Applied Mathematics & Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/amisl/100101

A Review: Alpha Power Transformation Family of Distributions

Maryam Mohiuddin and R. Kannan

Department of Statistics, Annamalai University, Annamalai Nagar, Chidambaram, Tamil Nadu-608 002, India

Received: 17 Sep. 2021, Revised: 10 Nov. 2021, Accepted: 13 Dec. 2021.
Published online: 1 Jan. 2022.

Abstract: In this study, a complete review of different families of distributions is presented. A total of nearly eighteen methods for developing new families of distributions are discussed and twenty-six similar distributions are examined. The study could provide a useful benchmark and promote the development of better distributions that can more appropriately model complicated phenomena.

Keywords: Gamma family, Generalized Gamma family, Alpha Power Transformation, Beta Generated Distribution.

1 Introduction

Probability distributions play a vital role in decision-making during uncertain circumstances. Their application in the fields of physics, computer science, public health, medical, insurance, reliability analysis, survival analysis, signal processing, communication and engineering, signify that classical distributions are not sufficient for modeling data sets, for this, it is important to extend the existing distributions. In statistical theory, extensions, over the distributions have become a prominent method. Generally, new distributions with the addition of parameters can be obtained either by combining the existing distributions or by the use of generators. The objective of adding new parameters to the distribution provides more flexibility to the standard distributions for valuable analysis of complicated data structure.

Distribution fitting has played a significant role in many fields of science. It is used to select a model that defines the pattern of a data set produced by some random process. Fitting distributions to data has the advantage of allowing for the development of more appropriate models for random process. In this regard, researchers proposed different methods of generating new continuous distributions to model lifetime data that have a high degree of skewness and kurtosis.

Statistical distributions are essential for parametric inferences and applications to fit real life phenomena. In the literature, many approaches for generating statistical distributions have been established. Some well-known techniques in the early days for generating uni-variate continuous distributions include a method based on differential equations which were introduced by Pearson [1], methods of translation was proposed by Johnson [2] and the techniques based on quantile functions was presented by Tukey [3]. The early generalization of gamma distribution can be traced back to Amoroso [4] discussed a generalized gamma distribution and applied it to fit income rates. Johnson et al. [5] introduced a four-parameter generalized gamma distribution which reduces to the generalized gamma distribution defined by Stacy [6]. Balakrishnan and Peng [7] applied gamma distribution to develop a generalized frailty model. Cardeiro et al. [8] derived other generalizations of Stacy gamma distribution using the exponentiated method and applied it to lifetime and survival analysis.

From last few decades, introducing a new or more flexible distributions has become an important aspect in statistical theory. Agarwal and Alsaleh [9] applied generalized gamma distribution to study the hazard rates of the distribution. Generalized gamma distribution is a continuous probability distribution with three parameters. It is a generalization of the two-parameter gamma distribution. The density function and distribution function of the generalized gamma distribution is given by

$$f(x;\alpha,\beta,p)=\frac{a\,x^{ap-1}\exp(-x/\beta)^{a}}{\beta^{ap}\,\Gamma p}\quad;\ x\geq 0,\ \alpha,\ p,\ \beta>0 \qquad (1)$$

and

*Corresponding author e-mail: masmariam7@gmail.com

$$F(x;\alpha,\beta,p)=\frac{\gamma\left(\alpha,\left(\frac{x}{\beta}\right)^{\alpha}\right)}{\Gamma p} \quad ; \ \alpha>0,\ p>0,\ \beta>0$$

Where, Γ (.) denotes the gamma function and γ (.) denotes the lower incomplete gamma function.

The beta distribution is a family of continuous probability distributions defined on the interval (0,1) and parameterized by two positive shape parameters that affect the shape of a distribution. When fitting symmetric and skewed models with different tail weights beta family of distributions offers a lot of versatility.

Donald [10] introduced a generalized beta distribution of the first and second kind. The probability density function (pdf) of the distribution is defined by

$$f(x;a,b,p,q)=\frac{a\,x^{ap-1}(1-(x/b)^{a})^{q=1}}{b^{ap}\,B(p,q)} \quad ; \ 0\le x\le b \tag{2}$$

$$f(x;a,b,p,q)=\frac{a\,x^{ap-1}}{b^{ap}\,B(p,q)(1+(x/b)^{a})^{p+q}} \quad ; \ 0\le x\le b \tag{3}$$

Azzalini [11] studied the skew-normal distribution by introducing an extra parameter to the normal distribution to bring more flexibility to the normal distribution. Skew-normal distribution takes the following form for the parameter $\lambda \epsilon \mathbb{R}$.

$$f(x;\lambda) = 2\,\varphi(x)\,\phi(\lambda x); \ \ x\,\epsilon\,\mathbb{R} \tag{4}$$

$\varphi(x)$ and $\phi(\lambda x)$ are the probability density function (pdf) and cumulative distribution function (cdf) of a standard normal distribution and $(\lambda)$is the skewness parameter. Skewed-normal distribution can be easily used for other symmetric distributions.

