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# Approximation of Functions belonging to Zygmund Class Associated with Hardy-Littlewood Series using Riesz Mean

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**Abstract:** Approximations of functions in the generalized Zygmund class associated with Fourier series have been studied by various researchers. In the present article, we have estimated the degree of approximation of functions of generalized Zygmund class associated with Hardy-Littlewood series using Riesz mean. Our result generalizes the result of Das et al.[2].

**Keywords:** Degree of approximation, Generalized Zygmund class, Hardy-Littlewood series, $(\overline{N}, q_n)$ -summability mean.

## 1 Introduction

Now-a-days the approximation of functions is a fundamental problem for engineers and scientists. The concept of approximating a function was first introduced by the great mathematician Weierstrass. Since then, a lot of results on degree of approximation of functions associated with Fourier series and conjugate Fourier series were studied by using different summability methods. The degree of approximation of functions belonging to Zygmund class (see, [1] and [4,5,6,7,8,9,10]) have been studied by many researchers but we didn't find any one who has studied the the approximation of functions in Zygmund class associated with Hardy-Littlewood series (HL-series). In 1998, Das et al.[2] have proved a result on degree of approximation of functions associated with HL-series in Hölder metric by using Borel's exponential mean. This motivated us to study the degree of approximation of functions of generalized Zygmund class associated with HL-series using Riesz mean.

## 2 Definitions and Notations

Let $h(x)$ be a periodic function of period $2\pi$, which is Lebesgue integrable in $[-\pi,\pi]$ and the Fourier series associated with $h(x)$ is given by

$$\sum_{n=0}^{\infty} U_n(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos nx + b_n \sin nx\Big). \tag{1}$$

Let $S_n^M(x)$ denote the $n$th modified partial sum of (1) given by

$$S_n^M(x) = \sum_{k=0}^{n-1} U_k(x) + \frac{U_n(x)}{2}.$$

Then the HL-series associated with $h(x)$ is

$$\frac{C_0}{2} + \sum_{n=1}^{\infty} \frac{S_n^M(x) - h(x)}{n} \tag{2}$$

where

$$C_0 = \frac{2}{\pi}\int_0^{\pi} \phi(x,u)\,\frac{u}{2}\cot\frac{u}{2}\,du$$

and

$$\phi(x,u) = h(x+u) + h(x-u) - 2h(x).$$

Let

$$\eta(u) = \int_u^{\pi} \phi(x,w)\,\frac{1}{2}\cot\frac{w}{2}\,dw\,. \tag{3}$$

Clearly, $\eta(u)$ is an even function and Lebesgue integrable. Also, the HL-series (2) is the Fourier series of

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$\eta(u)$ at $u=0$. We denote

$$L_l[-\pi,\pi]=\left\{\eta:[-\pi,\pi]\to R:\int_0^\pi|\eta(u)|^l du<\infty\right\}, l\geq 1.$$

Let us write

$$\xi_n^p(x)=\frac{2}{\pi}\int_0^\pi \eta(u)\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}\,du \tag{4}$$

which represents the $n$-th partial sum of $\eta(u)$. We define

$$\|\eta\|_l=\left(\frac{1}{\pi}\int_0^\pi|\eta(u)|^l\,du\right)^{\frac{1}{l}}, 1\leq l<\infty$$

and

$$\|\eta\|_l=ess\sup_{-\pi\leq u\leq\pi}|\eta(u)|,\ l=\infty.$$

Then the Zygmund modulus of continuity of $\eta(u)$ (see [11]) is defined as

$$m(\eta;r)=\sup_{0\leq r,u\in R}|\eta(u+w)+\eta(u-w)-2\eta(u)|.$$

Let $\boldsymbol{B}$ represents the Banach space of all $2\pi$ periodic functions which are continuous and defined over $[-\pi,\pi]$ under the supremum norm. We define

$$Z_{(\alpha)}$$
$$=\left\{\eta\in\boldsymbol{B}:|\eta(u+w)+\eta(u-w)-2\eta(u)|,\ 0<\alpha\leq 1\right\}.$$

