) = 1;$
2. $D_n^{*(1/n)}(t;x) = \frac{2nx+1}{2(n+1)};$
3. $D_n^{*(1/n)}(t^2;x) = \frac{3n^3x^2+9n^2x-3n^2x^2+3nx+n+1}{3(n+1)^3};$
4. $D_n^{*(1/n)}(t^3;x) = \frac{4(n^5+3n^4+2n^3)x^3+6(n^4+n^3-2n^2)x^2+4(n^3+9n^2+2n)x+(n+1)(n+2)}{4(n+1)^4(n+2)};$
5. $D_n^{*(1/n)}(t^4;x) = \frac{n^4x^4}{(n+1)^4} + \frac{x(1-x)(60x^2n^7+60x^2n^5+180n^6x-60n^5x+130n^5-10n^4)}{5n(n+1)^5(n+2)(n+3)} + \frac{nx}{(n+1)^4} + \frac{2n^2(n^3+10n^2-3n-10)x^2+8n^2(1+2n)x^2-12n^4x^3}{(n+1)^5(n+2)}.$

Lemma 2. *For $x \in I$ and $0 \leq \alpha \leq \beta$, we have the following recursive relation between $D_{n,\alpha,\beta}^{*(1/n)}(t^m;x), m = 0,1,2\ldots$ and $D_n^{*(1/n)}(t^i;x), i = 0,1,2\ldots$ where $f(t) = t^i$ is the test function as*

$$D_{n,\alpha,\beta}^{*(1/n)}(f;x) = \sum_{i=0}^{m} \binom{m}{i} \left(\frac{n}{n+\beta}\right)^i \left(\frac{\alpha}{n+\beta}\right)^{m-i} D_n^{*(1/n)}(t^i;x).$$

Proof. From equation (3), we have

$$D_{n,\alpha,\beta}^{*(1/n)}(f;x) = (n+\beta+1) \sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} f(t)dt.$$

We can rewrite this equation as

$$\begin{aligned} D_{n,\alpha,\beta}^{*(1/n)}(f;x) &= (n+\beta+1) \sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f\left(\frac{nt+\alpha}{n+\beta}\right) \cdot \frac{n+1}{n+\beta+1} dt \\ &= (n+1) \sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f\left(\frac{nt+\alpha}{n+\beta}\right) dt \\ &= (n+1) \sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \sum_{i=0}^{m} \binom{m}{i} \left(\frac{nt}{n+\beta}\right)^i \left(\frac{\alpha}{n+\beta}\right)^{m-i} dt. \end{aligned}$$

$$\begin{aligned} D_{n,\alpha,\beta}^{*(1/n)}(f;x) &= \sum_{i=0}^{m} \binom{m}{i} \left(\frac{n}{n+\beta}\right)^i \left(\frac{\alpha}{n+\beta}\right)^{m-i} (n+1) \sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} t^i dt \\ &= \sum_{i=0}^{m} \binom{m}{i} \left(\frac{n}{n+\beta}\right)^i \left(\frac{\alpha}{n+\beta}\right)^{m-i} D_n^{*(1/n)}(t^i;x). \end{aligned}$$

Lemma 3. *For the operators $D_{n,\alpha,\beta}^{*(1/n)}(f;x)$, we have*

1. $D_{n,\alpha,\beta}^{*(1/n)}(1;x) = 1;$
2. $D_{n,\alpha,\beta}^{*(1/n)}(t;x) = \frac{2n^2x+n+2\alpha(n+1)}{2(n+\beta)(n+1)};$

$3.D^{*(1/n)}_{n,\alpha,\beta}(t^2;x)=\frac{n^4(n-1)x^2}{(n+\beta)^2(n+1)^3}+\frac{\left(n^3(3n+1)+2n^2\alpha(n+1)^2\right)x}{(n+\beta)^2(n+1)^3}+\frac{n^2+3n\alpha(n+1)+3\alpha^2(n+1)^2}{3(n+\beta)^2(n+1)^2};$

$4.D^{*(1/n)}_{n,\alpha,\beta}(t^3;x)=\frac{1}{4(n+\beta)^3(n+1)^4(n+1)}\big(4n^8(n+2)x^3+\frac{6n^7x^2(1-x)}{n+1}+\frac{6n^6x(1-x)}{n+2}+6n^5x(1-x)+6n^4x+5n^3+3n^2\alpha(n+2)(n^3x^2+5n^2x-n^2x^2+3nx+2n+3)\big)+\frac{3n\alpha^2(nx+1)+\alpha^3(n+1)}{(n+\beta)^3(n+1)};$

