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$(L,\otimes)$-Fuzzy (S,T)-Soft Ideal and $(L,\otimes)$-Fuzzy (S,T)-Soft Ideal Base

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Abstract: The aim of our paper is to introduce the concepts of $(L,\otimes)$-fuzzy (S,T)-soft ideal and $(L,\otimes)$-fuzzy (S,T)-soft ideal base and study many of their characteristics. We point out that every $(L,\otimes)$-fuzzy (S,T)-soft ideal induces in a natural way an $(L,\otimes)$-fuzzy (S,T)-soft co-topology. We prove the existence of the products of $(L,\otimes)$-fuzzy (S,T)-soft ideals. Also, we investigate the image of $(L,\otimes)$-fuzzy (S,T)-soft ideals.

Keywords: $(L,\otimes)$-fuzzy (S,T)-soft ideal, $(L,\otimes)$-fuzzy (S,T)-soft ideal base, $(L,\otimes)$-fuzzy (S,T)-soft co-topology.

1 Introduction

Molodtsov [1] presented and studied the theory of soft sets. This theory was deemed as an important mathematical instrument to work on uncertainty. The concept of soft sets was applied in several areas such as: computer science, physics, engineering, medical science, social science and economics. Molodtsov [1,2] established the fundamental results of his new theory. Molodtsov et al. [3], introduced various applications of soft sets.

The concept of a fuzzy soft set was brought up by Maji et al. [4,5] and they examined its qualities. In the field of decision making, Maji et al. [6] and Roy,Maji [7] utilized soft sets and fuzzy soft sets to solve some obstacles in it. Many academics repeated their efforts to study the theory of fuzzy soft sets [8,9,10,11].

The notion of soft topological space was presented by Shabir and Naz [12] and they investigated many of its characteristics. Aygünoğlu et al. [13] came up with the concept of $(L,\otimes)$-fuzzy (S,T)-soft co-topology in Šostak's meaning [14]. Many topological notions are spread out to the theory of fuzzy soft sets ([15,16,17,18,19,20,21,22,23]).

Kuratowski [24] innovated the notion of ideal in a topological space and therefore Jankovic and Hamlet [25] used it to investigate the properties of ideal topological spaces. Sarkar [26] extended the idea of ideal to the theory of fuzzy set. Fuzzy ideal was expanded in many directions ([27,28,29,30,31,32,33]).

Ideal is considered as one of the most effective instruments in the theory of fuzzy sets, which inspires us to broaden the concept of ideal into the theory of fuzzy soft sets and explore their properties. The followings are the overall structure of this paper: In section 2, we review several basic concepts that will be useful in our research. In section 3, we present the idea of $(L,\otimes)$-fuzzy (S,T)-soft ideal and $(L,\otimes)$-fuzzy (S,T)-soft ideal base and study many of their characteristics. Also, We infer that every $(L,\otimes)$-fuzzy (S,T)-soft ideal induces in a natural way an $(L,\otimes)$-fuzzy (S,T)-soft co-topology. In section 4, we investigate more properties of $(L,\otimes)$-fuzzy (S,T)-soft ideal bases and we define the products of $(L,\otimes)$-fuzzy (S,T)-soft ideals. In the last section, we study the image of $(L,\otimes)$-fuzzy (S,T)-soft ideals and examine its characteristics.

2 Preliminaries

Let $L=(L,\leq,\vee,\wedge,0_L,1_L)$ be a completely distributive lattice where 0_L is the least element and 1_L is the greatest element. Throughout this work, the letter X stood for an initial universe and S, T are the sets of all parameters for X. L^X is the set of all L-fuzzy sets on X[34].

Definition 2.1.[35,36] A tripartite $(L,\leq,\otimes)$ is said to be

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strictly two-sided commutative quantale (for short, stsc-quantale) if it fulfills the following criteria:
(L1) $L=(L,\leq,0_L,1_L)$ is a complete lattice,
(L2) $(L,\otimes)$ is a commutative semigroup,
(L3) $v=v\otimes 1_L$, for every $v\in L$,
(L4) $(\bigvee_{i\in\Lambda} v_i)\otimes w=\bigvee_{i\in\Lambda}(v_i\otimes w)$, for each $v,w\in L$,

Definition 2.2.[36] For a stsc-quantale $(L,\leq,\otimes)$, a mapping $*:L\to L$ is said to be an order-reversing involution, if it fulfills the following criteria:
(i) $(u^*)^*=u$, for every $u\in L$.
(ii) If $u\leq v$ then, $v^*\leq u^*$, $u,v\in L$.
We suppose that $(L,\leq,\otimes,\oplus,*)$ is a stsc-quantale with an order-reversing involution $*$ and the binary operation $\oplus$ is obtained by $u\oplus v=(u^*\otimes v^*)^*$.

Lemma 2.3.[37] For each $u,v,w\in L$ the properties listed below are fulfilled
(i) if $v\leq w$, then $(u\otimes v)\leq(u\otimes w)$ and $(u\oplus v)\leq(u\oplus w)$,
(ii) $u\otimes v\leq u\wedge v\leq u\vee v\leq u\oplus v$.

Definition 2.4.[13] g is called an L-fuzzy soft sets on X, such that g is mapped each element of a parameter set T into element in L^X, i.e., $g_t:=g(t)$ is an L-fuzzy set on X, for each $t\in T$. We use the symbol $(L^X)^T$ to denote to the family of all L-fuzzy soft sets on X. Let $g_1,g_2\in(L^X)^T$, then we have the following properties:
(i) g_1 is a subset of g_2 and write $g_1\sqsubseteq g_2$ if $(g_1)_t\leq(g_2)_t$, $\forall t\in T$. $g_1=g_2$ iff $g_1\sqsubseteq g_2$ and $g_2\sqsubseteq g_1$.
(ii) $g=g_1\sqcap g_2$ is the intersection of $g_1,g_2\in(L^X)^T$, where $g_t=(g_1)_t\wedge(g_2)_t$, for each $t\in T$.
(iii) $g=g_1\sqcup g_2$ is the union of $g_1,g_2\in(L^X)^T$, where $g_t=(g_1)_t\vee(g_2)_t$, for each $t\in T$.
(iv) For $g_1,g_2\in(L^X)^T$, $g=g_1\otimes g_2$ is defined by $g_t=(g_1)_t\otimes(g_2)_t$, for each $t\in T$.
(v) For $g_1,g_2\in(L^X)^T$, $g=g_1\oplus g_2$ is defined by $g_t=(g_1)_t\oplus(g_2)_t$, for each $t\in T$.
(vi) The complement of g is g^*, where $g^*:T\to L^X$ is a map given by $g_t^*=(g_t)^*$, for each $t\in T$.
(vii) A null L-fuzzy soft set is 0_X where, $(0_X)_t(x)=0_L$, for each $t\in T,x\in X$.
(viii) An absolute L-fuzzy soft set is 1_X where, $(1_X)_t(x)=1_L$, for each $t\in T,x\in X$.

Definition 2.5.[13] Let T_1 and T_2 be parameters sets for the crisp sets X and Y, respectively and let $\phi:X\to Y$, $\psi:T_1\to T_2$ be two maps. Then $\phi_\psi:(L^X)^{T_1}\to(L^Y)^{T_2}$ is called a fuzzy soft mapping.
(i) For $h\in(L^X)^{T_1}$, $\phi_\psi(h)_t(y)=\bigvee_{x\in\psi^{-1}(\{y\})}(\bigvee_{e\in\psi^{-1}(\{t\})}h_e(x))$, $\forall t\in T_2,\forall y\in Y$,
(ii) For $f\in(L^Y)^{T_2}$, $\phi_\psi^{\leftarrow}(f)_e(x)=f_{\psi(e)}(\phi(x)),\forall e\in T_1,\forall x\in X$.
(iii) $\phi_\psi:(L^X)^{T_1}\to(L^Y)^{T_2}$ is called injective (resp. surjective, bijective) if ϕ and ψ are both injective (resp. surjective, bijective).

Lemma 2.6.[38] Let $\phi_\psi:(L^X)^{T_1}\to(L^Y)^{T_2}$. Then, for $h,h_i\in(L^X)^{T_1}$ and $f,f_i\in(L^Y)^{T_2}$, we have
(i) $f\sqsupseteq\phi_\psi(\phi_\psi^{\leftarrow}(f))$ and $f=\phi_\psi(\phi_\psi^{\leftarrow}(f))$ if ϕ_ψ is surjective,
(ii) $h\sqsubseteq\phi_\psi^{\leftarrow}(\phi_\psi(h))$ and $h=\phi_\psi^{\leftarrow}(\phi_\psi(h))$ if ϕ_ψ is injective,
(iii) if ϕ_ψ is injective,

$$\phi_\psi(h)_{e_2}(y)=\begin{cases}h_{e_1}(x), \text{ if } & x\in\phi^{-1}(y),e_1\in\psi^{-1}(e_2)\\ 0_L, & \text{otherwise.}\end{cases}$$

(iv) $\phi_\psi^{\leftarrow}(f^*)=(\phi_\psi^{\leftarrow}(f))^*$,
(v) $\phi_\psi^{\leftarrow}(\sqcup_{i\in\Lambda}f_i)=\sqcup_{i\in\Lambda}\phi_\psi^{\leftarrow}(f_i)$,
(vi) $\phi_\psi^{\leftarrow}(\sqcap_{i\in\Lambda}f_i)=\sqcap_{i\in\Lambda}\phi_\psi^{\leftarrow}(f_i)$,
(vii) $\phi_\psi(\sqcup_{i\in\Lambda}h_i)=\sqcup_{i\in\Lambda}\phi_\psi(h_i)$,
(viii) $\phi_\psi(\sqcap_{i\in\Lambda}h_i)\sqsubseteq\sqcap_{i\in\Lambda}\phi_\psi(h_i)$ the equality holds if ϕ_ψ is injective,
(ix) $\phi_\psi^{\leftarrow}(f_1\otimes f_2)=\phi_\psi^{\leftarrow}(f_1)\otimes\phi_\psi^{\leftarrow}(f_2)$,
(x) $\phi_\psi(h_1\otimes h_2)\sqsubseteq\phi_\psi(h_1)\otimes\phi_\psi(h_2)$ the equality holds if ϕ_ψ is injective.