Mudholkar and Srivastava [12] proposed exponentiated Weibull model having two shape parameters and a scale parameter. The exponentiated Weibull model is more flexible than the two-parameter Weibull model. For $\beta > 0$, the cumulative distribution function (cdf) of the proposed model is given by

$$F(x;\alpha,\beta,\lambda) \ = \ (1-\exp(-\lambda x^{\alpha})^{\beta} \ \ ; \ x>0, \lambda, \alpha, \beta \ > 0, \tag{5}$$

Where α and λ is the shape and scale parameter respectively. This family of distributions have scale and shape parameters similar to the gamma family and Weibull family.

Marshall and Olkin [13] proposed another method to introduce an additional parameter to any distribution function as follows, if $g(x)$ and $G(x)$ are the probability density function (pdf) and cumulative distribution function (cdf) respectively of a random variable X, then the newly proposed family of distributions has the following probability density function (pdf) for any parameter $\theta\,\epsilon\,(0,\infty)$.

$$f(x;\theta) = \frac{\theta\,g(x)}{(1-(1-\theta)(1-G(x)))^{2}} \ ; \quad x \in \mathbb{R} \tag{6}$$

Marshall and Olkin presented a general method for generating a new family of lifetime distributions defined in terms of the survival function. The probability density function (pdf) of the newly generated distribution can be obtained as

$$\overline{G}(x;\alpha)=\frac{\alpha\,\overline{F}(x)}{1-\overline{\alpha}\,\overline{F}(x)}$$

$$\overline{G}(x;\alpha)=\frac{\alpha\,\overline{F}(x)}{F(x)+\overline{\alpha}\,F(x)} \quad ;-\infty<x<\infty,\ \alpha>0 \tag{7}$$

Where, $\overline{F}(x)$ is a survival function of a random variable X.

Eugene et al. [14] proposed the beta-generated method that uses the beta distribution with parameters α and β as the generator to develop the beta-generated distributions. The probability density function (pdf) and cumulative distribution function (cdf) of a beta-generated random variable X is defined as

$$g(x;\alpha,\beta)=\frac{f(x)}{B(\alpha,\beta)}\,F^{\alpha-1}(x)\,(1-F(x))^{\beta-1} \quad ; \ \alpha,\ \beta>0 \tag{8}$$

and

$$G(x)=\int_{0}^{F(x)} b(t)\,dt\,,$$

Where, $b(t)$ is the pdf of beta random variable and $F(x)$ is the cdf of any random variable, X.

Ferreira and Steel [15] discussed a general framework for generating a family of skewed distributions based on symmetric distribution. The probability density function (pdf) of new family of distributions takes the form as follows

$$g(x/f,p)=f(x)\,p(F(x)) \tag{9}$$

Where, $F(.)$ *and* $f(.)$ are the cdf and pdf of a symmetric distribution over the range $x = (0,1)$.

Jones [16] introduced a general class of beta-generated distributions, that concentrates on cases where $F(.)$ is symmetric about zero with no free parameters other than location parameter and scale parameter and where, I is the whole real line.

The first distribution of beta-generated distributions was studied in depth by Eugene is a beta-normal distribution. denoted the standard normal distribution and density function by $\varphi(.)$ and $\phi(.)$, respectively, and let $\varphi^{-1}(U)$ with, $U \sim B(a,b)$ are the classical beta distribution. Then, X has a beta-normal distribution $BN\ (a,b,0,1)$ with density function $f(.)$.

$$f_{BN}(x;a,b,0,1)=B(a,b)^{-1}\,\phi(x)\,[\phi(x)]^{a-1}[1-\phi(x)]^{b-1} \quad ; -\infty<x<\infty \tag{10}$$

Where, a, b are the parameters of the beta-normal distribution, a is the location parameter and b is a scale parameter of the distribution.

The density function of beta-normal is symmetric when, $a = b$, for $a < b$, it is negatively skewed and positively skewed, when $a > b$. When $a = b > 1$ the beta-normal distribution has positive excess kurtosis and when $a = b < 1$ it has negative excess kurtosis, as demonstrated by Eugene et al. However, both skewness and kurtosis are very limited and the only way to gain even a modest degree of excess kurtosis is to skew the distribution as far as possible.