$5.D^{*(1/n)}_{n,\alpha,\beta}(t^4;x)=\frac{1}{(n+\beta)^4n(n+1)^5(n+2)(n+3)}\big(n^{11}(n+1)x^4+\frac{11n^{10}(n^2+1)x^3(1-x)}{(n+2)(n+3)}+\frac{6n^8(3n-1)x^2(1-x)}{(n+1)(n+2)}+\frac{5n^7(11n-1)x(1-x)}{(n+2)(n+3)}+10n^7(n+1)x^3+\frac{48n^8x^2(1-x)}{(n+2)}+\frac{60n^7x(1-x)}{(n+2)}+55n^6(n+1)x^2+70n^6x(1-x)+50n^5(n+1)x+24n^4(n+1)+4n^6(n+1)\alpha x^3+\frac{8n^7x^2(1-x)\alpha}{n+2}+\frac{11n^6x(1-x)\alpha}{n+2}+6n^5x(1-x)\alpha+11n^4(n+2)\alpha x+4n^3(n+3)\alpha x+6n^2\alpha^2(n+2)(n+3)(n^3x^2+2n^2x-n^2x^2+4nx+2n+2)\big)+\frac{6n\alpha^3(nx+1)+\alpha^4(n+1)}{(n+\beta)^4(n+1)}.$

*Proof.*From Lemma 1 and recursive relation in Lemma 2, we prove Lemma 3.

Lemma 4.*For $f\in C(I)$ (space of all real valued functions on I endowed with norm $\| f \|_{C(I)}=\sup_{x\in I}|f(x)|$), we have*

$$\| D^{*(1/n)}_{n,\alpha,\beta}(f) \|\leq\| f \| .$$

*Proof.*In view of (3) and Lemma 3, we get

$$\|D^{*(1/n)}_{n,\alpha,\beta}(f)\|\leq(n+\beta+1)\sum_{k=0}^{n}p^{(1/n)}_{n,k}(x)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}}|f(t)|dt$$
$$\leq\|f\|D^{*(1/n)}_{n,\alpha,\beta}(1;x)=\|f\|.$$

*Remark.*By simple applications of Lemma 3, we have

$$D^{*(1/n)}_{n,\alpha,\beta}((t-x);x)=\frac{n(1+2\alpha)+2\alpha-2(n+n\beta+\beta)x}{2(n+\beta)(n+1)}=\xi^{*(1/n)}_{n,\alpha,\beta}(x)$$

and

$$D^{*(1/n)}_{n,\alpha,\beta}((t-x)^2;x)=\frac{d_n(\beta)x^2}{(n+\beta)^2(n+1)^3}+\frac{d_n(\alpha,\beta)x}{(n+\beta)^2(n+1)^3}+\frac{n^2+3n\alpha(n+1)+3\alpha^2(n+1)^2}{3(n+\beta)^2(n+1)^2}=\zeta^{*(1/n)}_{n,\alpha,\beta}(x),$$

where $d_n(\beta)=-2n^4+n^3+n^3\beta^2+2n^3\beta+3n^2\beta^2+3n\beta^2+4n^2\beta+2n\beta+n^2+\beta^2$ and $d_n(\alpha,\beta)=3n^4+n^3+2n^2\alpha(n+1)^2-(n+2n\alpha+2\alpha)(n+\beta)(n+1)^2$.
Further,

$$D^{*(1/n)}_{n,\alpha,\beta}((t-x)^4;x)=O\left(\frac{1}{n^2}\right),\ as\ n\to\infty.$$

3 Direct Estimates

In this section we give some approximation results in several settings. For the reader's convenience we split up this section in more subsections.

Theorem 1.*Let $f\in C[0,1]$. Then $\lim_{n\to\infty}D^{*(1/n)}_{n,\alpha,\beta}(f;x)=f(x)$, uniformly in each compact subset of $[0,1]$.*

*Proof.*In view of Lemma 3, we get

$$\lim_{n\to\infty}D^{*(1/n)}_{n,\alpha,\beta}(e_i;x)=x^i,\ i=0,1,2,$$

uniformly in each compact subset of $[0,1]$. Applying Bohman-Korovkin theorem, it follows that $\lim_{n\to\infty}D^{*(1/n)}_{n,\alpha,\beta}(f;x)=f(x)$, uniformly in each compact subset of $[0,1]$.

3.1 Voronovskaja type theorem

A general Voronovskaja type theorem for a sequence of linear positive operators $(L_n)_n$, is a limit of the form:

$$\lim_{n\to\infty}\alpha_n\left(L_n(f;x)-f(x)\right)=E(x,f'(x),f'',\ldots).$$

For classical operators of approximation the usual value for α_n is $\alpha_n=n$.
Now, we prove Voronvoskaja type theorem for the operators $D^{*(1/n)}_{n,\alpha,\beta}$.