Definition 2.7.[13] A mapping $\mathfrak{F}:S\to L^{(L^X)^T}$ (where $\mathfrak{F}_s:=\mathfrak{F}(s):(L^X)^T\to L$ is a map for each $s\in S$) is called an $(L,\otimes)$-fuzzy (S,T)-soft co-topology (for short, $(L_\otimes,S_T)$-fs-co-topology) on X if it fulfills the following criteria for each $s\in S$:
(SCO1) $\mathfrak{F}_s(0_X)=\mathfrak{F}_s(1_X)=1_L$,
(SCO2) $\mathfrak{F}_s(g_1\oplus g_2)\geq\mathfrak{F}_s(g_1)\otimes\mathfrak{F}_s(g_2)$, $\forall g_1,g_2\in(L^X)^T$,
(SCO3) $\mathfrak{F}_s(\sqcap_{i\in\Lambda}g_i)\geq\bigwedge_{i\in\Lambda}\mathfrak{F}_s(g_i),\forall g_i\in(L^X)^T,i\in\Lambda$.
$(X,\mathfrak{F})$ is called an $(L,\otimes)$-fuzzy (S,T)-soft co-topological space (for short, $(L_\otimes,S_T)$-fs-co-topological space).

3 $(L,\otimes)$-fuzzy (S,T)-soft ideal

Definition 3.1. A mapping $\mathscr{I}:S\to L^{(L^X)^T}$ (where $\mathscr{I}_s:=\mathscr{I}(s):(L^X)^T\to L$ is a map for each $s\in S$) is called an $(L,\otimes)$-fuzzy (S,T)-soft ideal (for short, $(L_\otimes,S_T)$-fs-ideal) on X is if it fulfills the following criteria for each $s\in S$:
(SI1) $\mathscr{I}_s(0_X)=1_L$ and $\mathscr{I}_s(1_X)=0_L$,
(SI2) $\mathscr{I}_s(f\oplus g)\geq\mathscr{I}_s(f)\otimes\mathscr{I}_s(g)$, for each $f,g\in(L^X)^T$,
(SI3) If $f\sqsubseteq g$, then $\mathscr{I}_s(f)\geq\mathscr{I}_s(g)$.
$(X,\mathscr{I})$ is called an $(L,\otimes)$-fuzzy (S,T)-soft idealed set (for short, $(L_\otimes,S_T)$-fs-idealed set).

If $\mathscr{I}_1$ and $\mathscr{I}_2$ are two $(L_\otimes,S_T)$-fs-ideals on X. We point out that $\mathscr{I}_1$ is finer than $\mathscr{I}_2$ (or $\mathscr{I}_2$ is coarser than $\mathscr{I}_1$), symbolized by $\mathscr{I}_2\sqsubseteq\mathscr{I}_1$ iff $(\mathscr{I}_2)_s(g)\leq(\mathscr{I}_1)_s(g)$, $\forall g\in(L^X)^T$, $s\in S$.

We can get $(L_\otimes,S_T)$-fs-co-topology on X from an $(L_\otimes,S_T)$-fs-ideal $\mathscr{I}:K\to L^{(L^X)^T}$ as illustrated in the subsequent theorem:
Theorem 3.2. Let $(X,\mathscr{I})$ be an $(L_\otimes,S_T)$-fs-idealed set. If

we define $\mathfrak{F}_{\mathscr{I}} : S \to L^{(L^X)^T}$ (where $(\mathfrak{F}_{\mathscr{I}})_s := \mathfrak{F}_{\mathscr{I}}(s) : (L^X)^T \to L$ is a map for each $s \in S$) as:

$$(\mathfrak{F}_{\mathscr{I}})_s(g) = \begin{cases} \mathscr{I}_s(g), \text{ if } & g \neq 1_X \\ 1_L, \quad\text{ if } & g = 1_X. \end{cases}$$

Then, $(X, \mathfrak{F}_{\mathscr{I}})$ is an $(L_\otimes, S_T)$-fs-co-topological space.
Proof. (SCO1) and (SCO2) are unmistakable.

(SCO3) Let $\{g_i : i \in \Lambda\} \sqsubseteq (L^X)^T$. Then $\sqcap_{i\in\Lambda} g_i \sqsubseteq g_i$, $\forall i \in \Lambda$. By (SI3), we have

$$\mathscr{I}_s(\sqcap_{i\in\Lambda} g_i) \geq \mathscr{I}_s(g_i), \ \forall i \in \Lambda.$$

So,

$$(\mathfrak{F}_{\mathscr{I}})_s(\sqcap_{i\in\Lambda} g_i) \geq (\mathfrak{F}_{\mathscr{I}})_s(g_i), \ \forall i \in \Lambda.$$

Thus

$$(\mathfrak{F}_{\mathscr{I}})_s(\sqcap_{i\in\Lambda} g_i) \geq \bigwedge_{i\in\Lambda} (\mathfrak{F}_{\mathscr{I}})_s(g_i).$$

Hence, every $(L_\otimes, S_T)$-fs-idealed set provides an $(L_\otimes, S_T)$-fs-co-topological space.

Theorem 3.3. Let $\{\mathscr{I}_i\}_{i\in\Lambda}$ be a collection of $(L_\otimes, S_T)$-fs-ideals in a fixed set X. If a map $\mathscr{I} = \sqcup_{i\in\Lambda} \mathscr{I}_i : S \to L^{(L^X)^T}$, where $\mathscr{I} := \mathscr{I}_s = \bigvee_{i\in\Lambda} (\mathscr{I}_i)_s : (L^X)^T \to L$ defined by: $\mathscr{I}_s(g) = \bigvee_{i\in\Lambda} (\mathscr{I}_i)_s(g)$, $\forall g \in (L^X)^T$, $s \in S$, then $\mathscr{I}$ is an $(L_\otimes, S_T)$-fs-ideal on X.
Proof. (SI1) For each $s \in S$, we obtain

$$\mathscr{I}_s(0_X) = \bigvee_{i\in\Lambda} (\mathscr{I}_i)_s(0_X) = \bigvee_{i\in\Lambda} 1_L = 1_L$$

and

$$\mathscr{I}_s(1_X) = \bigvee_{i\in\Lambda} (\mathscr{I}_i)_s(1_X) = \bigvee_{i\in\Lambda} 0_L = 0_L.$$

(SI2) For every $g_1, g_2 \in (L^X)^T$, we have

$$\begin{aligned} \mathscr{I}_s(g_1) \otimes \mathscr{I}_s(g_2) &= \textstyle\bigvee_{i\in\Lambda} (\mathscr{I}_i)_s(g_1) \otimes \bigvee_{i\in\Lambda} (\mathscr{I}_i)_s(g_2) \\ &= \textstyle\bigvee_{i\in\Lambda} ((\mathscr{I}_i)_s(g_1) \otimes (\mathscr{I}_i)_s(g_2)) \\ &\leq \textstyle\bigvee_{i\in\Lambda} (\mathscr{I}_i)_s(g_1 \oplus g_2) \\ &= \mathscr{I}_s(g_1 \oplus g_2). \end{aligned}$$

(SI3) If $g_1 \sqsubseteq g_2$, then we have $(\mathscr{I}_i)_s(g_1) \geq (\mathscr{I}_i)_s(g_2)$, $\forall i \in \Lambda, s \in S$. Thus,

$$\mathscr{I}_s(g_1) = \bigvee_{i\in\Lambda} (\mathscr{I}_i)_s(g_1) \geq \bigvee_{i\in\Lambda} (\mathscr{I}_i)_s(g_2) = \mathscr{I}_s(g_2).$$

Hence, $\mathscr{I}$ is an $(L_\otimes, S_T)$-fs-ideal on X.

