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On the Generalized Laplace's Equation

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Abstract: In this paper, we study the properties and various results of the generalized Laplace Equation, using a previously defined and studied generalized derivative. We discuss the solution of this mathematical problem with conditions of the Dirichlet type and Neumann type. The results obtained are illustrated using various examples, by modeling the solutions under two variations: that of the kernel used and the order involved.

Keywords: Generalized derivatives and integral, Laplace's equation

1 Introduction

Probably one of the mathematical areas with the greatest expansion in the last hundred years, is that of Differential Equations in Partial Derivatives, due to its multiplicity of applications and the multiple theoretical relationships with various mathematical areas (using various operators and on different functional spaces). Over time, the number of researchers and the productions obtained have been increasing, you can consult in [1,2,3,4] different aspects of this increase and its overlaps with the development of Mathematics itself.

Fractional and generalized calculus is today as important as classical calculus. In the last 40 years, these work directions have become a center of interest for various mathematicians, due to their wide applications. In particular, that has led to different definitions of differential and integral operators, and its multiple links that have not yet been studied (see [5,6,7,8] for additional details and various formulations).

In [9] a generalized fractional derivative was defined in the following way.

Definition 1. *Given a function* $f:[0,+\infty)\to\mathbb{R}$. *Then the N-derivative of* f *of order* α *is defined by*

$$N_F^\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon F(t,\alpha))-f(t)}{\varepsilon} \tag{1}$$

for all $t>0$, $\alpha\in(0,1)$ *being* $F(\alpha,t)$ *an absolutely continuous function.*

If f *is* $\alpha-$*differentiable in some* $(0,\alpha)$, *and* $\lim_{t\to 0^+} N_F^\alpha f(t)$ *exists, then define* $N_F^\alpha f(0)=\lim_{t\to 0^+} N_F^\alpha f(t)$, *note that if* f *is differentiable, then* $N_F^\alpha f(t)=F(t,\alpha)f'(t)$ *where* $f'(t)$ *is the ordinary derivative.*

Remark. In the same article it is proved one of the most required properties of a derivative operator is the Chain Rule, to calculate the derivative of compound functions, which does not exist in the case of classical fractional derivatives $N_\Phi^\alpha(f\circ g)(t)=N_\Phi^\alpha f(g(t))=f'(g(t))N_\Phi^\alpha g(t)$.

This generalized derivative operator contains many of the known local operators (for example, the conformable derivative and the non-conformable of [10,11])) and has shown its usefulness in various applications, as it can be consulted, for example, in [12,13,14,15,16,17,18,19].

The following result is very easy to obtain.

Theorem 1. *Let* $a>0$ *and* $f:[a,b]\to\mathbb{R}$ *be a given function, if* f *is N-derivable in* $t_0\geq 0$, *then* f *is continuous in* t_0.

Now, we give the definition of a general fractional integral (cf. [20]). Throughout the work we will consider that the integral operator kernel T defined below is an absolutely continuous function.

Definition 2. *Let* I *be an interval* $I\subseteq\mathbb{R}$, $a,t\in I$ *and* $\alpha\in\mathbb{R}$. *The integral operator* $J_{T,a}^\alpha$, *is defined for every locally*

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integrable function f on I as

$$J^{\alpha}_{T,a}(f)(t) = \int_a^t \frac{f(s)}{T(s,\alpha)} ds, t > a \tag{2}$$

In the same study, the following properties are fundamental, and relate the previous integral operator with the generalized derivative.

Proposition 1. *Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$, $0 < \alpha \leq 1$ and f a α-differentiable function on I such that f' is a locally integrable function on I. Then, we have for all $t \in I$*

$$J^{\alpha}_{F,a}\big(N^{\alpha}_F(f)\big)(t) = f(t) - f(a).$$

Proposition 2. *Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$ and $\alpha \in (0,1]$.*

$$N^{\alpha}_F\big(J^{\alpha}_{F,a}(f)\big)(t) = f(t),$$

for every continuous function f on I and $a,t \in I$.

Remark. In [21] it is defined the integral operator $J^{\alpha}_{F,a}$ for the choice of the function F given by $F(t,\alpha) = t^{1-\alpha}$, and [21, Theorem 3.1] shows

$$T^{\alpha} J^{\alpha}_{t^{1-\alpha},a}(f)(t) = f(t),$$

for every continuous function f on I, $a,t \in I$ and $\alpha \in (0,1]$. Hence, Proposition 2 extends to any F this important equality.