Theorem 2.*Let f be a bounded and integrable function on $[0,1]$, second derivative of f exists at a fixed point $x\in[0,1]$, then*

$$\lim_{n\to\infty}n\left(D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)\right)=\left(\frac{(2\alpha+1)-2(\beta+1)x}{2}\right)f'(x)+x(1-x)f''(x).$$

*Proof.*Let $x\in[0,1]$ be fixed. Using Taylor's expansion formula of function f, it follows

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}(t-x)^2f''(x)+r(t,x)(t-x)^2,\quad(4)$$

where $r(t,x)$ is a continuous function on $[0,1]$ and $\lim_{t\to x}r(t,x)=0$.
Applying $D^{*(1/n)}_{n,\alpha,\beta}$ on both sides of (4), we get

$$n\left(D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)\right)=nf'(x)D^{*(1/n)}_{n,\alpha,\beta}((t-x);x)+\frac{1}{2}nf''(x)D^{*(1/n)}_{n,\alpha,\beta}((t-x)^2;x)+nD^{*(1/n)}_{n,\alpha,\beta}((t-x)^2r(t,x);x).$$

In view of Remark 2, we have

$$\lim_{n\to\infty}nD^{*(1/n)}_{n,\alpha,\beta}((t-x);x)=\frac{(2\alpha+1)-2(\beta+1)x}{2}\quad(5)$$

and

$$\lim_{n\to\infty} nD^{*(1/n)}_{n,\alpha,\beta}\left((t-x)^2;x\right) = 2x(1-x). \tag{6}$$

Now, we shall show that

$$\lim_{n\to\infty} nD^{*(1/n)}_{n,\alpha,\beta}\left(r(t,x)(t-x)^2;x\right) = 0.$$

By using Cauchy-Schwarz inequality, we have

$$D^{*(1/n)}_{n,\alpha,\beta}\left(r(t,x)(t-x)^2;x\right) \le \left(D^{*(1/n)}_{n,\alpha,\beta}(r^2(t,x);x)\right)^{1/2} \left(D^{*(1/n)}_{n,\alpha,\beta}((t-x)^4;x)\right)^{1/2}. \tag{7}$$

We observe that $r^2(x,x) = 0$ and $r^2(.,x) \in C[0,1]$. Then, it follows that

$$\lim_{n\to\infty} D^{*(1/n)}_{n,\alpha,\beta}(r^2(t,x);x) = r^2(x,x) = 0. \tag{8}$$

Now, from (7) and (8) we obtain

$$\lim_{n\to\infty} nD^{*(1/n)}_{n,\alpha,\beta}\left(r(t,x)(t-x)^2;x\right) = 0. \tag{9}$$

From (5), (6) and (9), we get the required result.

Next theorem uses the asymptotic formulae fulfilled by $D^{*(1/n)}_{n,\alpha,\beta}$ and $D^{*(1/n)}_{n}$ to state a sort of weak result that shows that for certain family of illustrative functions the new sequence approximates better than the previous operators.

Theorem 3.*Let $f \in C^2(I)$. Assume that there exists $n_0 \in N$, such that*

$$f(x) \le D^{*(1/n)}_{n,\alpha,\beta}(f;x) \le D^{*(1/n)}_{n}(f;x), \tag{10}$$

for all $n \ge n_0$ and $x \in (0,1)$. Then

$$x(1-x)f''(x) \ge (\alpha - \beta x)f'(x) \ge 0,\ x \in (0,1). \tag{11}$$

Particular $f'(x) \ge 0$ and $f''(x) \ge 0$.
Conversely, if (12) holds with strict inequalities at a given point $x \in (0,1)$, then there exists $n_0 \in N$ such that for $n \ge n_0$

$$f(x) < D^{*(1/n)}_{n,\alpha,\beta}(f;x) < D^{*(1/n)}_{n}(f;x).$$

*Proof.*From (10) we have that

$$0 \le n(D^{*(1/n)}_{n,\alpha,\beta}(f;x) - f(x)) \le n(D^{*(1/n)}_{n}(f;x) - f(x)),$$

for all $n \ge n_0$ and $x \in (0,1)$.
Then, using Theorem 2 and [7],

$$0 \le (\alpha - \beta x)f'(x) \le x(1-x)f''(x),$$

from which (12) follows directly.
Conversely, if (12) holds with strict inequalities for a given $x \in (0,1)$, then directly

$$0 < (\alpha - \beta x)f'(x) < x(1-x)f''(x),$$

and using again Theorem 2 and [7], the proof follows.