Theorem 3.4. Let $\phi : X \to Y$, $\psi : T_1 \to T_2$ and $\eta : S_1 \to S_2$. If $\mathscr{I}$ is an $(L_\otimes, S_{1T_1})$-fs-ideal on X and $\phi_{\psi,\eta}(\mathscr{I}) : S_2 \to L^{(L^Y)^{T_2}}$ (where $(\phi_{\psi,\eta}(\mathscr{I}))_s := \mathscr{I}_{\eta^{-1}(s)}(\phi_\psi^{\leftarrow})(h) : (L^Y)^{T_2} \to L$ is a map for each $s \in S_2$) defined by $(\phi_{\psi,\eta}(\mathscr{I}))_s(h) = \mathscr{I}_{\eta^{-1}(s)}((\phi_\psi)^{\leftarrow}(h))$, then $\phi_{\psi,\eta}(\mathscr{I})$ is an $(L_\otimes, S_{2T_2})$-fs-ideal on Y.
Proof. (SI1) It is obvious.
(SI2) For each $h_1, h_2 \in (L^Y)^{T_2}, s \in S_2$ we have,

$$\begin{aligned} &(\phi_{\psi,\eta}(\mathscr{I}))_s(h_1 \oplus h_2) \\ &= \mathscr{I}_{\eta^{-1}(s)}(\phi_\psi^{\leftarrow}(h_1 \oplus h_2)) \\ &= \mathscr{I}_{\eta^{-1}(s)}(\phi_\psi^{\leftarrow}(h_1) \oplus \phi_\psi^{\leftarrow}(h_2)) \\ &\geq \mathscr{I}_{\eta^{-1}(s)}(\phi_\psi^{\leftarrow}(h_1)) \otimes \mathscr{I}_{\eta^{-1}(s)}(\phi_\psi^{\leftarrow}(h_2)) \\ &= (\phi_{\psi,\eta}(\mathscr{I}))_s(h_1) \otimes (\phi_{\psi,\eta}(\mathscr{I}))_s(h_2). \end{aligned}$$

(SI3) If $h_1 \sqsubseteq h_2$, then

$$\begin{aligned} (\phi_{\psi,\eta}(\mathscr{I}))_s(h_1) &= \mathscr{I}_{\eta^{-1}(s)}((\phi_\psi)^{\leftarrow}(h_1)) \\ &\geq \mathscr{I}_{\eta^{-1}(s)}((\phi_\psi)^{\leftarrow}(h_2)) \\ &= (\phi_{\psi,\eta}(\mathscr{I}))_s(h_2). \end{aligned}$$

Hence, $\phi_{\psi,\eta}(\mathscr{I})$ is an $(L_\otimes, S_{2T_2})$-fs-ideal on Y.

Theorem 3.5. Let $\phi : X \to Y$, $\eta : S_1 \to S_2$ and $\psi : T_1 \to T_2$ be mappings. If $\mathscr{I}$ is an $(L_\otimes, S_{2T_2})$-fs-ideal on Y and $\phi^{\leftarrow}_{\psi,\eta}(\mathscr{I}) : S_1 \to L^{(L^X)^{T_1}}$ (where $(\phi^{\leftarrow}_{\psi,\eta}(\mathscr{I}))_s : (L^X)^{T_1} \to L$ is a map for each $s \in S_1$) defined by:

$$\phi^{\leftarrow}_{\psi,\eta}(\mathscr{I}_s)(h) = \begin{cases} \bigwedge\{\mathscr{I}_{\eta(s)}(f) : h = \phi_\psi^{\leftarrow}(f)\}, & \text{if } f \neq 0_X \\ 1_L, & \text{if } f = 0_X. \end{cases}$$

Then, $\phi^{\leftarrow}_{\psi,\eta}(\mathscr{I})$ is an $(L_\otimes, S_{1T_1})$-fs-ideal on X.
Proof. (SI1) It is easy.
(SI2) For every $g, h \in (L^X)^{T_1}$, we have

$$\begin{aligned} &\phi^{\leftarrow}_{\psi,\eta}(\mathscr{I}_s)(h) \otimes \phi^{\leftarrow}_{\psi,\eta}(\mathscr{I}_s)(g) \\ &= (\bigwedge\{\mathscr{I}_{\eta(s)}(h_1) : h = \phi_\psi^{\leftarrow}(h_1)\}) \otimes (\bigwedge\{\mathscr{I}_{\eta(s)}(g_1) \\ &\qquad : g = \phi_\psi^{\leftarrow}(g_1)\}) \\ &\leq \bigwedge\{\mathscr{I}_{\eta(s)}(h_1) \otimes \mathscr{I}_{\eta(s)}(g_1) : h \otimes g = \phi_\psi^{\leftarrow}(h_1) \otimes \phi_\psi^{\leftarrow}(g_1)\} \\ &\leq \bigwedge\{\mathscr{I}_{\eta(s)}(h_1 \oplus g_1) : h \oplus g = \phi_\psi^{\leftarrow}(h_1 \oplus g_1)\} \\ &\leq \phi^{\leftarrow}_{\psi,\eta}(\mathscr{I}_s)(h \oplus g). \end{aligned}$$

(SI3) If $h_1 \sqsubseteq h_2$, we have $\phi^{\leftarrow}_{\psi,\eta}(\mathscr{I}_s)(h_1) = \mathscr{I}_s(\phi_{\psi,\eta}(h_1)) \geq \mathscr{I}_s(\phi_{\psi,\eta}(h_2)) = \phi^{\leftarrow}_{\psi,\eta}(\mathscr{I}_s)(h_2)$.

Notation 3.6. Let $\mathscr{B} : S \to L^{(L^X)^T}$ (where $\mathscr{B}_s := \mathscr{B}(s) : (L^X)^T \to L$ is a map for each $s \in S$) and $h \in (L^X)^T$. For $s \in S$, the term $\langle \mathscr{B}_s \rangle(h)$ is defined by:

$$\langle \mathscr{B}_s \rangle(h) = \bigvee_{h \sqsubseteq f} \mathscr{B}_s(f).$$

Definition 3.7. A mapping $\mathscr{B} : S \to L^{(L^X)^T}$ (where $\mathscr{B}_s := \mathscr{B}(s) : (L^X)^T \to L$ is a map for each $s \in S$) is called an $(L, \otimes)$-fuzzy (S, T)-soft ideal base (for short, $(L_\otimes, S_T)$-fs-ideal base) on X if it fulfills the following criteria for each $s \in S$:

(SIB1) $\mathscr{B}_s(0_X) = 1_L$ and $\mathscr{B}_s(1_X) = 0_L$,

(SIB2) $\langle \mathscr{B}_s \rangle(g_1 \oplus g_2) \geq \mathscr{B}_s(g_1) \otimes \mathscr{B}_s(g_2)$, $\forall g_1, g_2 \in (L^X)^T$.

Theorem 3.8. Let $\mathscr{B}: S \to L^{(L^X)^T}$ be an $(L_\otimes, S_T)$-fs-ideal base. Then, $\langle \mathscr{B} \rangle$ is the coarsest $(L_\otimes, S_T)$-fs-ideal fulfills $\mathscr{B}_s(f) \le \langle \mathscr{B}_s \rangle(f), \forall f \in (L^X)^T, s \in S$.

Proof. The condition (SI1) and (SI2) are easily checked.
(SI3) Let $s \in S$ and suppose that there exist $f, g \in (L^X)^T$ such that:

$$\langle \mathscr{B}_s \rangle(f \oplus g) \not\ge \langle \mathscr{B}_s \rangle(f) \otimes \langle \mathscr{B}_s \rangle(g).$$

By the definition of $\langle \mathscr{B}_s \rangle$ and (L4) of Definition 2.1, there exist $f_1, g_1 \in (L^X)^T$ with $f \sqsubseteq f_1$, $g \sqsubseteq g_1$ such that:

$$\langle \mathscr{B}_s \rangle(f \oplus g) \not\ge \mathscr{B}_s(f_1) \otimes \mathscr{B}_s(g_1). \quad (1)$$

Since $\langle \mathscr{B} \rangle$ is an $(L_\otimes, S_T)$-fs-ideal base, we have

$$\langle \mathscr{B}_s \rangle(f_1 \oplus g_1) \ge \mathscr{B}_s(f_1) \otimes \mathscr{B}_s(g_1).$$

Since $f \oplus g \sqsubseteq f_1 \oplus g_1$, we have

$$\langle \mathscr{B}_s \rangle(f \oplus g) \ge \langle \mathscr{B}_s \rangle(f_1 \oplus g_1) \ge \mathscr{B}_s(f_1) \otimes \mathscr{B}_s(g_1).$$

It is a contradiction for (1). Thus for every $f, g \in (L^X)^T$,

$$\langle \mathscr{B}_s \rangle(f \oplus g) \ge \langle \mathscr{B}_s \rangle(f) \otimes \langle \mathscr{B}_s \rangle(g).$$

Thus, $\langle \mathscr{B} \rangle$ is an $(L_\otimes, S_T)$-fs-ideal base.
If $\mathscr{I}$ is an $(L_\otimes, S_T)$-fs-ideal fulfills $\mathscr{B}_s(f) \le \mathscr{I}_s(f), \forall f \in (L^X)^T, s \in S$. Thus,

$$\langle \mathscr{B}_s \rangle(f) = \bigvee_{f \sqsubseteq g} \mathscr{B}_s(g) \le \bigvee_{f \sqsubseteq g} \mathscr{I}_s(g) = \mathscr{I}_s(f).$$

Definition 3.9. Let $\phi: X \to Y$, $\eta: S_1 \to S_2$ and $\psi: T_1 \to T_2$ be mappings. If $(X, \mathscr{I})$ is an $(L_\otimes, S_{1T_1})$-fs-idealed set and $(Y, \mathscr{K})$ is an $(L_\otimes, S_{2T_2})$-fs-idealed set, then the map $\phi_\psi: (X, \mathscr{I}) \to (Y, \mathscr{K})$ is called:
(i) an L-fuzzy soft ideal map if $\mathscr{I}_s(\phi_\psi^{\leftarrow}(g)) \ge \mathscr{K}_{\eta(s)}(g)$, for each $g \in (L^Y)^{T_2}$, $s \in S_1$.
(ii) an L-fuzzy soft ideal preserving map if $\mathscr{I}_s(f) \le \mathscr{K}_{\eta(s)}(\phi_\psi^{\rightarrow}(f))$, for each $f \in (L^X)^{T_1}$, $s \in S_1$.