The following result summarizes some elementary properties of the integral operator $J^{\alpha}_{T,a}$.

Theorem 2. *Let I be an interval $I \subseteq \mathbb{R}$, $a,b \in I$ and $\alpha \in \mathbb{R}$. Suppose that f,g are locally integrable functions on I, and $k_1,k_2 \in \mathbb{R}$. Then we have*
(1) $J^{\alpha}_{T,a}\big(k_1 f + k_2 g\big)(t) = k_1 J^{\alpha}_{T,a} f(t) + k_2 J^{\alpha}_{T,a} g(t)$,
(2) if $f \geq g$, then $J^{\alpha}_{T,a} f(t) \geq J^{\alpha}_{T,a} g(t)$ for every $t \in I$ with $t \geq a$,
(3) $\left| J^{\alpha}_{T,a} f(t) \right| \leq J^{\alpha}_{T,a} |f|(t)$ for every $t \in I$ with $t \geq a$.

Theorem 3. *(Integration by parts) Let $f,g : [a,b] \to \mathbb{R}$ differentiable functions and $\alpha \in (0,1]$. Then, the following property hold*

$$J^{\alpha}_{F,a}((f)(N^{\alpha}_{F,a} g(t))) = [f(t)g(t)]^b_a - J^{\alpha}_{F,a}((g)(N^{\alpha}_{F,a} f(t))). \tag{3}$$

Proof. See [22].

Remark. As pointed out in the same article, many fractional integral operators can be obtained as particular cases of the previous one, under certain choices of the F kernel. For example, if $F(t-s,\alpha) = \Gamma(\alpha)(t-s)^{1-\alpha}$ the right Riemann-Liouville integral is obtained (similarly to the left), further details on Fractional Calculus and fractional integral operators linked to the generalized integral of the previous definition, can be found in [23,24,25,26,27,28,29,30].

Remark. We can define the function space $L^p_{\alpha}[a,b]$ as the set of functions over $[a,b]$ such that $(J^{\alpha}_{T,a}[f(t)]^p(b)) < +\infty$.

Taking into account the ideas of [31] we can define the generalized partial derivatives as follows.

Definition 3. *Given a real valued function $f : \mathbb{R}^n \to \mathbb{R}$ and $\overrightarrow{a} = (a_1,\cdots,a_n) \in \mathbb{R}^n$ a point whose ith component is positive. Then the generalized partial N-derivative of f of order α with respect to the variable x_i, in the point $\overrightarrow{a} = (a_1,\cdots,a_n)$ is defined by*

$$_iN^{\alpha}_F f(\overrightarrow{a}) = \lim_{\varepsilon\to 0} \frac{f(a_1,\cdots,a_i+\varepsilon F_i(a_i,\alpha),\cdots,a_n) - f(a_1,\cdots,a_i,\cdots,a_n))}{\varepsilon} \tag{4}$$

if it exists, is denoted $_iN^{\alpha}_F f(\overrightarrow{a})$, and called the ith generalized partial derivative of f of the order $\alpha \in (0,1]$ at $\overrightarrow{a}$.

Remark. If a real valued function f with n variables has all generalized partial derivatives of the order $\alpha \in (0,1]$ at $\overrightarrow{a}$, each $a_i > 0$, then the generalized α-gradient of f of the order $\alpha \in (0,1]$ at $\overrightarrow{a}$ is

$$\nabla^{\alpha}_N f(\overrightarrow{a}) = (_1N^{\alpha}_F f(\overrightarrow{a}),\cdots,_2N^{\alpha}_F f(\overrightarrow{a})) \tag{5}$$

Remark. The higher-order N-partial derivatives of a real valued function f with n variables, with $\alpha \in (0,1]$ at $\overrightarrow{a}$, are similarly defined by

$$_iN^{\alpha}_F(_iN^{\alpha}_F f) = {}_{(i,i)}N^{\alpha+\alpha}_F f. \tag{6}$$

Taking into account the above definitions, it is not difficult to prove the following result, on the equality of mixed partial derivatives.