3.2 Local approximation

This section deals with the local approximation properties for the defined operators.
For $C_B[0,\infty)$, let us consider the following K-functional:

$$K_2(f;\delta) = \inf_{x\in W^2_\infty}\{\| f-g \| + \delta \| g'' \|\},$$

where $\delta > 0$ and $W^2_\infty = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By p. 177, Theorem 2.4 in [19], there exists an absolute constant $M > 0$ such that

$$K_2(f;\delta) \le M\omega_2(f;\sqrt{\delta}), \tag{12}$$

where $\omega_2(f;\sqrt{\delta})$ is second order modulus of continuity defined by

$$\omega_2(f;\sqrt{\delta}) = \sup_{0<|h|\le\sqrt{\delta}} \sup_{x\in[0,\infty)} | f(x+2h) - 2f(x+h) + f(x) | .$$

The usual modulus of smoothness (or simply modulus of continuity of first order) for $f \in C_B[0,\infty)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and is defined as

$$\omega(f,\delta) = \sup_{0<|h|\le\delta} \sup_{x\in[0,\infty)} | f(x+h) - f(x) | .$$

Now, we present the direct local approximation theorem for the operators $D^{*(1/n)}_{n,\alpha,\beta}(f;x)$.

Theorem 4.*Let $f \in C[0,1]$. Then, for every $x \in [0,1]$, we have*

$$| D^{*(1/n)}_{n,\alpha,\beta}(f;x) - f(x) | \le M\omega_2\left(f, \chi^{*(1/n)}_{n,\alpha,\beta}(x)\right) + \omega\left(f, \xi^{*(1/n)}_{n,\alpha,\beta}\right),$$

where M is a positive constant and

$$\chi^{*(1/n)}_{n,\alpha,\beta}(x) = \left(\zeta^{*(1/n)}_{n,\alpha,\beta}(x) + \left(\xi^{*(1/n)}_{n,\alpha,\beta}\right)^2\right)^{1/2}.$$

*Proof.*For $x \in [0,1]$, we consider the auxiliary operators $\overline{D}^{*(1/n)}_{n,\alpha,\beta}$ defined by

$$\overline{D}^{*(1/n)}_{n,\alpha,\beta}(f;x) = D^{*(1/n)}_{n,\alpha,\beta}(f;x) - f\left(\frac{2n^2x + n + 2\alpha(n+1)}{2(n+\beta)(n+1)}\right) + f(x). \tag{13}$$

From Lemma 3, we observe that the operators $\overline{D}^{*(1/n)}_{n,\alpha,\beta}$ are linear and reproduce the linear functions.
Hence

$$\overline{D}^{*(1/n)}_{n,\alpha,\beta}((t-x);x) = 0. \tag{14}$$

Let $g \in W^2_\infty$ and $x,t \in [0,1]$. By Taylor's expansion we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv.$$

Applying the operator $\overline{D}^{*(1/n)}_{n,\alpha,\beta}$ on both sides of the above equation and using (14), we get

$$\overline{D}^{*(1/n)}_{n,\alpha,\beta}(g;x) - g(x) = \overline{D}^{*(1/n)}_{n,\alpha,\beta}\left(\int_x^t (t-v)g''(v)dv, x\right).$$

Thus, by (13) we get

$$|\overline{D}^{*(1/n)}_{n,\alpha,\beta}(g;x)-g(x)|$$

$$\le D^{*(1/n)}_{n,\alpha,\beta}\left(\left|\int_x^t (t-v)g''(v)dv\right|,x\right)$$

$$+\left|\int_x^{\frac{2n^2x+n+2\alpha(n+1)}{2(n+\beta)(n+1)}}\left(\frac{2n^2x+n+2\alpha(n+1)}{2(n+\beta)(n+1)}-v\right)g''(v)dv\right|$$

$$\le \left(\zeta^{*(1/n)}_{n,\alpha,\beta}(x)+\left(\xi^{*(1/n)}_{n,\alpha,\beta}(x)\right)^2\right)\parallel g''\parallel$$

$$\le \left(\chi^{*(1/n)}_{n,\alpha,\beta}(x)\right)^2\parallel g''\parallel. \quad (15)$$

On other hand, by (13) and Lemma 4, we have

$$|\overline{D}^{*(1/n)}_{n,\alpha,\beta}(f;x)|\le \parallel f\parallel. \quad (16)$$