Nadarajah and Kotz [17] studied the beta-generated distributions. If $G(.)$ denotes the cumulative distribution function (cdf) of a random variable, then a generalized class of distributions can be defined by

$$F(x;a,b)=I_{G(x)}(a,b) \quad ; \ a>0, \quad b>0 \tag{11}$$

Where, $$I_y(a,b)=\frac{B_y(a,b)}{B(a,b)}$$

denotes the incomplete beta function ratio and

$$B_y(a,b)=\int_0^y z^{a-1}(1-z)^{b-1}dz\,,$$

after simplification, $B_y(a,b)$ becomes,

$$B_y(a,b)=y^a\left(\frac{1}{a}+\frac{1-b}{1+a}x+\ldots+\frac{(1-b)\ldots(n-b)}{n!(a+n)}y^n+\ldots\right)$$

denotes the incomplete beta function.

The beta-Gumbel distribution was introduced by using the model proposed by Eugene. The Gumbel distribution is perhaps a widely applied statistical distribution for problems in engineering. The beta-Gumbel distribution is generated from the logit of a beta random variable. The probability density function (pdf) of beta-Gumbel distribution is defined by

$$f(x;a,b,\mu,\sigma)=\frac{u\exp(-u\,a)\{1-\exp(-u)\}^{b-1}}{\sigma\,B(a,b)} \quad ; \; -\infty<x<\infty,\; -\infty<\mu<\infty,\; a,\sigma,b>0$$

Where, $u=\left(-\frac{x-\mu}{\sigma}\right)$

Akinsete et al. [18] discussed the beta-pareto distribution by using inverse cumulative distribution function (cdf) to a beta distributed random variable X. $Let\ G(x)$ denotes the cumulative distribution function (cdf) of a random variable X. The cumulative distribution function (cdf) for a generalized class of distribution for the random variable X, is defined by Eugene et al., is generated by using the inverse cumulative distribution function (cdf) of a beta distributed random variable to obtain density function.

$$f(x;a,b)=\frac{1}{B(a,b)}[G(x)]^{a-1}[1-G(x)]^{b-1}g(x) \tag{12}$$

The probability density function (pdf) and cumulative distribution function (cdf) of the beta-pareto distribution is given by

$$f(x;k,\theta,a,b)=\frac{k}{\theta B(a,b)}\left(1-\left(\frac{x}{\theta}\right)^{-k}\right)^{a-1}\left(\frac{x}{\theta}\right)^{-k\theta-1} \quad ; \; x\geq 0,\, b,\, a,\, k,\, \theta>0$$

Where, k,θ is a shape and scale parameters of a distribution.

$$F(x;a,b)=\frac{\Gamma(a+b)}{\Gamma a+\Gamma b}\int_0^{G(x)} s^{a-1}(1-s)^{b-1}dt \quad ; a>0, b<\infty \tag{13}$$

After simplification, $F(x)$ becomes

$$F(x;a,b)=1-\frac{G(x)^b}{B(a,b)}\left(\frac{1}{b}+\frac{1-a}{b+1}F(x)+\ldots+\frac{(1-a)(2-a)\ldots(n-a)}{n!\,(n+b)}\right)$$

Zografos and Balakrishnan [19] proposed two novel families of uni-variate distributions generated by gamma random variables. For any baseline cumulative distribution function (cdf) $G(x)$, and $x\in R$, the probability density function (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$ is given by

$$f(x;a)=\frac{1}{\Gamma(a)}(-\log(1-G(x)))^{a-1}g(x) \tag{14}$$

Where, $a>0$ and $\Gamma a=\int_0^\infty t^{a-1}\exp(-t)\,dt$

$$F(x;a)=\frac{\gamma(a,-\log(1-G(x))}{\Gamma a} \tag{15}$$

Barreto-Souza et al. [20] studied beta-generated exponential distribution. The model was obtained by using the cumulative distribution function (cdf) of exponentiated exponential distribution introduced by Gupta and Kundu [21].

$$G\ (x;\alpha,\lambda)=(1-\exp(-\lambda x))^{\alpha}\quad ;\ x>0,\ \alpha,\ \lambda>0$$

Where, α, λ, are the shape and scale parameters of the distribution.

The Generalized Exponential density function varies significantly depending on the shape parameter α. Also, the hazard function is non-decreasing function when $\alpha > 1$, and non-increasing function when $\alpha < 1$. For, $\alpha = 1$, it is constant.

$$f(x;a,b)=\frac{1}{B(a,b)}\int_{0}^{G(x)} t^{a-1}(1-t)^{b-1}dt\quad ;0<x<1, a>0, b<\infty$$

The probability density function (pdf) of the beta-generated exponential distribution is given by

$$f(x;a,b,\alpha,\lambda)=\frac{\alpha\lambda}{B(a,b)}\exp(-\lambda x)(1-\exp(-\lambda x)(1-(1-\exp(-\lambda x)^{\alpha})^{b-1}\quad ;\ a>0, b>0, \alpha>0, \lambda>0, x>0$$

The shape of the probability density function (pdf) of the beta-generated exponential distribution is decreasing and uni-modal for $a, b < 1$ and $\lambda = \alpha = 1$.