Theorem 3.10. Let $\mathscr{B}$ be an $(L_\otimes, S_{1T_1})$-fs-ideal base on X and $\mathscr{K}$ be an $(L_\otimes, S_{2T_2})$-fs-ideal base on Y. If $\phi: X \to Y$, $\psi: T_1 \to T_2$ and $\eta: S_1 \to S_2$ are mappings, then we have the following properties:

(i) $\phi: (X, \langle \mathscr{B} \rangle) \to (Y, \langle \mathscr{K} \rangle)$ is an L-fuzzy soft ideal map iff $\mathscr{K}_{\eta(s)}(g) \le \langle \mathscr{B}_s \rangle(\phi_\psi^{\leftarrow}(g))$, for each $g \in (L^Y)^{T_2}$, $s \in S_1$.

(ii) $\phi: (X, \langle \mathscr{B} \rangle) \to (Y, \langle \mathscr{K} \rangle)$ is an L-fuzzy soft ideal preserving map iff $\mathscr{B}_s(f) \le \langle \mathscr{K}_{\eta(s)} \rangle(\phi_\psi^{\rightarrow}(f))$, for each $f \in (L^X)^{T_1}$, $s \in S_1$.

(iii) If $\mathscr{K}_{\eta(s)}(g) \le \mathscr{B}_s(\phi_\psi^{\leftarrow}(g))$, for each $g \in (L^Y)^{T_2}$, $s \in S_1$, then $\phi: (X, \langle \mathscr{B} \rangle) \to (Y, \langle \mathscr{K} \rangle)$ is an L-fuzzy soft ideal map.

(iv) If $\mathscr{B}_s(f) \le \mathscr{K}_{\eta(s)}(\phi_\psi^{\rightarrow}(f))$, for each $f \in (L^X)^{T_1}$, $s \in S_1$, then $\phi: (X, \langle \mathscr{B} \rangle) \to (Y, \langle \mathscr{K} \rangle)$ is an L-fuzzy soft ideal preserving map.

Proof. (i) Since $\phi: (X, \langle \mathscr{B} \rangle) \to (Y, \langle \mathscr{K} \rangle)$ is an L-fuzzy soft ideal map, then for each $g \in (L^Y)^{T_2}$, $s \in S_1$ we have, $\langle \mathscr{B}_s \rangle(\phi_\psi^{\leftarrow}(g)) \ge \langle \mathscr{K}_{\eta(s)} \rangle(g) \ge \mathscr{K}_{\eta(s)}(g)$.
Conversely, suppose that there exists $g \in (L^Y)^{T_2}$, $s \in S_1$ such that

$$\langle \mathscr{B}_s \rangle(\phi_\psi^{\leftarrow}(g)) \not\ge \langle \mathscr{K}_{\eta(s)} \rangle(g).$$

By definition of $\langle \mathscr{K}_{\eta(s)} \rangle(g)$, there exists g_1 with $g \sqsubseteq g_1$ such that:

$$\langle \mathscr{K}_{\eta(s)} \rangle(g) \ge \mathscr{K}_{\eta(s)}(g_1).$$

On the other hand, since $\langle \mathscr{B}_s \rangle(\phi_\psi^{\leftarrow}(g_1)) \ge \mathscr{K}_{\eta(s)}(g_1)$, we have

$$\langle \mathscr{B}_s \rangle(\phi_\psi^{\leftarrow}(g)) \ge \langle \mathscr{B}_s \rangle(\phi_\psi^{\leftarrow}(g_1)) \ge \mathscr{K}_{\eta(s)}(g_1).$$

It is a contradiction. Thus, $\langle \mathscr{B}_s \rangle(\phi_\psi^{\leftarrow}(g)) \ge \langle \mathscr{K}_{\eta(s)} \rangle(g)$, for each $g \in (L^Y)^{T_2}$, $s \in S_1$. Hence, $\phi: (X, \langle \mathscr{B} \rangle) \to (Y, \langle \mathscr{K} \rangle)$ is an L-fuzzy soft ideal map.
(ii), (iii) and (iv) are similarly proved.

4 More properties of $(L_\otimes, S_T)$-fs-ideal bases

Notation 4.1. Let $\mathscr{I}$ be an $(L_\otimes, S_T)$-fs-ideal and $\mathscr{B}$ be an $(L_\otimes, S_T)$-fs-ideal base on X. We denote $\mathscr{I}_s^0 = \{g \in (L^X)^T : \mathscr{I}_s(g) > 0_L\}$ and $\mathscr{B}_s^0 = \{g \in (L^X)^T : \mathscr{B}_s(g) > 0_L\}$.

Theorem 4.2. Let $\phi_i: X \to X_i$, $\psi_i: T \to T_i$ and $\eta_i: S \to S_i$ be mappings for every $i \in \Lambda$. Let $\{\mathscr{B}_i\}_{i \in \Lambda}$ be a family of $(L_\otimes, S_{iT_i})$-fs-ideal bases on X_i fulfill the following condition:

(C) If $g_i \in (\mathscr{B}_i)^0_{\eta_i(s)}$, for every $i \in \Lambda$, $s \in S$, we have $\oplus_{i \in J} \phi_{i\psi_i}^{\leftarrow}(g_i) \ne 1_X$ for each finite subset $J \subseteq \Lambda$. We define a map $\bigsqcup_{i \in \Lambda} \phi_{i\psi_i,\eta_i}^{\leftarrow}(\mathscr{B}_i) : S \to L^{(L^X)^T}$ (where $(\bigsqcup_{i \in \Lambda} \phi_{i\psi_i,\eta_i}^{\leftarrow}(\mathscr{B}_i)_s := (\bigsqcup_{i \in \Lambda} \phi_{i\psi_i,\eta_i}^{\leftarrow}(\mathscr{B}_i))(s) : (L^X)^T \to L)$ as:

$$(\bigsqcup_{i \in \Lambda} \phi_{i\psi_i,\eta_i}^{\leftarrow}(\mathscr{B}_i))_s(f) = \begin{cases} \bigvee \{\otimes_{i \in J} (\mathscr{B}_i)_{\eta_i(s)}(g_i)\}, & \text{if } \ f = \oplus_{i \in J} \phi_{i\psi_i}^{\leftarrow}(g_i), \\ & g_i \in (\mathscr{B}_i)^0_{\eta_i(s)}, s \in S \\ 0_L, & \text{otherwise} \end{cases}$$

where $\bigvee$ is applied to each finite subset $J \subseteq \Lambda$ such that: $f = \oplus_{i \in J} \phi_{i\psi_i}^{\leftarrow}(g_i)$.

Suppose that $\mathscr{B} = \bigsqcup_{i \in \Lambda} \phi_{i\psi_i,\eta_i}^{\leftarrow}(\mathscr{B}_i)$ is obtained, then:

(i) $\mathscr{B}$ is an $(L_\otimes, S_T)$-fs-ideal base on X.

(ii) $\langle \mathscr{B} \rangle$ is the coarsest $(L_\otimes, S_T)$-fs-ideal on X such that each mapping $\phi_{i\psi_i} : (X, \langle \mathscr{B} \rangle) \to (X_i, \langle \mathscr{B}_i \rangle)$ is an L-fuzzy soft ideal map.

(iii) A map $\phi_\psi : (Y, \mathscr{D}) \to (X, \langle \mathscr{B} \rangle)$ is an L-fuzzy soft ideal map if and only if for every $i \in \Lambda$, $\phi_{i\psi_i} \circ \phi_\psi : (Y, \mathscr{D}) \to (X_i, \langle \mathscr{B}_i \rangle)$ is an L-fuzzy soft ideal map.

(iv) $\langle\bigsqcup_{i\in\Lambda}\phi^{\leftarrow}_{i_{\psi_i},\eta_i}(\mathscr{B}_i)\rangle=\langle\bigsqcup_{i\in\Lambda}\phi^{\leftarrow}_{i_{\psi_i},\eta_i}(\langle\mathscr{B}_i\rangle)\rangle$.