Using (15) and (16) in (13), we obtain

$$|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|$$

$$\le |\overline{D}^{*(1/n)}_{n,\alpha,\beta}(f-g;x)|+|(f-g)(x)|+|\overline{D}^{*(1/n)}_{n,\alpha,\beta}(g;x)-g(x)|$$

$$+\left|f\left(\frac{2n^2x+n+2\alpha(n+1)}{2(n+\beta)(n+1)}\right)-f(x)\right|$$

$$\le 2\parallel f-g\parallel+\left(\chi^{*(1/n)}_{n,\alpha,\beta}(x)\right)^2\parallel g''\parallel$$

$$+\left|f\left(\frac{n^2x+n(\alpha+1)+2\alpha}{(n+\beta)(n+2)}\right)-f(x)\right|.$$

Taking infimum over all $g\in W^2_\infty$, we get

$$|\,D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)\,|$$

$$\le K_2\left(f,(\chi^{*(1/n)}_{n,\alpha,\beta}(x))^2\right)+\omega\left(f,\xi^{*(1/n)}_{n,\alpha,\beta}(x)\right).$$

In view of (12), we get

$$|\,D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)\,|$$

$$\le M\omega_2\left(f,\chi^{*(1/n)}_{n,\alpha,\beta}(x)\right)+\omega\left(f,\xi^{*(1/n)}_{n,\alpha,\beta}(x)\right),$$

which completed the proof.

Next, we obtain the local direct estimate of the operators defined in (3), using the Lipschitz-type maximal function of order η introduced by B. Lenze [20] as follows:

$$\widetilde{\omega}_\eta(f,x)=\sup_{t\ne x,t\in[0,1]}\frac{|f(t)-f(x)|}{|t-x|^\eta},x\in[0,1],\eta\in(0,1] \quad (17)$$

Here, an upper bound can be obtained for the defined operators (3) with the function in the terms of Lipschitz Maximal function.

Theorem 5. *Let $f\in C[0,1]$ and $0<\eta\le 1$. Then, for all $x\in[0,1]$, we have*

$$|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|\le \widetilde{\omega}_\eta(f,x)\left(\zeta^{*(1/n)}_{n,\alpha,\beta}(x)\right)^{\eta/2}.$$

Proof. In view of (17), we have

$$|f(t)-f(x)|\le \widetilde{\omega}_\eta(f,x)|t-x|^\eta$$

and

$$|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|\le \widetilde{\omega}_\eta(f,x)D^{*(1/n)}_{n,\alpha,\beta}(|t-x|^\eta;x).$$

Applying the Hölder's inequality with $p=\frac{2}{\eta}$ and $\frac{1}{q}=1-\frac{1}{p}$, we get

$$|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|\le \widetilde{\omega}_\eta(f,x)D^{*(1/n)}_{n,\alpha,\beta}((t-x)^2;x)^{\eta/2}$$

$$\le \widetilde{\omega}_\eta(f,x)\left(\zeta^{*(1/n)}_{n,\alpha,\beta}(x)\right)^{\eta/2}.$$

Thus, the proof is completed.

Özarslan and Aktuğlu [21] defined a new type of Lipschitz-space having two parameters. Let $a,b>0$ be fixed numbers, then Lipschitz-type-space is defined by:

$$Lip_M^{(a,b)}(\eta)=\left(f\in C[0,1]:|f(t)-f(x)|\le M\frac{|t-x|^\eta}{(t+ax^2+bx)^{\eta/2}}\right),$$

where M is a positive constant $x,t\in(0,1)$ and $0<\eta\le 1$. Using the above definition, we have the local approximation result:

Theorem 6. *Let $f\in Lip_M^{(a,b)}(\eta)$. Then, for all $x\in(0,1]$, we have*

$$|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|\le M\left(\frac{\zeta^{*(1/n)}_{n,\alpha,\beta}(x)}{ax^2+bx}\right)^{\eta/2}.$$

Proof. First, we prove the result for the case $\eta=1$. Then, for $f\in Lip_M^{(a,b)}(1)$, and $x\in(0,1]$, we have

$$|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|\le D^{*(1/n)}_{n,\alpha,\beta}(|f(t)-f(x)|;x)$$

$$\le MD^{*(1/n)}_{n,\alpha,\beta}\left(\frac{|t-x|}{(t+ax^2+bx)^{1/2}};x\right)$$

$$\le \frac{M}{(ax^2+bx)^{1/2}}D^{*(1/n)}_{n,\alpha,\beta}(|t-x|;x).$$

Applying Cauchy-Schwarz inequality, we get

$$|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|\le \frac{M}{(ax^2+bx)^{1/2}}\left(D^{*(1/n)}_{n,\alpha,\beta}((t-x)^2;x)\right)^{1/2}$$

$$\le M\left(\frac{\zeta^{*(1/n)}_{n,\alpha,\beta}(x)}{ax^2+bx}\right)^{1/2}.$$

Thus the result holds for $\eta=1$.
Now, we prove that the result is true for the case $0<\eta<1$. Then, for $f\in Lip_M^{(a,b)}(\eta)$, and $x\in(0,1]$, we get

$$|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|\le \frac{M}{(ax^2+bx)^{\eta/2}}D^{*(1/n)}_{n,\alpha,\beta}(|t-x|^\eta;x).$$