Theorem 4. *Under assumptions of Definiton 3, assume that $f(t_1,t_2)$ it is a function for which, mixed generalized partial derivatives exist and are continuous, $N^{\alpha+\beta}_{F_{1,2},t_1,t_2}(f(t_1,t_2))$ and $N^{\beta+\alpha}_{F_{2,1},t_2,t_1}(f(t_1,t_2))$ over some domain of $\mathbb{R}^2$ then*

$$_{(1,2)}N^{\alpha+\beta}_F(f(t_1,t_2)) = {}_{(2,1)}N^{\beta+\alpha}_F(f(t_1,t_2)) \tag{7}$$

Using the previously defined, the following definition is clear.

Definition 4. *Let $0 < \alpha < 1$. We will say that the function $f(x)$ is N-differentiable, with N-differential denoted by $d^F f = N^{\alpha}_F f dt$. If f is differentiable on $(0,+\infty)$, then we have $d^F f = F(t,\alpha)f'(t)dt$.*

Laplace's Equation is used as an indicator of equilibrium, in applications such as conduction, dissipation and heat transfer (see [32]). Some results concerning the Laplace's Equation in the framework of local derivatives can be found in [33,34,35]. It is known that the Fourier series is one of the most important

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methods used in engineering and physical sciences to find the analytical solution of initial and boundary problems. For the case of conformable derivatives, the method is introduced in [36]. Here we generalize this method to any local derivative, and we obtain the solution of the generalized Laplace equation, at the end of the work several examples are presented where the strengths of the results obtained are shown, in particular , the known results for the case of the conformable derivative, are particular cases of those presented here.

2 Main Results

2.1 The generalized Fourier Series

Let F be the function of Definition 1, let's define $\mathbb{F}(t,\alpha)=\int_0^t \frac{1}{F(s,\alpha)}ds$. Consider $0<\alpha\leq 1$ and $g:[0,\infty)\to\mathbb{R}$ be any function. Let's define the f function as follows $f:[0,\infty)\to\mathbb{R}$ and $f(t)=(g\circ\mathbb{F})(t)=g(\mathbb{F}(t,\alpha))$.

Definition 5. *A function $f(t)$ is called generalized α-periodical with period p if we have $f(t)=g(\mathbb{F}(t,\alpha))=g(\mathbb{F}(t,\alpha)+\mathbb{F}(p,\alpha))$, for all $t\in[0,+\infty)$.*

Remark. If $F\equiv 1$ in the previous definition, that is, we are considering the derivative and the classical integral, the above concept coincides with the classical periodic function. If we can $F(t,\alpha)=t^{1-\alpha}$, that is, we consider the derivative and conformable integral of Khalil (see [21]), then this Definition contains as a particular case, the α-periodic function of [33,34,36]. If we consider the function $f(t)=cost$, and the general kernel, it will be α-periodic generalized, provided that $\mathbb{F}(p,\alpha)=2\pi$; if we put the conformable kernel, it is easy to get the example from [36].

Definition 6. *A family of functions, $f_1(t),f_2(t),\cdots,f_n(t)$ is said α-ortogonal generalized on $[0,b]$, if*

$$J^{\alpha}_{F,0}(f_if_j)(b)=0, \tag{8}$$

for all $i\neq j$.

Remark. If $F\equiv 1$ we have the classic orthogonality, and if $F(t,\alpha)=t^{1-\alpha}$, we obtain the α-orthogonality of [33,34,36].

To solve the generalized Laplace equation, using the generalized Fourier method, we must first consider the generalized Fourier coefficients that can be calculated using the Definition 2 of the generalized integral. The calculation of the coefficients of the generalized Fourier series involves the kernel F, so it is somewhat more complex than that of the classical Fourier series. This generalized series can be written like this:

$$S_Ff(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\left(n\mathbb{F}(t,\alpha)\right)+b_n\sin\left(n\mathbb{F}(t,\alpha)\right)\right\}, \tag{9}$$

where a_0, a_n and b_n are the generalized Fourier coefficients defined by

$$a_0=\frac{1}{p}J^{\alpha}_F(f)(p)=\frac{1}{p}\int_0^p\frac{f(t)}{F(t,\alpha)}dt \tag{10}$$