Clearly,

$$Z_{(\alpha)}=O\left(|w|^\alpha\right)$$

and is a Banach space under the norm $\|.\|_{(\alpha)}$ defined by

$$\|\eta\|_{(\alpha)}=\sup_{-\pi\leq u\leq\pi}|\eta(u)|$$
$$+\sup_{u,w\neq 0}\frac{|\eta(u+w)+\eta(u-w)-2\eta(u)|}{|v|^\alpha}.$$

For $\eta\in L_l[-\pi,\pi],(l\geq 1)$, the integral Zygmund modulus of continuity is defined by

$$m_l(\eta;r)$$
$$=\sup_{0<w\leq r}\left\{\frac{1}{\pi}\int_0^\pi|\eta(u+w)+\eta(u-w)-2\eta(u)|^l\,du\right\}^{\frac{1}{l}}$$

and for $\eta\in\boldsymbol{B},\ l=\infty$,

$$m_\infty(\eta;r)=\sup_{0<w\leq r}\max_u|\eta(u+w)+\eta(u-w)-2\eta(u)|.$$

Clearly,

$$m_l(\eta;r)\to 0\quad\text{as}\quad l\to 0.$$

Suppose

$$Z_{(\alpha),\,l}=$$
$$\left\{\eta\in L_l[-\pi,\pi]:\left(\int_0^\pi|\eta(u+w)+\eta(u-w)-2\eta(u)|^l\,du\right)^{\frac{1}{l}}.\right.$$

Then $Z_{(\alpha),\,l}$ is a Banach space under the norm $\|.\|_{(\alpha),\,l}$ for $0<\alpha\leq 1$ and $l\geq 1$. Clearly,

$$\|\eta\|_{(\alpha),\,l}=\|\eta\|_l+\sup_{w\neq 0}\frac{\|\eta(.+w)+\eta(.-w)-2\eta(.)\|_l}{|w|^\alpha}.$$

For the Zygmund modulus of continuity $m$ satisfying

(a) $m(0)=0$

(b) $m(w_1+w_2)\leq m(w_1)+m(w_2)$,

we write

$$Z^{(m)}=\left\{\eta\in\mathbf{B}:|\eta(u+w)+\eta(u-w)-2\eta(u)|=O\left(m(w)\right)\right\}$$

and

$$Z_l^{(m)}$$
$$=\left\{\eta\in L_l:1\leq l<\infty,\sup_{w\neq 0}\frac{\|\eta(.+w)+\eta(.-w)-2\eta(.)\|_l}{m(w)}<\infty\right\}.$$

As $L_l,(l\geq 1)$ is complete, $Z_l^{(m)}$ is complete.

Clearly, $Z_l^{(m)}$ is a Banach space under the norm $\|.\|_l^{(m)}$, where

$$\|\eta\|_l^{(m)}=\|\eta\|_l+\sup_{w\neq 0}\frac{\|\eta(.+w)+\eta(.-w)-2\eta(.)\|_l}{m(w)}, l\geq 1.$$

Let $m(w)$ and $\mu(w)$ represent the Zygmund moduli of continuity such that $\left(\frac{m(w)}{\mu(w)}\right)$ is positive and non-decreasing then

$$\|\eta\|_l^{(\mu)}\leq\max.\left(1,\frac{m(2\pi)}{\mu(2\pi)}\right)\|\eta\|_l^{(m)}\leq\infty \tag{5}$$

Clearly,

$$Z_l^{(m)}\subseteq Z_l^{(\mu)}\subseteq L_l,(l\geq 1).$$

Let $\sum U_n$ be an infinite series with sequence of partial sums $\{s_n\}$ and $\{q_n\}$ be the sequence of non-negative numbers such that

$$Q_n=\sum_{k=0}^n q_k\to\infty\quad\text{as}\quad n\to\infty. \tag{6}$$

Let us write

$$t_n^N=\frac{1}{Q_n}\sum_{k=0}^n q_k s_k,\quad n=0,1,2,\ldots\ . \tag{7}$$

Then, $t_n^N$ represents the $(\overline{N},q_n)$ mean of $\{s_n\}$ generated by the sequence $\{q_n\}$. If

$$\lim_{n\to\infty}t_n^N\to s,$$

then the series $\sum u_n$ is said to be summable $(\overline{N},q_n)$.
We know, $(\overline{N},q_n)$ method is regular [3].
The following notations are used in the rest part of our paper:

$$\Omega(u,w)=\eta(u+w)+\eta(u-w)-2\eta(u) \tag{8}$$

$$\kappa_n^N(w)=\frac{1}{\pi\,Q_n}\sum_{k=0}^n q_k\frac{\sin\left(k+\frac{1}{2}\right)w}{\sin\left(\frac{w}{2}\right)} \tag{9}$$