Taking $p=\frac{1}{\eta}$ and $q=\frac{1}{1-\eta}$, applying the Hölders inequality, we have

$$|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|\leq\frac{M}{(ax^2+bx)^{\eta/2}}\left(D^{*(1/n)}_{n,\alpha,\beta}(|t-x|;x)\right)^{\eta}.$$

Finally by Cauchy-Schwarz inequality, we get

$$|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|\leq M\left(\frac{\zeta^{*(1/n)}_{n,\alpha,\beta}(x)}{ax^2+bx}\right)^{\eta/2}.$$

Thus, the proof is completed.

3.3 Pointwise estimates

In the present section, we obtain some pointwise estimates of the rate of convergence of the operators $D^{*(1/n)}_{n,\alpha,\beta}$. First, we give the relationship between the local smoothness of f and local approximation.
We know that a function $f\in C[0,1]$ is in $\mathrm{Lip}_{M_f}(\eta)$ on E, $\eta\in(0,1]$, E$\subset[0,1]$ if it satisfies the condition

$$|f(t)-f(x)|\leq M_f|t-x|^{\eta},\ t\in E \text{ and } x\in[0,1],$$

where M_f is a constant depending only on η and f.

Theorem 7. *Let $f\in C[0,1]\cap Lip_{M_f}(\eta)$, $\eta\in(0,1]$ and E be any bounded subset of the interval $[0,1]$. Then, for each $x\in[0,1]$, we have*

$$|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|\leq M_f\left(\left(\zeta^{*(1/n)}_{n,\alpha,\beta}(x)\right)^{\eta/2}+2(d(x,E))^{\eta}\right),$$

where M_f is a constant depending on η and f and $d(x,E)$ is the distance between x and E defined as

$$d(x,E)=\inf\{|t-x|:t\in E\}.$$

Proof. Let $\overline{E}$ be the closure of E in $[0,1]$. Then, there exists at least one point $x_0\in\overline{E}$ such that

$$d(x,E)=|x-x_0|.$$

From the triangle inequality, we have

$$|f(t)-f(x)|\leq|f(t)-f(x_0)|+|f(x)-f(x_0)|.$$

Using the definition of $\mathrm{Lip}_{M_f}(\eta)$, we get

$$\begin{aligned}&|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|\\&\leq D^{*(1/n)}_{n,\alpha,\beta}(|f(t)-f(x_0)|;x)+D^{*(1/n)}_{n,\alpha,\beta}(|f(x)-f(x_0)|;x)\\&\leq M_f\left(D^{*(1/n)}_{n,\alpha,\beta}(|t-x_0|^{\eta};x)+|x-x_0|^{\eta}\right)\\&\leq M_f\left(D^{*(1/n)}_{n,\alpha,\beta}(|t-x|^{\eta};x)+2|x-x_0|^{\eta}\right).\end{aligned}$$

Now, applying Hölder's inequality with $p=\dfrac{2}{\eta}$ and $\dfrac{1}{q}=1-\dfrac{1}{p}$, we obtain

$$\begin{aligned}&|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|\\&\leq M_f\left(\{D^{*(1/n)}_{n,\alpha,\beta}(|t-x|^2;x)\}^{\eta/2}+2(d(x,E))^{\eta}\right),\end{aligned}$$

from which the desired result immediate.

3.4 Rate of convergence

Let $\omega_a(f,\delta)$ denote the usual modulus of continuity of f on the closed interval $[0,a], a>0$, and defined as

$$\omega_a(f,\delta)=\sup_{|t-x|\leq\delta}\ \sup_{x,t\in[0,a]}|f(t)-f(x)|.$$

We observe that for a function $f\in C_B[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero.
Now, we give a rate of convergence theorem for the operators $D^{*(1/n)}_{n,\alpha,\beta}$.

Theorem 8. *Let $f\in C_B[0,\infty)$ and $\omega_{a+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,a+1]\subset[0,\infty)$, where $a>0$. Then, we have*

$$\begin{aligned}&|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|\\&\leq 6M_f(1+a^2)\zeta^{*(1/n)}_{n,\alpha,\beta}(a)+2\omega_{a+1}\left(f,\sqrt{\zeta^{*(1/n)}_{n,\alpha,\beta}(a)}\right),\end{aligned}$$

where $\zeta^{*(1/n)}_{n,\alpha,\beta}(a)$ is defined in Remark 2 and M_f is a constant depending only on f.