$$\begin{aligned}a_n&=\frac{2}{p}J^{\alpha}_F(fcos(n\mathbb{F}(t,\alpha)))(p)\\&=\frac{2}{p}\int_0^p\frac{f(t)cos(n\mathbb{F}(t,\alpha))}{F(t,\alpha)}dt\end{aligned} \tag{11}$$

$$\begin{aligned}b_n&=\frac{2}{p}J^{\alpha}_F(fsin(n\mathbb{F}(t,\alpha)))(p)\\&=\frac{2}{p}\int_0^p\frac{f(t)\sin(n\mathbb{F}(t,\alpha))}{F(t,\alpha)}dt\end{aligned} \tag{12}$$

with $n=1,2,3,..$ and $p=\mathbb{F}^{-1}(2\pi,\alpha)$.

Remark. This definition of generalized series makes sense because the set

$$\{Sin\left(nF\left(t,\alpha\right)\right)/n\in N\}\cup\{Cos\left(nF\left(t,\alpha\right)\right)/n\in N\} \tag{13}$$

is α-orthogonal.

Proof.

$$\begin{aligned}&J^{\alpha}_F\left(\cos\left(n\mathbb{F}\left(t,\alpha\right)\right).\cos\left(m\mathbb{F}\left(t,\alpha\right)\right)\right)(p)\\&=\int_0^p\frac{\cos\left(n\mathbb{F}\left(t,\alpha\right)\right).\cos\left(m\mathbb{F}\left(t,\alpha\right)\right)}{F\left(t,\alpha\right)}dt\end{aligned} \tag{14}$$

We do the change $u=\mathbb{F}(t,\alpha)$, then $du=\frac{dt}{F(t,\alpha)}$ and we obtain

$$\begin{aligned}&J^{\alpha}_F\left(\cos\left(n\mathbb{F}\left(t,\alpha\right)\right).\cos\left(m\mathbb{F}\left(t,\alpha\right)\right)\right)(p)\\&=\int_0^{\mathbb{F}(p,\alpha)}\cos\left(nu\right)\cos\left(mu\right)du=0\end{aligned} \tag{15}$$

if $n\neq m$.

The same calculation for $\sin\left(n\mathbb{F}\left(t,\alpha\right)\right)$ and for $\sin\left(n\mathbb{F}\left(t,\alpha\right)\right)$ with $\cos\left(n\mathbb{F}\left(t,\alpha\right)\right)$

Remark. If we consider the kernel $F\equiv 1$, we obtain the classical coefficients of the Fourier Series and if we take $F(t,\alpha)=t^{1-\alpha}$, we obtain the α-fractional Fourier series of [33,34,36].

The following result can be proved, following the classical case.

Theorem 5. *The generalized Fourier series of a generalized periodic function, piecewise continuous, converges pointwise to the average limit of the function at each point of discontinuity, and to the function at each point of continuity.*

2.2 Generalized Laplace's Partial Differential Equation

Now we are going to apply the generalized Fourier series to solve the following generalized Laplace Equation ($0 < \alpha \leq 1$):

$$_{(1,1)}N_F^{\alpha+\alpha}u(t,x)) +_{(2,2)} N_F^{\alpha+\alpha}u(t,x)) = 0. \quad (16)$$

As in the classical case, we will study this equation with boundary conditions at the boundary of the enclosure where the equation is fulfilled, which must have a certain regularity, these conditions can be of two types:
i) Dirichlet conditions: these are conditions in the function $u(t,x)$.
ii) Neumann conditions: these are conditions imposed on the generalized partial derivatives of $u(t,x)$ of the order $_{(1,1)}N_F^{\alpha+\alpha}u(t,x)$ or $_{(2,2)}N_F^{\alpha+\alpha}u(t,x)$.

In solving both, we will use the generalized Variables Separation Method, which contains as particular cases the classical case and the conformable of [33,34,36].