Proof. (i) We will prove that, $\mathscr{B}=\bigsqcup_{i\in\Lambda}\phi^{\leftarrow}_{i_{\psi_i}}(\mathscr{B}_i)$ is an $(L_\otimes,S_T)$-fs-ideal base on X. Since $\mathscr{B}_i$ is a nonzero function, there exists $g_i\in(\mathscr{B}_i)^0_{\eta_i(s)}$ such that $\mathscr{B}(\phi^{\leftarrow}_{i_{\psi_i},\eta_i}(g_i))\geq(\mathscr{B}_i)_{\eta_i(s)}(g_i)>0$. Thus $\mathscr{B}$ is nonzero function.

(SIB1) It is insignificant that $\mathscr{B}_s(0_X)=1_L$ and $\mathscr{B}_s(1_X)=0_L$.

(SIB2) For each finite index subsets $M,N\subseteq\Lambda$ with $f=\oplus_{m\in M}\phi^{\leftarrow}_{m_{\psi_m}}(f_m)$, $g=\oplus_{n\in N}\phi^{\leftarrow}_{n_{\psi_n}}(g_n)$ we get,

$$f\oplus g=(\oplus_{m\in M}\phi^{\leftarrow}_{m_{\psi_m}}(f_m))\oplus(\oplus_{n\in N}\phi^{\leftarrow}_{n_{\psi_n}}(g_n)).$$

Put $j\in M\cup N$ such that:

$$h_j=\begin{cases} f_j, & \text{if } \ j\in M-(M\cap N)\\ g_j, & \text{if } \ j\in N-(M\cap N)\\ f_j\oplus g_j, & \text{if } \ j\in M\cap N\end{cases}$$

For each $j\in M\cap N, s\in S$ we have:

$$\langle(\mathscr{B}_j)_{\eta_j(s)}\rangle(f_j\oplus g_j)\geq(\mathscr{B}_j)_{\eta_j(s)}(f_j)\oplus(\mathscr{B}_j)_{\eta_j(s)}(g_j).$$

From the definition of $\langle(\mathscr{B}_j)_{\eta_j(s)}\rangle(f_j\oplus g_j)$, there exists $W_j\in(L^{X_j})^{T_j}$ with $f_j\oplus g_j\leq W_j$ such that:

$$\langle(\mathscr{B}_j)_{\eta_j(s)}\rangle(f_j\oplus g_j)\geq(\mathscr{B}_j)_{\eta_j(s)}(W_j).$$

Since,

$$\begin{aligned} f\oplus g&=(\oplus_{m\in M}\phi^{\leftarrow}_{m_{\psi_m}}(f_m))\oplus(\oplus_{n\in N}\phi^{\leftarrow}_{n_{\psi_n}}(g_n))\\ &=\oplus_{j\in M\cup N}\phi^{\leftarrow}_{j_{\psi_j}}(h_j)\\ &\leq(\oplus_{j\in M\cup N-M\cap N}\phi^{\leftarrow}_{j_{\psi_j}}(h_j))\oplus(\oplus_{j\in M\cap N}\phi^{\leftarrow}_{j_{\psi_j}}(w_j)),\end{aligned}$$

there exists a finite index $M\cup N$ such that:

$$\langle\mathscr{B}_s\rangle(f\oplus g)\geq(\otimes_{j\in M\cup N-M\cap N}\mathscr{B}_j(h_j))\otimes(\otimes_{j\in M\cap N}\mathscr{B}_j(w_j)).$$

By Definition 2.1(L4), we get

$$\langle\mathscr{B}_s\rangle(f\oplus g)\geq\mathscr{B}_s(f)\otimes\mathscr{B}_s(g).$$

(ii) From Theorem 3.10(iii) we only show that $\mathscr{B}_s(\phi^{\leftarrow}_{i_{\psi_i}}(g_i))\geq(\mathscr{B}_i)_{\eta_i(s)}(g_i)$, for each $i\in\Lambda$.

If $(\mathscr{B}_i)_{\eta_i(s)}(g_i)=0_L$, it is trivial.

If $(\mathscr{B}_i)_{\eta_i(s)}(g_i)>0_L$, for one family $\{g_i\in((\mathscr{B}_i)_{\eta_i(s)})^0\}$, we have $\mathscr{B}_s(\phi^{\leftarrow}_{i_{\psi_i}}(g_i))\geq(\mathscr{B}_i)_{\eta_i(s)}(g_i)$.

Let $\mathscr{I}$ be another $(L_\otimes,S_T)$-fs-ideal on X such that $\forall i\in\Lambda$, the map $\phi_{i_{\psi_i}}:(X,\mathscr{I})\to(X_i,\langle\mathscr{B}_i\rangle)$ is an L-fuzzy soft ideal map. Then,

$$\mathscr{I}_s(\phi^{\leftarrow}_{i_{\psi_i}}(g_i))\geq(\langle\mathscr{B}_i\rangle)_{\eta_i(s)}(g_i),\text{for each } g\in(L^{X_i})^{T_i}, s\in S.$$

For every finite subset $J\subseteq\Lambda$ with $f\leq\oplus_{i\in J}\phi^{\leftarrow}_{i_{\psi_i}}(g_i)$, and for each $s\in S$ we have:

$$\begin{aligned}\mathscr{I}_s(f)&\geq\mathscr{I}_s(\oplus_{i\in J}\phi^{\leftarrow}_{i_{\psi_i}}(g_i))\\ &\geq\otimes_{i\in J}\mathscr{I}_s(\phi^{\leftarrow}_{i_{\psi_i}}(g_i))\\ &\geq\otimes_{i\in J}(\langle\mathscr{B}_i\rangle)_{\eta_i(s)}(g_i)\\ &\geq\otimes_{i\in J}(\mathscr{B}_i)_{\eta_i(s)}(g_i).\end{aligned}$$

By definition of $\langle\mathscr{B}\rangle$ we get $\langle\mathscr{B}\rangle\sqsubseteq\mathscr{I}$.

(iii) Because the composition of L-fuzzy soft ideal maps is an L-fuzzy soft ideal map, the composition condition is obvious.

Conversely, assume that $\phi_\psi:(Y,\mathscr{D})\to(X,\langle\mathscr{B}\rangle)$ is not an L-fuzzy soft ideal map. Then for $s\in S$, there exists $f\in(L^X)^T$ and $r\in L_0$ with

$$\langle\mathscr{B}_s\rangle(f)>\mathscr{D}_s(\phi^{\leftarrow}_\psi(f)) \tag{2}$$

By definition of $\langle\mathscr{B}\rangle$, there exists a finite index subset $J\subseteq\Lambda$ with $f\sqsubseteq\oplus_{j\in J}\phi^{\leftarrow}_{j_{\psi_j}}(g_j)$ such that

$$\langle\mathscr{B}_s\rangle(f)\geq\otimes_{j\in J}(\mathscr{B}_j)_{\eta_j(s)}(g_j),\forall j\in J \tag{3}$$

On the other hand, since $\forall i\in\Lambda$, $\phi_{i_{\psi_i}}\circ\phi_\psi:(Y,\mathscr{D})\to(X_i,\langle\mathscr{B}_i\rangle)$ is an L-fuzzy soft ideal map we have

$$\langle(\mathscr{B}_i)_{\eta_i(s)}\rangle(g_i)\leq\mathscr{D}_s(\phi_{i_{\psi_i}}\circ\phi_\psi)^{\leftarrow}(g_i)=\mathscr{D}_s(\phi^{\leftarrow}_\psi(\phi^{\leftarrow}_{i_{\psi_i}}(g_i))).$$

Then,

$$\mathscr{D}_s(\phi^{\leftarrow}_\psi(\phi^{\leftarrow}_{j_{\psi_j}}(g_j)))\geq\langle(\mathscr{B}_j)_{\eta_j(s)}\rangle(g_j),\forall j\in J.$$

Since $\phi^{\leftarrow}_\psi(f)\sqsubseteq\oplus_{j\in J}\phi^{\leftarrow}_\psi(\phi^{\leftarrow}_{j_{\psi_j}}(g_j))$, we have

$$\begin{aligned}\mathscr{D}_s(\phi^{\leftarrow}_\psi(f))&\geq\mathscr{D}_s(\oplus_{j\in J}\phi^{\leftarrow}_\psi(\phi^{\leftarrow}_{j_{\psi_j}}(g_j)))\\ &\geq\otimes_{j\in J}\mathscr{D}_s(\phi^{\leftarrow}_\psi(\phi^{\leftarrow}_{j_{\psi_j}}(g_j)))\\ &\geq\otimes_{i\in J}\langle(\mathscr{B}_i)_{\eta_i(s)}\rangle(g_i)\\ &\geq\otimes_{i\in J}(\mathscr{B}_i)_{\eta_i(s)}(g_i).\end{aligned}$$

By using (3) and (L4), we obtain

$$\mathscr{D}_s(\phi^{\leftarrow}_\psi(f))\geq\langle\mathscr{B}_s\rangle(f).$$

It is a contradiction with (2). Hence $\langle\mathscr{B}_s\rangle(f)\leq\mathscr{D}_s(\phi^{\leftarrow}_\psi(f))$ and therefore ϕ_ψ is an L-fuzzy soft ideal map.