Jones, Larsen [22] and Arnold et al. [23] introduced the multivariate beta-generated class of distributions. Some practical applications have been considered, e.g. Jones and Larsen fitted skewed t and $\log F(.)$ to temperature data, Akinsete et al. fitted the beta Pareto distribution to flood data.

Razzaghi [24] applied the beta-normal distribution to dose-response modeling and risk assessment for quantitative responses. However, the classical beta generator has only two parameters, so it can add only a limited structure to the parent distribution. For many choices of parent distributions, the computations of quantiles and moments of a beta-generated distributions can become rather complex. Also, when $a = b$ (the skewness is zero if $F(.)$ is symmetric) the beta generator typically induces Meso-kurtosis, in that the BG distribution has a lower kurtosis than the parent distribution.

Jones [25] advocated replacing the beta generator with the Kumaraswamy [26] distribution, commonly termed the Minimax distribution. He extended the beta generated family of distributions by using Kumuraswamy distribution as $b(t)=\alpha\beta t^{\alpha-1}(1-t^{\alpha})^{\beta-1}\quad ; t\in(0,1)$ instead of beta distribution. The probability density function (pdf) for Kumaraswamy generated (Kw-G) family of distributions is defined by

$$g(x;\alpha,\beta) = \alpha\beta f(x)\ F^{\alpha-1}(x)\ (1-F(x))^{\beta-1}\quad ;\alpha>0,\ \beta>0 \tag{16}$$

Alexander et al. [27] studied the generalized beta X family of distribution by considering pdf as

$$b(t;a,b) = C\,B(a,b)\ t^{ac}(1-t^{c})^{b-1}\quad ;\ 0<t<1 \tag{17}$$

The generalized beta of the first kind was introduced by Donald. The new family of distributions is called a generalized beta-generated (GBG) family of distributions. The probability density function (pdf) of the GBG family of distributions is given by

$$g(x;r,a,b,c)=C\beta(a,b)^{-1} f(x;r)\,F(x;r)^{ac-1}(1-F(x;r)^{c})^{b-1} \tag{18}$$

2 Transformed-Transformer Family of Distributions

Alzaatreh et al. [28] introduced a new method for generating families of continuous distributions called $T - X$ family by replacing the beta probability density function (pdf) with a probability density function (pdf) $r(t)$, of a random variable and applying a function $W(F(x))$ that satisfies some conditions of distribution to develop $T - X$ family. The $T - X$ family of distributions was generated by using the function $W(.)$ which satisfies the following conditions.

- $W(F(x))\in(a,b)$
- W is differentiable and monotonically non-decreasing

- $W(F(x)) \to a$ as $x \to -\infty$ and $W(F(x)) \to b$ as $x \to \infty$

Where, [a, b] is the support of the random variable T $-\infty \le a < b \le \infty$.

Let $F(x)$ be the cumulative distribution function (cdf) of a continuous random variable X and $r(t)$ be its probability density function (pdf). The function $W(F(x))$ of the cdf $F(x)$ is monotonic and continuous functions, then, according to Alzaatreh, et al. the cumulative distribution function (cdf) of the $T-X$ family of distributions is defined by

$$G(x) = \int_a^{W(F(x))} r(t)\,dt = \Pr\left(T \le W(F(x))\right) = R(W(F(x)) \quad (19)$$

Where, R is the cdf T.

The corresponding probability density function (pdf) of the $T-X$ family of distributions is given by

$$g(x) = \left\{\frac{d}{dx} W(F(x))\right\} \left\{r(W(F(x)))\right\} \quad (20)$$

The most interesting property of $T-X$ family is that it can be used to generate a variety of probability distributions on considering different combinations T, X and $W(.)$.

Alzaatreh et al. [29] considered the function $W(F(x))$ to the quantile function of the random variable Y and defined the $T-R(Y)$ family of distributions. Let T, R *and* Y be a random variable with cdf, $F_T(x) = P\,(T \le x)$, $F_R(x) = P\,(R \le x)$, $F_Y(x) = P\,(Y \le x)$ and corresponding quantile function $Q_T(p)$, $Q_R(p)$ *and* $Q_Y(p)$ where, the quantile functions is defined as $Q_Z(p) = \inf(Z\colon F_Z(z) \ge p$. If the density function of the distribution exists then it can defined as $f_Z(x)$, for $z = T, R$ *and* Y. Assume that the random variables $T, Y \in (a, b)$ for $-\infty \le a < b \le \infty$. The random variable X in $T-R(Y)$ family of distributions is defined as