## 3 Known Result

Using Borel's exponential mean, Das et al.[2] proved the following theorem:

**Theorem 3.1** *If $\xi_n^p(x)$ is the nth partial sum of the HL-series (2) and $B_p(T;x)$ is the Borel's exponential mean of $\{\xi_n^p(x)\}$ then for $0 \le \beta < \alpha \le 1$ and $h \in H_\alpha$,*

$$\left\|B_p(T;x) - \eta\left(\frac{\pi}{p}\right)\right\|_\beta = O\Big(p^{\beta-\alpha}(logp)^{1+\frac{\beta}{\alpha}}\Big).$$

## 4 Main Theorem

In the present article, we establish the following theorem on the degree of approximation of the HL-series of a function using $(\overline{N}, q_n)$ mean.

**Theorem 4.1** *Let $\xi_n^p(x)$ be the nth partial sum of the HL-series (2) and $\tau_n{}^{\overline{N}}$ be the $(\overline{N}, q_n)$ mean of $\{\xi_n^p(x)\}$. If $\eta \in Z_l^{(m)}$ then degree of approximation is given by*

$$E_n(\eta) = \inf_{\tau_n{}^{\overline{N}}} \|\tau_n{}^{\overline{N}} - \eta\|_l^\mu = O\left(\int_{\frac{1}{(n+1)}}^{\pi} \frac{m(w)}{w\,\mu(w)} dw\right). \quad (10)$$

*where $m(w)$ and $\mu(w)$ are the Zygmund moduli of continuity and $\frac{m(w)}{w\,\mu(w)}$ is positive and non-decreasing.*

## 5 Lemmas

In order to establish the main theorem, we require following lemmas.

**Lemma 5.1** *If $\kappa_n^{\overline{N}}(w)$ is as defined in (9), then*

$$(i)\ |\kappa_n^{\overline{N}}(w)| = O(n) \quad \texttt{for} \quad 0 \le w \le \frac{1}{(n+1)}.$$

$$(ii)\ |\kappa_n^{\overline{N}}(w)| = O\Big(\frac{1}{w}\Big) \quad \texttt{for} \quad \frac{1}{(n+1)} \le w \le \pi.$$

**Lemma 5.2 [4]** *If $\eta \in Z_l^{(m)}$ then for $0 < w \le \pi$,*

$$(i) \|\Omega(.,w)\|_l = O\Big(m(w)\Big).$$

$$(ii) \|\Omega(.+t,w) + \Omega(.-t,w) - 2\Omega(.,w)\|_l = O\Big(m(w)\Big) \text{ or } O\Big(m(t)\Big).$$

*(iii)If $m(w)$ and $\mu(w)$ are as defined in the Theorem 4.1, then*

$$\|\Omega(.+t,w) + \Omega(.-t,w) - 2\Omega(.,w)\|_l = O\Big(\mu(t)\frac{m(w)}{\mu(w)}\Big).$$

**Proof of Lemma 5.1**

For $0 \le w \le \frac{1}{n+1}$, we have $\sin nw \le n \sin w$. Then

$$\begin{aligned}
|\kappa_n^{\overline{N}}(w)| &= \left| \frac{1}{\pi\, Q_n} \sum_{k=0}^{n} q_k \frac{\sin\left(k+\frac{1}{2}\right)w}{\sin\frac{w}{2}} \right| \\
&\le \left| \frac{1}{\pi\, Q_n} \sum_{k=0}^{n} q_k \frac{(2k+1)\sin\frac{w}{2}}{\sin\frac{w}{2}} \right| \\
&= \frac{(2n+1)}{\pi} \left| \frac{1}{Q_n} \sum_{k=0}^{n} q_k \right| \\
&= O(n) \qquad (11)
\end{aligned}$$

This completes the proof of (i).