Case i). Let us discuss the solution of the equation (16), on the region $0 \leq t \leq a$, $0 \leq x \leq b$. with the following boundary conditions:

$$u(t,0) = u(t,b) = 0, 0 \leq t \leq a, \quad (17)$$
$$u(0,x) = 0, 0 \leq x \leq b, \quad (18)$$
$$u(a,x) = f(x), 0 \leq x \leq b. \quad (19)$$

Let's find the solution in the form $u(t,x) = T(t)X(x)$, when substituting in the equation (16) we obtain:

$$_{(1,1)}N_F^{\alpha+\alpha}T(t)X(x) +_{(2,2)} N_F^{\alpha+\alpha}X(x)T(t) = 0. \quad (20)$$

As we look for non-trivial solutions, that is, $u \neq 0$, assuming that $T(t) \neq 0$ and $X(x) \neq 0$ we obtain from the previous equation

$$\frac{1}{T(t)} \cdot_{(1,1)} N_F^{\alpha+\alpha}T(t) = -\frac{1}{X(x)} \cdot_{(2,2)} N_F^{\alpha+\alpha}X(x). \quad (21)$$

Since t and x are independent variables, from the above we have

$$\frac{1}{T(t)} \cdot_{(1,1)} N_F^{\alpha+\alpha}T(t) = -\frac{1}{X(x)} \cdot_{(2,2)} N_F^{\alpha+\alpha}X(x) = c \quad (22)$$

for some real constant c, to be determined. This leads us to the following generalized linear differential equations:

$$\frac{1}{T(t)} \cdot_{(1,1)} N_F^{\alpha+\alpha}T(t) = c, \quad (23)$$
$$\frac{1}{X(x)} \cdot_{(2,2)} N_F^{\alpha+\alpha}X(x) = -c. \quad (24)$$

The following definition (see [9]) will allow us to obtain the solution we are looking for.

Definition 7. *The generalized exponential function is defined for every $t \geq 0$ by:*

$$E_\alpha^N(k,t) = exp(k\mathscr{F}(t)) \quad (25)$$

where $k \in \mathbb{R}$, $0 < \alpha < 1$ and $\mathscr{F}(t) = J_{F,u}^\alpha(1)(t) = \int_u^t d_\alpha\zeta = \int_u^t \frac{1}{F(\zeta,\alpha)}d\zeta$ and $u \in \mathbb{R}^+$.

Using the Chain Rule we have the simple identity $N_F^\alpha(exp(k\mathscr{F}(t)) = k(exp(kF(t))$, which allows us to obtain the solution of the equations (24) (with boundary conditions $u(t,0) = u(t,b) = 0$) and (23) (with boundary condition $u(0,x) = 0$), and write the general solution of the equation (16) as (using a procedure similar to the classic case):

$$u_n(t,x) = C_n \sinh\left[\frac{n\pi\mathbb{F}(t,\alpha)}{\mathbb{F}(a,\alpha)}\right] \sin\left[\frac{n\pi\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right] \quad (26)$$

where C_n is a certain real constant, which depends on n and which appeared in the solution process. Since the equation is linear, any linear combination of solutions is another solution; therefore, we can consider it as a formal general solution:

$$u(t,x) = \sum_{n=1}^{\infty} C_n \sinh\left[\frac{n\pi\mathbb{F}(t,\alpha)}{\mathbb{F}(a,\alpha)}\right] \sin\left[\frac{n\pi\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right]. \quad (27)$$

We can obtain the solution in terms of a generalized fourier series, using $u(a,x) = f(x)$, so we have

$$u(a,x) = \sum_{n=1}^{\infty} C_n \sinh(n\pi) \sin\left[\frac{n\pi\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right] = f(x) \quad (28)$$
$$= \sum_{n=1}^{\infty} D_n \sin\left[\frac{n\pi F(x,\alpha)}{F(b,\alpha)}\right] = f(x). \quad (29)$$

Finally, we can calculate the value of the coefficients $D_n = C_n \sinh(n\pi)$, if we consider the previous expression, as a generalized Fourier Fourier, in sines, of $f(x)$, therefore, we have

$$D_n = \frac{2}{\mathbb{F}(b,\alpha)} J_{F,0}^\alpha\left(f(x)sin\left[\frac{n\pi\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right]\right)(b)$$
$$= \frac{2}{\mathbb{F}(b,\alpha)} \int_0^b \left(f(x)sin\left[\frac{n\pi\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right]\right) \frac{dx}{F(x,\alpha)}.$$

Remark. If we consider the kernel $F \equiv 1$, the results obtained coincide with those known from the classic case. If we take $F = t^{1-\alpha}$, then the previous results cover those obtained in [33,36] and [34] for the Dirichlet-type boundary conditions.