$$F_X(x) = \int_a^{Q_Y(F_R(x)} f_T(t)\,dt = F_T(Q_Y(F_R(x))) \quad (21)$$

The corresponding probability density function (pdf) associated with cdf is

$$f_X(x) = f_T(Q_Y F_R(x))\; Q'_Y F_R(x)\, f_R(x) \quad (22)$$

The cumulative distribution function (cdf) can be re-written as

$$f_X(x) = f_R(x)\frac{f_T\,(Q_Y\,(F_R(x)))}{f_Y\,(Q_Y\,(F_R(x)))} \quad (23)$$

The hazard function of the random variable X can be written as

$$h_X(x) = h_R(x)\frac{h_T\,(Q_Y\,(F_R(x)))}{h_Y\,(Q_Y\,(F_R(x)))} \quad (24)$$

The generalizations of gamma distribution that fall in the $T-R(Y)$ family of distributions of the generalized gamma generated distribution was studied by Zografes and Balakrishnan. The Gamma-Pareto distribution by Alzaatreh et al. and Gamma-Normal distribution. Alzaatreh et al. studied $T-R(exponential), T-Normal\,(Y)$, and $T-Gamma\,(Y)$ families of distributions etc. Various generalizations of the gamma distribution can be seen as members of $T-G(Y)$ family. When $T\sim\beta(a,b)$ and $Y\sim U(0,1)$, the $T-G(Y)$ family distribution reduces to beta-gamma distribution (Kong et al.). When $T\sim Power(a)$ and $Y\sim U(0,1)$ then $T-G(Y)$ reduced to the exponentiated gamma distribution (Nadarajah and Kotz), [30].

Transmuted Family of distributions was introduced by Shaw and Buckley [31] to modify the existing probability distributions to capture the quadratic behaviour of the data by using the technique of quadratic rank transmutation map. The novelty of transmuted is used to introduce the skewness and kurtosis to the symmetric and other type models.

The theory of weighted distributions provides collective access to the problems of model specification and data interpretation. Weighted distributions provide a technique for fitting the models to the weight functions when the samples are taken from the original distribution and developed distributions.

3 Alpha Power Transformation Family of Distributions

The Alpha power transformation family of distributions was developed by Mahdavi and Kundu [32]. The alpha power model provides the extensions or generalizations to the existing models. The resulting distributions we obtained so-called alpha power transformation distributions. The alpha power transformation distributions used a cumulative distribution function (cdf) and probability density function (pdf) of the existing (baseline) distributions into the APT model. The alpha power transformation model is suitable for the distributions which are non-symmetrical in nature.

According to Mahdavi and Kundu, for a given cumulative distribution function (cdf) and probability density function (pdf) of a baseline distribution, defined the cumulative distribution function (cdf) of Alpha power transformation family of distribution as follows

$$F_{APT}(x;\alpha)=\begin{cases}\left(\dfrac{\alpha^{F(x)}-1}{\alpha-1}\right) & ;\ if\ \alpha>0,\alpha\neq1\\ F(x) & ;\ if\ \alpha=1\end{cases} \tag{25}$$

and its corresponding probability density function (pdf) is defined by

$$f_{APT}(x;\alpha)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\alpha^{F(x)}f(x) & ;\ if\ \alpha>0,\alpha\neq1\\ f(x) & ;\ if\ \alpha=1\end{cases} \tag{26}$$

Mahdavi and Kundu suggested a new model for generating distributions with an application to exponential distribution by incorporating a shape parameter to an exponential distribution by using Alpha power transformation (APT) method to bring more flexibility in the model. The APT model was applied to an exponential distribution having scale parameter to generate a two-parameter exponential distribution known as Alpha Power Exponential (APE) distribution. Using the APT to one parameter exponential distribution, the resulting distribution has the following density function and distribution function.

$$F_{APT}(x;\lambda,\alpha)=\begin{cases}\left(\dfrac{\alpha^{1-e^{-\lambda x}}-1}{\alpha-1}\right) & ;\ x>0,\alpha>0,\lambda>0,\ \alpha\neq1\\ 1-e^{-\lambda x} & ;\ \alpha=1\end{cases}$$

and

$$f_{APT}(x;\lambda,\alpha)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\alpha^{1-e^{-\lambda x}}\lambda\,\exp(-\lambda x) & ;\quad \alpha>0,\ \lambda>0,\ \alpha\neq 1\\ \lambda\,\exp(-\lambda x) & ;\quad \lambda>0,\ \alpha=1\end{cases}$$

The density function $f(x,\alpha,\lambda)$ of the APE distribution is a decreasing function when, $\alpha \leq e$, $x > 0$, and if $\alpha > e$, the function $f(x,\alpha,\lambda)$ is uni-model with mode $\log(log\ \alpha)/\lambda$. When shape parameter of the APE distribution is $\alpha \leq 1$, then the hazard rate of the distribution is decreasing of $x > 0$ and for $\alpha > 1$ the hazard function is an increasing of x. The hazard function decreases from $(\lambda\ log\ \alpha/\alpha - 1)\ to\ \lambda$ for $\alpha < 1$ and for $\alpha > 1$, it increases from λ to $(\lambda\ log\ \alpha/\alpha - 1)$. Some properties of distribution are discussed such as, entropies, order statistics moments, order statistics. The method of maximum likelihood estimation is established to determine the unknown parameters.