For $\frac{1}{n+1} \le w \le \pi$, using Jordan's lemma

$$\sin\left(\frac{w}{2}\right) \ge \frac{w}{\pi}.$$

Since, $sin\ nw \le 1$ for all $w$,

$$\begin{aligned}
|\kappa_n^{\overline{N}}(w)| &= \left| \frac{1}{\pi\, Q_n} \sum_{k=0}^{n} q_k \frac{\sin\left(k+\frac{1}{2}\right)w}{2\sin\frac{w}{2}} \right| \\
&\le \left| \frac{1}{\pi\, Q_n} \sum_{k=0}^{n} \frac{\pi}{w}\, q_k \right| \\
&= \left| \frac{1}{w\, Q_n} \sum_{k=0}^{n} q_k \right| \\
&= O\Big(\frac{1}{w}\Big). \qquad (12)
\end{aligned}$$

This completes the proof of (ii).

**Proof of Main Theorem**

Let $\tau_n^{\overline{N}}$ denote the $(\overline{N}, q_n)$ mean of $\{\xi_n^p(x)\}$ then

$$\tau_n^{\overline{N}} = \frac{1}{Q_n} \sum_{k=0}^{n} q_k\, \xi_k^p$$

and hence

$$\begin{aligned}
\tau_n^{\overline{N}} - \eta(u) &= \frac{2}{\pi} \int_0^\pi \Omega(u,w) \Big\{ \frac{1}{Q_n} \sum_{k=0}^{n} q_k \frac{\sin\left(k+\frac{1}{2}\right)w}{2\sin\frac{w}{2}} \Big\}\, dw \\
&= \int_0^\pi \Omega(u,w)\, \kappa_n^{\overline{N}}(w)\, dw \\
&= \sigma_n(u)\ \texttt{(say)}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\sigma_n(u+t) + \sigma_n(u-t) - 2\sigma_n(u) \\
&= \int_0^\pi \Big\{ \Omega(u+t,w) + \Omega(u-t,w) - 2\Omega(u,w) \Big\}\, \kappa_n^{\overline{N}}(w)\, dw
\end{aligned}$$

Using Minkowski's inequality, we have

$$\|\sigma_n(.+t)+\sigma_n(.-t)-2\sigma_n(.)\|_l$$
$$=\left\{\frac{1}{\pi}\int_0^\pi |\sigma_n(u+t)+\sigma_n(u-t)-2\sigma_n(u)|^l\,du\right\}^{\frac{1}{l}}$$
$$=\frac{1}{\pi}\left[\int_0^\pi \left|\int_0^\pi \left\{\Omega(u+t,w)+\Omega(u-t,w)-2\Omega(u,w)\right\}\times\kappa_n^{\overline{N}}(w)\,dw\right|^l du\right]^{\frac{1}{l}}$$
$$\le \int_0^\pi |\kappa_n^{\overline{N}}(w)|\left\{\frac{1}{\pi}\int_0^\pi |\Omega(u+t,w)+\Omega(u-t,w)-2\Omega(u,w)|^l du\right\}^{\frac{1}{l}} dw$$
$$=\int_0^\pi \|\Omega(.+t,w)+\Omega(.-t,w)-2\Omega(.,w)\|_l|\kappa_n^{\overline{N}}(w)|\,dw$$
$$=\int_0^{\frac{1}{n+1}} \|\Omega(.+t,w)+\Omega(.-t,w)-2\Omega(.,w)\|_l|\kappa_n^{\overline{N}}(w)|\,dw$$
$$+\int_{\frac{1}{n+1}}^\pi \|\Omega(.+t,w)+\Omega(.-t,w)-2\Omega(.,w)\|_l|\kappa_n^{\overline{N}}(w)|\,dw$$
$$=\Gamma_1+\Gamma_2,\ \texttt{(say)} \tag{13}$$

Using Lemma 5.1(i), Lemma 5.2 and monotonicity of $\left(\frac{m(w)}{\mu(w)}\right)$ with respect to $'w'$, we have

$$\Gamma_1$$
$$=\int_0^{\frac{1}{n+1}} \|\Omega(.+t,w)+\Omega(.-t,w)-2\Omega(.,w)\|_l|\kappa_n^{\overline{N}}(w)|\,dw$$
$$=\int_0^{\frac{1}{n+1}} O\left(\mu(t)\frac{m(w)}{\mu(w)}\right)O(n)\,dw$$