Proof. For $x\in[0,a]$ and $t>a+1$. Since $t-x>1$, we have

$$\begin{aligned}|f(t)-f(x)|&\leq M_f(2+x^2+t^2)\\&\leq M_f(t-x)^2(2+3x^2+2(t-x)^2)\\&\leq 6M_f(1+a^2)(t-x)^2.\end{aligned}$$

For $x\in[0,a]$ and $t\leq a+1$, we have

$$|f(t)-f(x)|\leq\omega_{a+1}(f,|t-x|)\leq\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta)$$

with $\delta>0$.
From the above, we have

$$|f(t)-f(x)|\leq 6M_f(1+a^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta),$$

for $x\in[0,a]$ and $t\geq0$.
Thus

$$\begin{aligned}&|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|\\&\leq 6M_f(1+a^2)(D^{*(1/n)}_{n,\alpha,\beta}(t-x)^2;x)\\&\quad+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}(D^{*(1/n)}_{n,\alpha,\beta}(t-x)^2;x)^{\frac{1}{2}}\right).\end{aligned}$$

Applying Cauchy-Schwarz's inequality, we get
$|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|$

$$\leq 6M_f(1+a^2)\zeta^{*(1/n)}_{n,\alpha,\beta}(a)+2\omega_{a+1}\left(f,\sqrt{\zeta^{*(1/n)}_{n,\alpha,\beta}(a)}\right),$$

on choosing $\delta=\sqrt{\zeta^{*(1/n)}_{n,\alpha,\beta}(a)}$. This completes the proof of theorem.

3.5 Weighted approximation

In this section we give some weighted approximation properties of the operators $D^{*(1/n)}_{n,\alpha,\beta}$. We do this for the following class of continuous functions defined on $[0,1]$.
Let $B_\nu[0,1]$ denote the weighted space of real-valued functions f defined on $[0,1]$ with the property $|f(x)|\leq M_f\nu(x)$ for all $x\in[0,1]$, where $\nu(x)=1+x^2$ is a weight function and M_f is a constant depending on the function f. We also consider the weighted subspace $C_\nu[0,1]$ of $B_\nu[0,1]$ given by $C_\nu[0,1]=\{f\in B_\nu[0,1]: f$ is continuous on $[0,1]\}$ and $C^*_\nu[0,1]$ denotes the subspace of all functions $f\in C_\nu[0,1]$ for which $\lim\limits_{|x|\to\infty}\frac{f(x)}{\nu(x)}$ exists finitely.
It is obvious that $C^*_\nu[0,1]\subset C_\nu[0,1]\subset B_\nu[0,1]$. The space $B_\nu[0,1]$ is a normed linear space with the following norm:

$$\| f\|_\nu=\sup_{x\in[0,1]}\frac{|f(x)|}{\nu(x)}.$$

Theorem 9. *For each $f\in C^*_\nu[0,1]$, we have*

$$\lim_{n\to\infty}\| D^{*(1/n)}_{n,\alpha,\beta}(f)-f\|_\nu=0.$$

Proof. From [22], we know that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\| D^{*(1/n)}_{n,\alpha,\beta}(e_i)-e_i\|_\nu=0,\ i=0,1,2. \tag{18}$$

Since $D^{*(1/n)}_{n,\alpha,\beta}(1;x)=1$, the condition in (18) holds true for $i=0$.
By Lemma 3, we have

$$\| D^{*(1/n)}_{n,\alpha,\beta}(t)-x\|_\nu = \sup_{x\in[0,1]}\frac{|D^{*(1/n)}_{n,\alpha,\beta}(t;x)-x|}{1+x^2} \leq \left|\frac{2n+(n+1)(2\alpha+\beta)}{2(n+\beta)(n+1)}\right|$$

which implies that $\lim\limits_{n\to\infty}\| D^{*(1/n)}_{n,\alpha,\beta}(t)-x\|_\nu=0$.
Again by Lemma 3, we have

$$\| D^{*(1/n)}_{n,\alpha,\beta}(t^2)-x^2\|_\nu = \sup_{x\in[0,1]}\frac{|D^{*(1/n)}_{n,\alpha,\beta}(t^2;x)-x^2|}{1+x^2}$$

$$\leq\left|\frac{n^4(n-1)}{(n+\beta)^2(n+1)^3}-1\right|+\left|\frac{(n^3(3n+1)+2n^2\alpha(n+1)^2)}{(n+\beta)^2(n+1)^3}\right| +\left|\frac{n^2+3n\alpha(n+1)+3\alpha^2(n+1)^2}{3(n+\beta)^2(n+1)^2}\right|,$$

which implies that $\lim\limits_{n\to\infty}\| D^{*(1/n)}_{n,\alpha,\beta}(t^2)-x^2\|_\nu=0$.
This completes the proof of theorem.