Alpha Power Transformation family, properties and Applications was discussed by Mead, Cordeiro, Afify and Al-Mofleh [33]. The density function of the Alpha power transformation family as the linear representation of the model is defined by

$$f_{APT}(x)=\frac{1}{M}\,w(x,\alpha)\,g(x) \tag{25}$$

Where, $w(x,\alpha) = \alpha^{G(x)}$ is weight function of $f_{APT}(x)$ and $M = E(w(x,\alpha))$.

Here, the density function of $f_{APT}(x)$ is defined in terms of X

$$\alpha^{z}=\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k}}{k!}z^{k} \tag{26}$$

$$f(x)=\frac{1}{\alpha-1}\sum_{k=0}^{\infty}\left(\frac{(\log\alpha)^{k+1}}{k!}\right)\big(G(x)\big)^{k}g(x) \tag{27}$$

Let $h_{k+1}(x) = (k+1)g(x)\,G(x)^{k}$ be the exponentiated (Exp-G) density function with power parameter $k+1$ for $k \geq 0$. Hence, for the APT family density function can be re-written as a linear combination of (Exp-G) densities.

$$f_{APT}(x)=\sum_{k=0}^{\infty}b_k\,h_{k+1}(x) \tag{28}$$

Where, $b_k=\left(\log\alpha\right)^{k+1}\left((\alpha-1)(k+1)\right)$

$$F_{APT}(x)=\sum_{k=0}^{\infty}b_k H_{k+1}(x) \tag{29}$$

Where, $H_{k+1}(x)$ is the cdf of Exp-G with power parameter $k+1$.

The author(s) reveal that the probability density function (pdf) of the Alpha power transformation is the weighted function of the probability density function (pdf) of the baseline distribution, here $\alpha^{G(x)}$ is the weight function for the model $g(x)$. The statistical properties of the APT model are studied such as, linear representation of the model, moments, incomplete moments, moment generating function and order statistics. The density function and distribution function of the Alpha power Exponentiated Weibull is defined by

$$f(x;\theta,\beta,\lambda,\alpha)=\begin{cases}\dfrac{\log\alpha}{(\alpha-1)\exp(\theta x^{\beta})}\lambda\theta\beta x^{\beta-1}\left(1-e^{-\lambda x^{\beta}}\right)^{\theta-1}\alpha^{(1-\exp(-\lambda x^{\beta}))^{\theta}} & ;\quad x>0,\beta,\ \lambda,\ \theta>0\alpha\neq 1\\ \lambda\theta\beta x^{\beta-1}\left(1-e^{-\lambda x^{\beta}}\right)^{\theta-1}e^{-\theta x^{\beta}} & ;\quad x>0,\ \alpha=1,\ \beta,\ \lambda,\ \theta>0\end{cases}$$

and

$$F(x;\theta,\beta,\lambda,\alpha)=\begin{cases}\dfrac{\alpha^{(1-\exp(-\lambda x^{\beta}))^{\theta}}-1}{\alpha-1} & ;\ x>0,\ \beta,\lambda,\ \theta>0,\alpha\neq 1\\ (1-\exp(-\lambda x^{\beta}))^{\theta} & ;\ x>0,\beta,\ \lambda,\ \theta>0,\ \alpha=1\end{cases}$$

The statistical properties of the APEW model is studied. This includes linear representation of the model, moments, incomplete moments, moment generating function and order statistics. A simulation study was also carried out to investigate the behaviour of the MLEs for different sample sizes.