(iv) Since $(\mathscr{B}_i)_{\eta_i(s)}(g_i)\leq\mathscr{B}_s(\phi^{\leftarrow}_{i_{\psi_i}}(g_i))$, By Theorem 3.10(iii), we have $\phi_{i_{\psi_i}}:(X,\langle\mathscr{B}\rangle)\to(X_i,\langle\mathscr{B}_i\rangle)$ is an L-fuzzy soft ideal map. Let $\mathscr{I}=\langle\bigsqcup_{i\in\Lambda}\phi^{\leftarrow}_{i_{\psi_i},\eta_i}(\langle\mathscr{B}_i\rangle)\rangle$. From (iii), the identity map $id_X:(X,\langle\mathscr{B}\rangle)\to(X,\mathscr{I})$ is an L-fuzzy soft ideal map. Thus

$$\mathscr{I}_s(f)\leq\langle\mathscr{B}_s\rangle(Id^{\leftarrow}_X)(f)=\langle\mathscr{B}_s\rangle(f),\quad\forall f\in(L^X)^T, s\in S.$$

From the definition of $\mathscr{I}$, $\langle(\mathscr{B}_i)_{\eta_i(s)}\rangle(g_i)\leq\mathscr{I}_s(\phi^{\leftarrow}_{i_{\psi_i}}(g_i))$, that is $\phi_{i_{\psi_i}}:(X,\mathscr{I})\to(X_i,\langle\mathscr{B}_i\rangle)$ is an L-fuzzy soft ideal map. From (ii), we get

$$\langle\mathscr{B}_s\rangle(f)\leq\mathscr{I}_s(f),\quad\forall f\in(L^X)^T, s\in S.$$

Thus, $\mathscr{I}=\langle\mathscr{B}\rangle$.

Theorem 4.3. Let $\{\mathscr{I}_i\}_{i\in\Lambda}$ be a family of $(L_\otimes,S_T)$-fs-ideals on X fulfilling the following condition:

(C) If $g_i\in(\mathscr{I}_i)^0_s$ for every $i\in\Lambda$, $s\in S$, then $\oplus_{i\in J}g_i\neq 1_X$ for every finite subset $J\subseteq\Lambda$.

Define the map $\bigsqcup_{i\in\Lambda}\mathscr{I}_i : S \to L^{(L^X)^T}$ (where $(\bigsqcup_{i\in\Lambda}\mathscr{I}_i)_s := (\bigsqcup_{i\in\Lambda}\mathscr{I}_i)(s) : (L^X)^T \to L$)as:

$$(\bigsqcup_{i\in\Lambda}\mathscr{I}_i)_s(f) = \begin{cases} \bigvee\{\otimes_{i\in J}(\mathscr{I}_i)_s(g_i)\}, & \text{if } f = \oplus_{i\in J} g_i,\ g_i \in (\mathscr{I}_i)^0_s, s\in S \\ 0_L, & \text{otherwise} \end{cases}$$

where $\bigvee$ is applied to any finite subset $J \subseteq \Lambda$ with $f = \oplus_{i\in J} g_i$. Then, $\mathscr{I}$ is the coarsest $(L_\otimes, S_T)$-fs-ideal finer than $\mathscr{I}_i$ for each $i\in\Lambda$.

Proof. By Theorem 4.2, put $(X_i,\mathscr{B}_i) = (X,\mathscr{I}_i)$ and $g_i = id_X$, where id_X is an identity map $\forall i\in\Lambda$. Suppose that $\mathscr{I} = \bigsqcup_{i\in\Lambda}\mathscr{I}_i$ is given. It suffices to demonstrate that $\mathscr{I} = \langle\mathscr{I}\rangle$.
It is trivially show that:

$$\mathscr{I} \sqsubseteq \langle\mathscr{I}\rangle. \tag{4}$$

Assume that there is $f \in (L^X)^T$ with:

$$\mathscr{I}_s(f) < \langle\mathscr{I}_s\rangle(f). \tag{5}$$

From definition of $\langle\mathscr{I}_s\rangle(f)$, there is $g \in (L^X)^T$ with $f \sqsubseteq g$ such that

$$\langle\mathscr{I}_s\rangle(f) \geq \mathscr{I}_s(g).$$

From the definition of $\mathscr{I}$, there is a finite index subset $J \subseteq \Lambda$ with $g = \oplus_{i\in J} g_i$ such that:

$$\mathscr{I}_s(g) \geq \otimes_{j\in J}(\mathscr{I}_j)_s(g_j).$$

On the other hand, we have $g = g\otimes f = \oplus_{i\in J}(g_j\otimes f)$, then

$$\begin{aligned}\mathscr{I}_s(g) &\geq \otimes_{j\in J}(\mathscr{I}_j)_s(g_j\otimes f)\\ &\geq \otimes_{j\in J}(\mathscr{I}_j)_s(g_j).\end{aligned}$$

By Definition 2.1(L4), we have $\mathscr{I}_s \geq \langle\mathscr{I}_s\rangle$. It is a contradiction for (5). Thus,

$$\mathscr{I} \sqsupseteq \langle\mathscr{I}\rangle. \tag{6}$$

From (4) and (6) we have $\mathscr{I} = \langle\mathscr{I}\rangle$.

Theorem 4.4. Let $\{\mathscr{B}_i\}_{i\in\Lambda}$ be a collection of $(L_\otimes, S_{iT_i})$-fs-ideal bases on X_i. Let $X = \Pi_{i\in\Lambda}X_i$ be a product set and $\pi_i : X \to X_i$ be a projection map, $\psi_i : E \to E_i$ and $\eta_i : S \to S_i$ be maps for every $i\in\Lambda$. The map $\bigsqcup_{i\in\Lambda}\pi^{\leftarrow}_{i\psi_i,\eta_i}(\mathscr{B}_i) : S \to L^{(L^X)^T}$ (where $(\bigsqcup_{i\in\Lambda}\pi^{\leftarrow}_{i\psi_i,\eta_i}(\mathscr{B}_i)_s := (\bigsqcup_{i\in\Lambda}\pi^{\leftarrow}_{i\psi_i,\eta_i}(\mathscr{B}_i))(s) : (L^X)^T \to L)$ is define by:

$$(\bigsqcup_{i\in\Lambda}\pi^{\leftarrow}_{i\psi_i,\eta_i}(\mathscr{B}_i))_s(f) = \begin{cases} \bigvee\{\otimes_{i\in J}(\mathscr{B}_i)_{\eta_i(s)}(g_i)\}, & \text{if } f = \oplus_{i\in J}\pi^{\leftarrow}_{i\psi_i}(g_i),\ g_i \in (\mathscr{B}_i)^0_{\eta_i(s)}, s\in S \\ 0_L, & \text{otherwise} \end{cases}$$

where $\bigvee$ is applied to any finite subset $J\subseteq\Lambda$ with $f = \oplus_{i\in J}\pi^{\leftarrow}_{i\psi_i}(g_i)$. Let $\mathscr{B} = \bigsqcup_{i\in\Lambda}\pi^{\leftarrow}_{i\psi_i,\eta_i}(\mathscr{B}_i)$ be given. Then:

(i) $\langle\mathscr{B}\rangle$ is the coarsest $(L_\otimes, S_T)$-fs-ideal on X such that each projection $\pi_{i\psi_i} : (X,\langle\mathscr{B}\rangle) \to (X_i,\langle\mathscr{B}_i\rangle)$ is an L-fuzzy soft ideal map.

(ii) A map $\phi_\psi : (Y,\mathscr{D}) \to (X,\langle\mathscr{B}\rangle)$ is an L-fuzzy soft ideal map iff $\forall i\in\Lambda$, $\pi_{i\psi_i}\circ\phi_\psi : (Y,\mathscr{D}) \to (X_i,\langle\mathscr{B}_i\rangle)$ is an L-fuzzy soft ideal map.

Proof. By Theorem 4.2, we only prove that a collection $\{\mathscr{B}_i\}_{i\in\Lambda}$ satisfies the condition (C). Let $g_i \in (\mathscr{B}_i)^0_{\eta_i(s)}$ for each $i\in J\subseteq\Lambda$, J is finite. Since $(\mathscr{B}_i)_{\eta_i(s)}(g_i) > 0_L$, for each $i\in J$, we have $g_i \neq 1_X$. For every $i\in J$, there is $x_i \in X_i$ such that $g_i(x_i) < 1_L$. Let $x\in X$ such that $\pi_i(x) = x_i$ for each $i\in J$. Then, $\oplus_{i\in J}\pi^{\leftarrow}_{i\psi_i}(g_i) = \oplus_{i\in J} g_i(\pi_i(x)) < 1_L$. Thus $\oplus_{i\in J}\phi^{\leftarrow}_{i\psi_i}(g_i) \neq 1_X$.

From Theorem 4.4, we can define the product of $(L_\otimes, S_T)$-fs-ideal spaces as:

Definition 4.5. Let $\{\mathscr{I}_i\}_{i\in\Lambda}$ be a family of $(L_\otimes, S_{iT_i})$-fs-ideals on X_i. Let $X = \Pi_{i\in\Lambda}X_i$ be a product set and $\pi_i : X\to X_i$ be a projection map, $\psi_i : T\to T_i$ and $\eta_i : S\to S_i$ be maps $\forall i\in\Lambda$. The structure $\langle\bigsqcup_{i\in\Lambda}\pi^{\leftarrow}_{i\psi_i,\eta_i}(\mathscr{B}_i)\rangle$ is said to be a product $(L_\otimes, S_T)$-fs-ideals on X.