Now we give the following theorem to approximate all functions in C^*_ν. Such type of results are given in [23] for locally integrable functions.

Theorem 10. *For each $f\in C^*_\nu$ and $\vartheta>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,1]}\frac{|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^{1+\vartheta}}=0.$$

Proof. For any fixed $x_0\in[0,1]$,
$\sup_{x\in[0,1]}\frac{|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^{1+\vartheta}}$

$$\leq\sup_{x\in[0,x_0]}\frac{|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^{1+\vartheta}}+\sup_{x\in[x_0,1]}\frac{|D^{*(1/n)}_{n,\alpha,\beta}(f;x)-f(x)|}{(1+x^2)^{1+\vartheta}}$$

$$\leq\| D^{*(1/n)}_{n,\alpha,\beta}(f)-f\|_{C[0,x_0]}+\| f\|_\nu\sup_{x\in[x_0,1]}\frac{|D^{*(1/n)}_{n,\alpha,\beta}(1+t^2;x)|}{(1+x^2)^{1+\vartheta}} +\sup_{x\in[x_0,1]}\frac{|f(x)|}{(1+x^2)^{1+\vartheta}}$$

$$=J_1+J_2+J_3,\text{(say)} \tag{19}$$

Since $\|f\|_\nu=\sup_{x\in[0,1]}\frac{|f(x)|}{(1+x^2)}\implies|f(x)|\leq\|f\|_\nu(1+x^2)$, we obtain
$J_3=\sup_{x\in[x_0,1]}\frac{|f(x)|}{(1+x^2)^{1+\vartheta}}$

$$\leq\sup_{x\in[x_0,1]}\frac{\|f\|_\nu}{(1+x^2)^\vartheta}\leq\sup_{x\in[x_0,1]}\frac{\|f\|_\nu}{(1+x_0^2)^\vartheta} \tag{20}$$

Let $\varepsilon>0$ be arbitrary. In view of Remark(2), there exists a $n_0\in\mathbb{N}$ such that
$\|f\|_\nu\frac{|D^{*(1/n)}_{n,\alpha,\beta}((1+t^2);x)|}{(1+x^2)^{1+\vartheta}}$

$$<\frac{\|f\|_\nu}{(1+x^2)^{1+\vartheta}}\left((1+x^2)+\frac{\varepsilon}{3\|f\|_\nu})\right),\forall\ n\geq n_0$$

$$<\frac{\|f\|_\nu}{(1+x^2)^\vartheta}+\frac{\varepsilon}{3(1+x^2)^{1+\vartheta}},\qquad\forall\ n\geq n_0.$$

Hence, $J_2=\|f\|_\nu\sup_{x\in[x_0,1]}\frac{|D^{*(1/n)}_{n,\alpha,\beta}((1+t^2);x)|}{(1+x^2)^{1+\vartheta}}<\frac{\|f\|_\nu}{(1+x_0^2)^\vartheta}+\frac{\varepsilon}{3},\quad\forall\ n\geq n_0$.
Choose x_0 large enough, so that $\frac{\|f\|_\nu}{(1+x_0^2)^\vartheta}<\frac{\varepsilon}{6}$. Then, we get

$$J_2+J_3<\frac{2\varepsilon}{3},\quad\forall n\geq n_0. \tag{21}$$

$$J_1 = \|D^{*(1/n)}_{n,\alpha,\beta}(f) - f\|_{C[0,x_0]} < \frac{\varepsilon}{3} \quad \forall n \geq n_1. \tag{22}$$

Let $n' = \max\{n_0, n_1\}$. From (21) and (22) we have

$$\lim_{n\to\infty} \sup_{x\in[0,1]} \frac{|D^{*(1/n)}_{n,\alpha,\beta}(f;x) - f(x)|}{(1+x^2)^{1+\vartheta}} < \varepsilon \qquad \forall n \geq n'.$$

This completes the proof.

Acknowledgement

The authors would like to express their deep gratitude to the anonymous learned referee(s) and the editor for their valuable suggestions and constructive comments, which resulted in the subsequent improvement of this research article.

Conflict of Interest

The authors declare that they have no conflict of interest

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