Alpha power transformed power Lindley distribution was suggested by Hassan et al. [34]. The probability density function (pdf) and cumulative distribution function (cdf) is defined by

$$f(x;\theta,\beta,\alpha)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\left(\dfrac{\beta\theta^{2}}{\theta+1}\right)x^{\beta-1}(1+x^{\beta})\exp(-\theta x^{\beta})\,\alpha^{1-\exp(-\theta x^{\beta})\,(1+\theta x^{\beta}/\theta+1)} & ;\ x,\theta,\beta,\alpha>0,\alpha\neq 1\\ \left(\dfrac{\beta\theta^{2}}{\theta+1}\right)x^{\beta-1}(1+x^{\beta})\exp(-\theta x^{\beta}) & ;\ x,\theta,\beta>0,\alpha=1\end{cases}$$

and

$$F(x;\theta,\beta,\alpha)=\begin{cases}\dfrac{\alpha^{1-\exp(-\theta x^{\beta})\,(1+\theta x^{\beta}/\theta+1)}-1}{\alpha-1} & ;\ x,\theta,\beta,\alpha>0,\alpha\neq 1\\ 1-\left(1+\dfrac{\theta x^{\beta}}{\theta+1}\right)e^{-\theta x^{\beta}} & ;\ x,\theta,\beta,\alpha>0,\alpha=1\end{cases}$$

Some statistical properties of the model are derived such as quantile function, moments, probability weighted moments and stochastic ordering. The parameters of the model are estimated by maximum likelihood estimators and maximum product of spacing estimators, ordinary, weighted least-squares estimators of the population parameters. A simulation study is employed to study the behaviour of the parameters. The performance of the model was analyzed by using two data sets and make a comparison with Lindley, power Lindley, Exponential and extended Lindley distribution.

Ghosh et al. [35] introduced and studied the properties of the Alpha power transformation Lindley distribution. The density function and distribution function of the model is defined by

$$f(x;\theta,\alpha)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\left(\dfrac{\theta^{2}}{\theta+1}\right)(1+x)\exp(-\theta x)\,\alpha^{1-(1+\theta+\theta x/\theta+1)} & ;\ x,\theta,\ \alpha>0,\alpha\neq 1\\ \left(\dfrac{\theta^{2}}{\theta+1}\right)(1+x)\exp(-\theta x) & ;\ x,\theta,\alpha>0,\alpha=1\end{cases}$$

and

$$F(x;\theta,\alpha)=\begin{cases}\dfrac{\alpha^{1-\left(\frac{1+\theta+\theta x}{\theta+1}\right)}-1}{\alpha-1} & ;\quad x,\theta,\alpha>0,\alpha\neq 1\\ 1-\left(\frac{1+\theta+\theta x}{\theta+1}\right) & ;\quad x,\theta,\alpha>0,\alpha=1\end{cases}$$

The properties of the model were obtained and discussed. These properties include ordinary moments, incomplete and conditional moments, mean residual lifetime, mean deviations, L moments, moment generating function, cumulant generating function, characteristic function, Bonferroni and Lorenz curves, entropies, stress-strength reliability, stochastic ordering, statistics and distribution of sums, differences, ratios and products. The maximum likelihood estimation was considered for the estimation of model parameters. A simulation study is employed to study the behaviour of the parameters. The performance of the model was analyzed by using two data sets.

Alpha power transformation Inverse Lindley distribution was suggested by Dey et al. [36]. The distribution has following the probability density function (pdf) and cumulative distribution function (cdf).

$$f(x;\lambda,\alpha)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\left(\dfrac{\lambda^2}{\lambda+1}\right)\left(\dfrac{1+x}{x^3}\right)\exp(-\lambda/x)\alpha^{(1+\lambda/(\lambda+1)x)e^{-\lambda/x}} & ;\ x>0,\lambda,\alpha>0,\alpha\neq 1\\ \left(\dfrac{\lambda^2}{\lambda+1}\right)\left(\dfrac{1+x}{x^3}\right)\exp(-\lambda/x) & ;\ x>0,\ \lambda,\alpha>0,\alpha=1\end{cases}$$

and

$$F(x;\lambda,\alpha)=\begin{cases}\dfrac{\alpha^{\left(1+\frac{\lambda}{(\lambda+1)x}\right)e^{-\lambda/x/}}-1}{\alpha-1} & ;\ x>0,\ \alpha\neq 1,\lambda,\alpha>0\\ \left(1+\dfrac{\lambda}{(\lambda+1)x}\right)e^{-\frac{\lambda}{x}} & ;\ x>0,\ \alpha=1,\ \lambda,\ \alpha>0\end{cases}$$

Various properties of the proposed model are obtained including explicit expressions for the mode, moments, conditional moments, mean residual lifetime, Bonferroni and Lorenz curve, entropies, stochastic ordering, stress–strength reliability and order statistics. The parameters of the model were obtained by using the maximum likelihood estimation. The approximate confidence interval of the model parameters was also obtained. A simulation study was carried out to examine the performance of the model parameters.