5 The image of $(L_\otimes, S_T)$-fs-ideals

Theorem 5.1. Let $\phi_i : X_i \to X$, $\psi_i : T_i \to T$ and $\eta_i : S_i \to S$ be mappings for every $i\in\Lambda$. Let $\{\mathscr{I}_i\}_{i\in\Lambda}$ be a family of $(L_\otimes, S_{iT_i})$-fs-ideals on X_i fulfill the following condition:

(C) If $g_i \in (\mathscr{I}_i)^0_{\eta_i^{-1}(s)}$ for every $i\in\Lambda$, $s\in S$, we have $\oplus_{i\in J}\phi^{\rightarrow}_{i\psi_i}(g_i) \neq 1_X$ for every finite subset $J\subseteq\Lambda$. The map $\bigsqcup_{i\in\Lambda}\phi^{\rightarrow}_{i\psi_i,\eta_i}(\mathscr{I}_i) : S \to L^{(L^X)^T}$ (where $(\bigsqcup_{i\in\Lambda}\phi^{\rightarrow}_{i\psi_i,\eta_i}(\mathscr{I}_i)_s := (\bigsqcup_{i\in\Lambda}\phi^{\rightarrow}_{i\psi_i,\eta_i}(\mathscr{I}_i))(s) : (L^X)^T \to L)$ define by:

$$(\bigsqcup_{i\in\Lambda}\phi^{\rightarrow}_{i\psi_i,\eta_i}(\mathscr{I}_i))_s(f) = \begin{cases} \bigvee\{\otimes_{i\in J}(\mathscr{I}_i)_{\eta_i^{-1}(s)}(g_i)\}, & \text{if } f = \oplus_{i\in J}\phi^{\rightarrow}_{i\psi_i}(g_i),\ g_i \in (\mathscr{I}_i)^0_{\eta_i^{-1}(s)}, s\in S \\ 0_L, & \text{otherwise} \end{cases}$$

where $\bigvee$ is applied to any finite subset $J\subseteq\Lambda$ with $f = \oplus_{i\in J}\phi^{\rightarrow}_{i\psi_i}(g_i)$. Let $\mathscr{I} = \bigsqcup_{i\in\Lambda}\phi^{\rightarrow}_{i\psi_i,\eta_i}(\mathscr{I}_i)$ be given, Then:

(i) $\mathscr{I}$ is the coarsest $(L_\otimes, S_T)$-fs-ideal on X such that each mapping $\phi_{i\psi_i} : (X_i,\mathscr{I}_i) \to (X,\mathscr{I})$ is an L-fuzzy soft ideal preserving map.

(ii) A map $\phi_\psi : (X,\mathscr{I}) \to (Y,\mathscr{H})$ is an L-fuzzy soft ideal preserving map iff for every $i\in\Lambda$, $\phi_\psi\circ\phi_{i\psi_i} : (X_i,\mathscr{I}_i) \to (Y,\mathscr{H})$ is an L-fuzzy soft ideal preserving map.

Proof. (i) Firstly, we will show that $\mathscr{I}$ is an $(L_\otimes, S_T)$-fs-ideal on X. Since $\mathscr{I}_i$ is nonzero function, there exists $g_i \in (\mathscr{I}_i)^0_{\eta_i^{-1}(s)}$ such that:

$$\mathscr{I}_s(\phi^{\rightarrow}_{i\psi_i}(g_i)) \geq (\mathscr{I}_i)_{\eta_i^{-1}(s)}(g_i) > 0_L.$$

Thus, $\mathscr{I}$ is nonzero function.

(SI1) It is easy.

(SI2) For each finite index subsets $M, N \subseteq \Lambda$ with $f = \oplus_{m\in M}\phi^{\rightarrow}_{m\psi_m}(f_m)$, $g = \oplus_{n\in N}\phi^{\rightarrow}_{n\psi_n}(g_n)$ we get,

$$\mathscr{I}_s(f) \geq \otimes_{m\in M}(\mathscr{I}_m)_{\eta_m^{-1}(s)}(f_m),$$

$$\mathscr{I}_s(g) \geq \otimes_{n\in N}(\mathscr{I}_n)_{\eta_n^{-1}(s)}(g_n)$$

Put $j \in M\cup N$ such that:

$$h_j = \begin{cases} f_j, & \text{if} \quad j \in M-(M\cap N) \\ g_j, & \text{if} \quad j \in N-(M\cap N) \\ f_j \oplus g_j, & \text{if} \quad j \in M\cap N \end{cases}$$

Since

$$\begin{aligned} f\oplus g &= (\oplus_{m\in M}\phi^{\rightarrow}_{m\psi_m}(f_m)) \oplus (\oplus_{n\in N}\phi^{\rightarrow}_{n\psi_n}(g_n)) \\ &= \oplus_{j\in M\cup N}\phi^{\rightarrow}_{j\psi_j}(h_j), \end{aligned}$$

there exist a finite subset $M\cup N \subseteq \Lambda$ with :

$$\mathscr{I}_s(f\oplus g) \geq (\otimes_{m\in M}(\mathscr{I}_m)_{\eta_m^{-1}(s)}(f_m)) \otimes (\otimes_{n\in N}(\mathscr{I}_n)_{\eta_n^{-1}(s)}(g_n)).$$

By Definition 2.1(L4), we have

$$\mathscr{I}_s(f\oplus g) \geq \mathscr{I}_s(f) \otimes \mathscr{I}_s(g).$$

(SI3) Assume that there exist $f, g \in (L^Y)^T$, $s \in S$ with $f \sqsubseteq g$. There exist a finite subset $J \subseteq \Lambda$ with $g = \oplus_{i\in J} g_i$ such that:

$$\mathscr{I}_s(g) \geq \otimes_{i\in J}(\mathscr{I}_i)_{\eta_i^{-1}(s)}(g_i).$$

On the other hand, since $f = f\otimes f \sqsubseteq f \otimes g = \oplus_{i\in J}(f\otimes g_i)$ we have

$$\begin{aligned} \mathscr{I}_s(f) &\geq \otimes_{i\in J}(\mathscr{I}_i)_{\eta_i^{-1}(s)}(f\otimes g_i) \\ &\geq \otimes_{i\in J}(\mathscr{I}_i)_{\eta_i^{-1}(s)}(g_i). \end{aligned}$$

By Definition 2.1(L4), we get $\mathscr{I}_s(f) \geq \mathscr{I}_s(g)$.

We will show that $\mathscr{I}_s(\phi^{\rightarrow}_{i\psi_i}(g_i)) \geq (\mathscr{I}_i)_{\eta_i^{-1}(s)}(g_i)$, for each $i \in \Lambda$, $g \in (L^{X_i})^{T_i}, s \in S$.

If $(\mathscr{I}_i)_{\eta_i^{-1}(s)}(g_i) = 0_L$, it is trivial.

If $(\mathscr{I}_i)_{\eta_i^{-1}(s)}(g_i) > 0_L$, for one family $\{g_i \in ((\mathscr{I}_i)_{\eta_i^{-1}(s)})^0\}$, we have $\mathscr{I}_s(\phi^{\rightarrow}_{i\psi_i}(g_i)) \geq (\mathscr{I}_i)_{\eta_i^{-1}(s)}(g_i)$.

Let $\mathscr{K}$ be another $(L_\otimes, S_T)$-fs-ideal on X such that for every $i \in \Lambda$, the map $\phi_{i\psi_i} : (X_i, \mathscr{I}_i) \to (X, \mathscr{I})$ is an L-fuzzy soft ideal preserving map. Then,

$$\mathscr{K}_s(\phi^{\rightarrow}_{i\psi_i}(g_i)) \geq (\mathscr{I}_i)_{\eta_i^{-1}(s)}(g_i), \text{for each } g \in (L^{X_i})^{T_i}, s \in S.$$

By the definition of $\mathscr{I}$, for every finite subset $J \subseteq \Lambda$ with $f = \oplus_{j\in J}\phi^{\rightarrow}_{j\psi_j}(g_j)$, we have:

$$\mathscr{I}_s(f) \geq \otimes_{i\in J}(\mathscr{I}_i)_{\eta_i^{-1}(s)}(g_i).$$

On the other hand, since

$$\mathscr{K}_s(\phi^{\rightarrow}_{j\psi_j}(g_j)) \geq (\mathscr{I}_j)_{\eta_j^{-1}(s)}(g_j), \text{for each } j \in J,$$

we have

$$\begin{aligned} \mathscr{K}_s(f) &\geq \mathscr{K}_s(\oplus_{j\in J}\phi^{\rightarrow}_{j\psi_j}(g_j)) \\ &\geq \otimes_{j\in J}\mathscr{K}_s(\phi^{\rightarrow}_{j\psi_j}(g_j)) \quad \text{(By (SI2))} \\ &\geq \otimes_{j\in J}(\mathscr{I}_j)_{\eta_j^{-1}(s)}(g_j). \end{aligned}$$

From the definition of $\mathscr{I}$ and Definition 2.1(L4) we have $\mathscr{I} \sqsubseteq \mathscr{K}$.