By using the second mean value theorem of integral, we have

$$\Gamma_1\le O\left(n\,\mu(t)\,\frac{m\left((n+1)^{-1}\right)}{\mu\left((n+1)^{-1}\right)}\int_0^{\frac{1}{n+1}} dw\right)$$
$$=O\left(\frac{n}{n+1}\,\mu(t)\,\frac{m\left((n+1)^{-1}\right)}{\mu\left((n+1)^{-1}\right)}\right)$$
$$=O\left(\mu(t)\,\frac{m\left((n+1)^{-1}\right)}{\mu\left((n+1)^{-1}\right)}\right) \tag{14}$$

Again, by using Lemma 5.1(ii) and Lemma 5.2, we get

$$\Gamma_2$$
$$=\int_{\frac{1}{n+1}}^\pi \|\Omega(.+t,w)+\Omega(.-t,w)-2\Omega(.,w)\|_l|\kappa_n^{\overline{N}}(w)|\,dw$$
$$\le\int_{\frac{1}{n+1}}^\pi O\left(\mu(t)\frac{m(w)}{\mu(w)}\right)\frac{1}{w}\,dw$$
$$=O\left(\mu(t)\int_{\frac{1}{n+1}}^\pi \frac{m(w)}{w\,\mu(w)}\,dw\right) \tag{15}$$

By (13), (14) and (15), we have

$$\|\sigma_n(.+t)+\sigma_n(.-t)-2\sigma_n(.)\|_l$$
$$=O\left(\mu(t)\frac{m\left((n+1)^{-1}\right)}{\mu\left((n+1)^{-1}\right)}\right)+O\left(\mu(t)\int_{\frac{1}{n+1}}^\pi \frac{m(w)}{w\,\mu(w)}\,dw\right)$$

Therefore, we have

$$\sup_{t\ne 0}\frac{\|\sigma_n(.+t)+\sigma_n(.-t)-2\sigma_n(.)\|_l}{\mu(t)}$$
$$=O\left(\frac{m\left((n+1)^{-1}\right)}{\mu\left((n+1)^{-1}\right)}\right)+O\left(\int_{\frac{1}{n+1}}^\pi \frac{m(w)}{w\,\mu(w)}\,dw\right) \tag{16}$$

Applying Minkowski's inequality to (8), we get

$$\|\Omega(u,w)\|_l=\|\eta(u+w)+\eta(u-w)-2\eta(u)\|_l$$
$$=O\left(m(w)\right) \tag{17}$$

Also, by using Lemma 5.1 and (17), we have

$$\|\sigma_n(.)\|_l\le\left(\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^\pi\right)\|\Omega(.,w)\|_l\ \ |\kappa_n^{\overline{N}}(w)|\,dw$$
$$=O\left(n\int_0^{\frac{1}{n+1}} m(w)\,dw\right)+O\left(\int_{\frac{1}{n+1}}^\pi \frac{m(w)}{w}dw\right)$$
$$=O\left(m\left((n+1)^{-1}\right)\right)+O\left(\int_{\frac{1}{n+1}}^\pi \frac{m(w)}{w}dw\right) \tag{18}$$

From (16) and (18), we have

$$\|\sigma_n(.)\|_l^\mu=\|\sigma_n(.)\|_l+\sup_{t\ne 0}\frac{\|\sigma_n(.+t)+\sigma_n(.-t)-2\sigma_n(.)\|_l}{\mu(t)}$$
$$=O\left(m\left((n+1)^{-1}\right)\right)+O\left(\int_{\frac{1}{n+1}}^\pi \frac{m(w)}{w}dw\right)$$
$$+O\left(\frac{m\left((n+1)^{-1}\right)}{\mu\left((n+1)^{-1}\right)}\right)+O\left(\int_{\frac{1}{n+1}}^\pi \frac{m(w)}{w\,\mu(w)}\,dw\right)$$
$$=\sum_{j=1}^4 G_j$$

In view of monotonicity of $\mu(w)$ for $0<w\le\pi$, we have

$$m(w)=\frac{m(w)}{\mu(w)}.\mu(w)\le\mu(\pi).\frac{m(w)}{\mu(w)}=O\left(\frac{m(w)}{\mu(w)}\right)$$

Therefore,

$G_1=O(G_3)$.