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Case ii) We are going to discuss the solutions of the equation (16), subject to the conditions:

$$ {}_1N_F^\alpha u(t,0) = u(t,0) = 0, 0 \le t \le a, \quad (30)$$
$$ {}_1N_F^\alpha u(t,b) = u(t,b) = 0, 0 \le t \le a, \quad (31)$$
$$ {}_2N_F^\alpha u(0,x) = f(x), 0 \le t \le b, \quad (32)$$
$$ {}_2N_F^\alpha u(a,x) = 0, 0 \le t \le b \quad (33)$$

We will use, again, the Separation of Variables Method looking for the solution in the form $u(t,x) = T(t)X(x)$ which leads us to consider the following two generalized linear differential equations:

$$\frac{1}{T(t)} \cdot {}_{(1,1)}N_F^{\alpha+\alpha} T(t) = c, \quad (34)$$
$$\frac{1}{X(x)} \cdot {}_{(2,2)}N_F^{\alpha+\alpha} X(x) = -c. \quad (35)$$

By having boundary conditions on the generalized partial derivatives, it obviously leads us to some variations of the solution obtained in the previous case. So we have (following the classic case and considering the three possible cases for c, positive, negative or zero):

$$u_n(t,x) = C_n \cos\left(n\pi \frac{\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right) \times \left[\exp\left(n\pi \frac{\mathbb{F}(t,\alpha)}{\mathbb{F}(a,\alpha)}\right) + \exp\left(-n\pi \frac{\mathbb{F}(t,\alpha)}{\mathbb{F}(a,\alpha)}\right)\exp(2n\pi)\right]. \quad (36)$$

The formal solution sought is the linear combination of all the solutions obtained in the case $c = 0$, so we can write:

$$u(t,x) = K + \sum_{n=1}^{\infty} C_n \cos\left(n\pi \frac{\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right) \times \left[\exp\left(n\pi \frac{\mathbb{F}(t,\alpha)}{\mathbb{F}(a,\alpha)}\right) + \exp\left(-n\pi \frac{\mathbb{F}(t,\alpha)}{\mathbb{F}(a,\alpha)}\right)\exp(2n\pi)\right], \quad (37)$$

with K some real constant.

The values of C_n The values of C_n will be determined with the help of the generalized Fourier Series, using the boundary condition (32). So, we have

$$ {}_1N_F^\alpha u(t,x) = \sum_{n=1}^{\infty} \frac{C_n n\pi}{\mathbb{F}(b,\alpha)} \cos\left(n\pi \frac{\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right) \left[\exp\left(n\pi \frac{\mathbb{F}(t,\alpha)}{\mathbb{F}(a,\alpha)}\right) + \exp\left(-n\pi \frac{\mathbb{F}(t,\alpha)}{\mathbb{F}(a,\alpha)}\right)\exp(2n\pi)\right] \quad (38)$$

where do we get

$$N_{F_2}^\alpha u(t,0) = \sum_{n=1}^{\infty} \frac{C_n n\pi}{\mathbb{F}(b,\alpha)} \cos\left(n\pi \frac{\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right) \times \left[1 - \exp\left(2n\pi \frac{\mathbb{F}(a,\alpha)}{\mathbb{F}(b,\alpha)}\right)\right] = \sum_{n=1}^{\infty} D_n \cos\left(n\pi \frac{\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right),$$

from the above, we get

$$D_n = \frac{C_n n\pi}{\mathbb{F}(b,\alpha)}\left[1 - \exp\left(2n\pi \frac{\mathbb{F}(a,\alpha)}{\mathbb{F}(b,\alpha)}\right)\right] = \frac{2}{\mathbb{F}(b,\alpha)} J_{F,0}^\alpha \left[f(x) cos\left(n\pi \frac{\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right)\right](a) = \frac{2}{\mathbb{F}(b,\alpha)} \int_0^a \left[f(x) cos\left(n\pi \frac{\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right)\right] \frac{dx}{F(x,\alpha)}.$$

with

$$C_n = \frac{2}{n\pi\left[1 - \exp\left(2n\pi \frac{\mathbb{F}(a,\alpha)}{\mathbb{F}(b,\alpha)}\right)\right]} J_{F,0}^\alpha \left[f(x) cos\left(n\pi \frac{\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right)\right](a) = \frac{2}{n\pi\left[1 - \exp\left(2n\pi \frac{\mathbb{F}(a,\alpha)}{\mathbb{F}(b,\alpha)}\right)\right]} \int_0^a \left[f(x) cos\left(n\pi \frac{\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right)\right] \frac{dx}{F(x,\alpha)}.$$

We must point out that, in this case, the independent term of the generalized Fourier Series, $\frac{D_0}{2}$, must be zero, that is, $D_0 = \frac{2}{\mathbb{F}(b,\alpha)} \int_0^a f(x) \frac{dx}{F(x,\alpha)} = 0$.