(iii) Since the composition of L-fuzzy soft ideal preserving maps is an L-fuzzy soft ideal preserving map, the composition condition is obvious.

Conversely, suppose that $\phi_\psi : (X, \mathscr{I}) \to (Y, \mathscr{H})$ is not an L-fuzzy soft preserving map. Then for $s \in S$, there exists $f \in (L^X)^T$ such that

$$\mathscr{I}_s(f) > \mathscr{H}_s(\phi^{\rightarrow}_\psi(f)) \tag{7}$$

From the definition of $\mathscr{I}$, there is a finite index subset $J \subseteq \Lambda$ with $f = \oplus_{j\in J}\phi^{\rightarrow}_{j\psi_j}(g_j)$ such that

$$\mathscr{I}_s(f) \geq \otimes_{j\in J}(\mathscr{I}_j)_{\eta_j^{-1}(s)}(g_j). \tag{8}$$

On the other hand, since for every $i \in \Lambda$, $\phi_\psi \circ \phi_{i\psi_i} : (X_i, \mathscr{I}_i) \to (Y, \mathscr{H})$ is an L-fuzzy soft ideal preserving map, we get

$$(\mathscr{I}_i)_{\eta_i^{-1}(s)}(g_i) \leq \mathscr{H}_s(\phi_\psi \circ \phi_{i\psi_i})^{\rightarrow}(g_i) = \mathscr{H}_s(\phi^{\rightarrow}_\psi(\phi^{\rightarrow}_{i\psi_i}(g_i))).$$

Then,

$$\mathscr{H}_s(\phi^{\rightarrow}_\psi(\phi^{\rightarrow}_{j\psi_j}(g_j))) \geq (\mathscr{I}_j)_{\eta_j^{-1}(s)}(g_j), \forall j \in J.$$

Since $\phi^{\rightarrow}_\psi(f) = \oplus_{j\in J}\phi^{\rightarrow}_\psi(\phi^{\rightarrow}_{j\psi_j}(g_j))$, we have

$$\begin{aligned} \mathscr{H}_s(\phi^{\rightarrow}_\psi(f)) &\geq \mathscr{H}_s(\oplus_{j\in J}\phi^{\rightarrow}_\psi(\phi^{\rightarrow}_{j\psi_j}(g_j))) \\ &\geq \otimes_{j\in J}\mathscr{H}_s(\phi^{\rightarrow}_\psi(\phi^{\rightarrow}_{j\psi_j}(g_j))) \\ &\geq \otimes_{j\in J}(\mathscr{I}_j)_{\eta_j^{-1}(s)}(g_j). \end{aligned}$$

By using (8) and Definition 2.1(L4), we obtain

$$\mathscr{H}_s(\phi^{\rightarrow}_\psi(f)) \geq \mathscr{I}_s(f).$$

It is a contradiction with (7). Hence $\mathscr{H}_s(\phi_\psi^{\rightarrow}(f)) \geq \mathscr{I}_s(f), \forall f \in (L^X)^T$ and therefore ϕ_ψ is an L-fuzzy soft ideal preserving map.

Corollary 5.2. Let $\phi_i : X_i \to X$, $\psi_i : T_i \to T$ and $\eta_i : S_i \to S$ be mappings for every $i \in \Lambda$. Let $\{\mathscr{B}_i\}_{i\in\Lambda}$ be a collection of $(L_\otimes, S_{iT_i})$-fs-ideal bases on X_i fulfill the following condition:

(C) If $g_i \in (\mathscr{B}_i)^0_{\eta_i^{-1}(s)}$ for every $i \in \Lambda$, $s \in S$, we have $\oplus_{i\in J}\phi_{i_{\psi_i}}^{\rightarrow}(g_i) \neq 1_X$ for every finite subset $J \subseteq \Lambda$.

Define the map $\bigsqcup_{i\in\Lambda} \phi_{i_{\psi_i,\eta_i}}^{\rightarrow}(\mathscr{B}_i) : S \to L^{(L^X)^T}$ (where $(\bigsqcup_{i\in\Lambda} \phi_{i_{\psi_i,\eta_i}}^{\rightarrow}(\mathscr{B}_i))_s := (\bigsqcup_{i\in\Lambda} \phi_{i_{\psi_i,\eta_i}}^{\rightarrow}(\mathscr{B}_i))(s) : (L^X)^T \to L$)as:

$$(\bigsqcup_{i\in\Lambda} \phi_{i_{\psi_i,\eta_i}}^{\rightarrow}(\mathscr{B}_i))_s(f) = \begin{cases} \bigvee\{\otimes_{i\in J}(\mathscr{B}_i)_{\eta_i^{-1}(s)}(g_i)\}, & \text{if } \quad f = \oplus_{i\in J}\phi_{i_{\psi_i}}^{\rightarrow}(g_i), \\ & g_i \in (\mathscr{B}_i)^0_{\eta_i^{-1}(s)}, s \in S \\ 0_L, & \text{otherwise} \end{cases}$$

where $\bigvee$ is applied to any finite subset $J \subseteq \Lambda$ with $f = \oplus_{i\in J}\phi_{i_{\psi_i}}^{\rightarrow}(g_i)$.

Assume that $\mathscr{B} = \bigsqcup_{i\in\Lambda} \phi_{i_{\psi_i,\eta_i}}^{\rightarrow}(\mathscr{B}_i)$ is obtained, then:

(i) $\mathscr{B}$ is an $(L_\otimes, S_T)$-fs-ideal base on X for which each mapping $\phi_{i_{\psi_i}} : (X_i, \langle\mathscr{B}_i\rangle) \to (X, \langle\mathscr{B}\rangle)$ is an L-fuzzy soft ideal preserving map.

(ii) A map $\phi_\psi : (X, \langle\mathscr{B}\rangle) \to (Y, \mathscr{H})$ is an L-fuzzy soft ideal preserving map iff for every $i \in \Lambda$, $\phi_\psi \circ \phi_{i_{\psi_i}} : (X_i, \langle\mathscr{B}_i\rangle) \to (Y, \mathscr{H})$ is an L-fuzzy soft ideal preserving map.

(iii) $\langle\mathscr{B}\rangle = \bigsqcup_{i\in\Lambda} \phi_{i_{\psi_i,\eta_i}}^{\rightarrow}(\langle\mathscr{B}_i\rangle)$.

Proof. (i) and (ii) are similarly proved as Theorem 5.1.

(iii) Let $\mathscr{I} = \bigsqcup_{i\in\Lambda} \phi_{i_{\psi_i,\eta_i}}^{\rightarrow}(\langle\mathscr{B}_i\rangle)$. Since $(\mathscr{B}_i)_{\eta_i^{-1}(s)}(g_i) \leq \mathscr{B}(\phi_{i_{\psi_i}}^{\rightarrow}(g_i))$, by Theorem 3.10(iv), $\phi_{i_{\psi_i}} : (X_i, \langle\mathscr{B}_i\rangle) \to (X, \langle\mathscr{B}\rangle)$ is an L-fuzzy soft ideal preserving map. Since $id_X \circ \phi_{i_{\psi_i}} : (X_i, \langle\mathscr{B}_i\rangle) \to (X, \langle\mathscr{B}\rangle)$ is an L-fuzzy soft ideal preserving map, by Theorem 5.1(ii), the identity map $id_X : (X, \mathscr{I}) \to (X, \langle\mathscr{B}\rangle)$ is an L-fuzzy soft ideal preserving map. Thus

$$\mathscr{I} \sqsubseteq \langle\mathscr{B}\rangle.$$

Since

$$\{g_i \in (L^{X_i})^{T_i} : g_i \in (\mathscr{B}_i)^0_{\eta_i^{-1}(s)}\} \sqsubseteq \{g_i \in (L^{X_i})^{T_i} : g_i \in (\langle\mathscr{B}_i\rangle)^0_{\eta_i^{-1}(s)}\},$$

we have $\mathscr{I} \sqsupseteq \mathscr{B}$. By Theorem 3.8, we obtain

$$\mathscr{I} \sqsupseteq \langle\mathscr{B}\rangle.$$

Thus, $\mathscr{I} = \langle\mathscr{B}\rangle$.

6 Conclusion

In our study, we presented the notions of $(L_\otimes, K_E)$-fs-ideal, $(L_\otimes, K_E)$-fs-ideal bases, L-fuzzy soft ideal map and L-fuzzy soft ideal preserving map and illustrated several of their properties. We showed that every $(L,\otimes)$-fuzzy (S,T)-soft ideal induced an $(L,\otimes)$-fuzzy (S,T)-soft co-topology. Also, we defined the products of $(L,\otimes)$-fuzzy (S,T)-soft ideals. In the final, we examined the image of $(L,\otimes)$-fuzzy (S,T)-soft ideals and studied its properties.

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Conflict of Interest

The author declare that there is no conflict of interest regarding the publication of this article.

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