Again, by using monotonicity of $\mu(w)$,

$$G_2=\int_{\frac{1}{n+1}}^\pi \frac{m(w)}{w}dv$$
$$=\int_{\frac{1}{n+1}}^\pi \frac{m(w)}{w\,\mu(w)}\,\mu(w)dw\le\mu(\pi)\int_{\frac{1}{n+1}}^\pi \frac{m(w)}{w\,\mu(w)}dw=O(G_4)$$

Since, $\frac{m(w)}{\mu(w)}$ is positive and increasing

$$G_4=\int_{\frac{1}{n+1}}^\pi \frac{m(w)}{w\,\mu(w)}dw$$
$$=\frac{m\left((n+1)^{-1}\right)}{\mu\left((n+1)^{-1}\right)}\int_{\frac{1}{n+1}}^\pi \frac{dw}{w}\ge\frac{m\left((n+1)^{-1}\right)}{\mu\left((n+1)^{-1}\right)}$$

Therefore,

$G_3 = O(G_4)$.

Thus,

$$\|\sigma_n(.)\|_l^\mu = O(G_4) = O\Big(\int_{\frac{1}{n+1}}^{\pi} \frac{m(w)}{w\,\mu(w)}dw\Big)$$

Hence,

$$E_n(\eta) = \inf_n \|\sigma_n(.)\|_l^\mu = O\Big(\int_{\frac{1}{n+1}}^{\pi} \frac{m(w)}{w\,\mu(w)}dw\Big)$$

This completes the proof of the Main Theorem.

## 6 Conclusion

It can be seen that our main theorem is the generalization of the result due to Das et al.[2]. The result obtained by us is clearly the best approximation even though our result seems to be similar with the results obtained earlier by many researchers, as we have used the HL-series instead of Fourier series and single mean instead of product mean. Our result can be further generalized by taking product summability mean, which can be treated as the future scope for the researchers.

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## Conflict of Interest

The authors declare that they have no conflict of interest

## References

[1] A.A. Das, S.K. Paikray, T. Pradhan and H. Dutta, Approximation of signals in the weighted Zygmund class via Euler-Huasdroff product summabilitymean of Fourier series, *J. Indian Math. Soc.*, **87**(1-2), 22-36, (2020).

[2] G. Das, A.K. Ojha and B.K. Ray, Degree of approximation of functions associated with Hardy-Littlewood series in the Hölder metric by Borel Means, *J. Math. Anal. Appl.*, **219**, 279-293, (1998).

[3] G.H. Hardy, *Divergent Series*, Oxford University Press, First Edition, (1949).

[4] S. Lal and Shireen, Best approximation of functions of generalized Zygmund class by Matrix-Euler summability mean of Fourier series, *Bull. Math. Anal. Appl.*, **5**(4),1-13, (2013).

[5] L. Leindler, Strong approximation and generalized Zygmund class, *Acta. Sci. Math.*, **43**, 301-309 (1981).

[6] F. Moricz, Enlarged Lipschitz and Zygmund classes of functions and Fourier transforms, *East. J. Approx.*, **16**(3), 259-271, (2010).

[7] F. Moricz and J. Nemeth, Generalized Zygmund classes of functions and strong approximation of Fourier series, *Acta. Sci. Math.*, **73**, 637-647, (2007).

[8] H.K. Nigam, On approximation in generalized Zygmund class, *Demonstr. Math.*, **52**, 370-387, (2019).

[9] T. Pradhan, S.K. Paikray, A.A. Das and H. Dutta, On approximation of signals in the generalized Zygmund class via $(E,1)(N,p_n)$ summability means of conjugate Fourier series, *Proyecciones J. Math.*, **38**(5), 981-998, (2019).

[10] M.V. Singh, M. L. Mittal and B. E. Rhoades, Approximation of functions in the generalized Zygmund class using Hausdroff means, *J. Inequal. Appl.*, **101**, doi: 10.1186/s13600-017-1361-8, (2017).

[11] A. Zygmund, *Trigonometric series*, 2nd revised Edition, Cambridge university Press, (1959).

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