2.3 Examples

*Example 1.*Recall that the solution of the generalized Laplace's equation is

$$u(t,x) = \sum_{n=1}^{\infty} C_n \sinh\left[\frac{n\pi \mathbb{F}(t,\alpha)}{\mathbb{F}(a,\alpha)}\right] \sin\left[\frac{n\pi \mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right]. \quad (39)$$

and using the initial condition $u(a,x) = f(x)$ we obtain the coefficients

$$C_n = \frac{2}{\mathbb{F}(b,\alpha) sinh(n\pi)} J_{F,0}^\alpha \left(f(x) sin\left[\frac{n\pi \mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right]\right)(b) = \frac{2}{\mathbb{F}(b,\alpha) sinh(n\pi)} \int_0^b \left(f(x) sin\left[\frac{n\pi \mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right]\right) \frac{dx}{F(x,\alpha)}.$$

Choosing the kernel $F(t,\alpha) = t^{1-\alpha}$ then $\mathbb{F}(t,\alpha) = \int_0^t \frac{1}{F(s,\alpha)} ds = t^\alpha$ we obtain the coefficients

$$C_n = \frac{2b^{-\alpha}}{sinh\left(\frac{n\pi a^\alpha}{b^\alpha}\right)} J_{F,0}^\alpha \left(f(x) sin\left[\frac{n\pi x^\alpha}{b^\alpha}\right]\right)(b) = \frac{2b^{-\alpha}}{sinh\left(\frac{n\pi a^\alpha}{b^\alpha}\right)} \int_0^b \left(f(x) sin\left[\frac{n\pi x^\alpha}{b^\alpha}\right]\right) x^{\alpha-1} dx$$

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*Example 2.*If we take $f(x) = C$ a real constant, then the coefficients of the solution are

$$C_n = \frac{2}{\mathbb{F}(a,\alpha)sinh(n\pi)} J^{\alpha}_{F,0}\left(sin\left[\frac{n\pi\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right]\right)(b)$$
$$= \frac{2\mathbb{F}(b,\alpha)}{n\pi\mathbb{F}(a,\alpha)sinh(n\pi)}\times$$
$$\int_0^{n\pi}(sin(u))\,du = \frac{2\mathbb{F}(b,\alpha)((-1)^n-1)}{n\pi\mathbb{F}(a,\alpha)sinh(n\pi)}.$$

*Example 3.*Now we choose $f(x) = \mathbb{F}(a,\alpha)$ then the coefficients are

$$C_n = \frac{2}{\mathbb{F}(a,\alpha)sinh(n\pi)} J^{\alpha}_{F,0}\left(\mathbb{F}(x,\alpha)sin\left[\frac{n\pi\mathbb{F}(x,\alpha)}{\mathbb{F}(b,\alpha)}\right]\right)(b)$$
$$= \frac{2\mathbb{F}(b,\alpha)}{n\pi\mathbb{F}(a,\alpha)sinh(n\pi)}$$
$$\int_0^{n\pi}(usin(u))\,du = \frac{2\mathbb{F}(b,\alpha)(-1)^n}{\mathbb{F}(a,\alpha)sinh(n\pi)}.$$

3 Conclusions

In this paper, we have solved the generalized Laplace Equation, subject to Dirichlet and Newman boundary conditions. For this, we have used generalized derivative and integral operators, previously defined. From these notions, we have defined the generalized Fourier series and the concrete expressions of the coefficients of said series.

The results obtained indicate that they allow a broader application than the classic and conformable cases, which are covered by these. Finally, some examples are presented to show the strength and breadth of the results obtained, which will allow their application in future applied research in various areas.

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Conflict of interests:

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