Alpha power transformed power Inverse Lindley distribution was proposed by Mahmoud Eltehiwy [37]. The density function and distribution function of the distribution is defined by

$$f(x;\beta,\lambda,\alpha)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\left(\dfrac{\beta}{\lambda}\right)\left(1+\dfrac{x}{\lambda}\right)^{-\beta-1}e^{1-\left(1+\frac{x}{\lambda}\right)^{-\beta}} & ;\ x>0,\lambda,\beta,\alpha>0,\ \alpha\neq 1\\ \left(\dfrac{\beta}{\lambda}\right)\left(1+\dfrac{x}{\lambda}\right)^{-\beta-1} & ;\ x>0,\ \lambda,\ \beta,\ \alpha>0,\alpha=1\end{cases}$$

and

$$F(x;\beta,\lambda,\alpha)=\begin{cases}\dfrac{\alpha^{1-\left(1+\frac{x}{\lambda}\right)^{-\beta}}-1}{\alpha-1} & ;\ x>0,\ \lambda,\beta,\ \alpha>0,\alpha\neq 1\\[2ex] 1-\left(1+\dfrac{x}{\lambda}\right)^{-\beta} & ;\ x>0,\lambda,\beta,\ \alpha>0,\alpha=1\end{cases}$$

Numerous properties of the model are studied including moments, incomplete moments, quantiles, entropy, and stochastic ordering. Different methods have been employed for parameter estimation such as maximum likelihood, maximum products of spacings, and ordinary and weighted least squares methods of estimation.

Alpha power transformed Quasi Lindley distribution was introduced by Patrick and Harrison [38]. The density and distribution function of the proposed model is given by

$$f(x;\theta,\beta,\alpha)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\left(\dfrac{\theta(\beta+\theta x)}{\beta+1}\right)\exp(-\theta x)\alpha^{1-(1+\beta+\theta x/\beta+1)\exp(-\theta x)} & ;\ \alpha,\theta>0,\beta>-1,\ x>0,\alpha\neq 1\\[2ex] \left(\dfrac{\theta(\beta+\theta x)}{\beta+1}\right)\exp(-\theta x) & ;\ \alpha,\ \theta>0,\beta>-1,\ x>0,\alpha=1\end{cases}$$

and

$$F(x;\ \theta,\beta,\alpha)=\begin{cases}\dfrac{\alpha^{1-\left(\frac{1+\beta+\theta x}{\beta+1}\right)e^{-\theta x}}-1}{\alpha-1} & ;\ \alpha,\theta>0,\beta>-1,\ x>0,\alpha\neq 1\\[2ex] 1-\left(\dfrac{1+\beta+\theta x}{\beta+1}\right)e^{-\theta x} & ;\ \theta>0,\beta>-1,\ x>0,\alpha=1\end{cases}$$

The statistical properties of the model were studied and discussed by the author(s). These include survival function, hazard function, quantile function, moments, moment generating function, stochastic ordering, and renyi entropy. The model parameters were investigated by using the method of maximum likelihood estimation. A simulation study was conducted to investigate the behaviour of the maximum likelihood estimates. The importance of the model was illustrated by using real-life data sets.

Alpha power transformed Weibull-G family of distribution was proposed by Golam Kibria [39] with application to failure data. The density and distribution function of the distribution is defined by

$$f(x;\theta,\beta,\alpha)=\log\alpha\,\theta\,\beta\exp(-\beta x)\frac{(1-\exp(-\beta x))^{\theta}}{(\exp(-\beta x))^{\theta+1}}\exp(-(\exp(\beta x)-1)^{\theta})\,\alpha^{1-\exp(-\exp(\beta x-1))^{\theta}};\alpha>0,\alpha\neq 1$$

and

$$F(x;\theta,\beta,\alpha)=\frac{\alpha^{1-e(^{-e^{-\beta x}}-1)^{\theta}}-1}{\alpha-1}$$

The distribution is generated by combining the two families of distributions APT-G family and Weibull-G family. The statistical properties of the APTW-G are derived and discussed. The sub-models of APTW-G are Alpha power transformation Weibull exponential distribution, Alpha power transformation Weibull Rayleigh distribution and Alpha power transformation Weibull Lindley distribution.

Alpha power transformed Pareto distribution was introduced by Shiktivel et al. [40]. The probability density function (pdf) and cumulative distribution function (cdf) is given by

$$f(x;\kappa,\beta,\alpha)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}\,\dfrac{\beta}{x^{\beta+1}}\,\alpha^{(1-k/x)^{\beta}} & ;\ \beta,\alpha\ \kappa>0\,,x\geq k,\ \alpha\neq 1\\[2ex] \dfrac{\beta}{x^{\beta+1}}\,\alpha^{(1-k/x)^{\beta}} & ;\ \beta,\alpha\ \kappa>0\,,x\geq k\,,\alpha=1\end{cases}$$

and

$$F(x;\kappa,\beta,\alpha)=\begin{cases}\dfrac{\alpha^{(1-k/x)^{\beta}}-1}{\alpha-1} & ;\ \beta,\alpha\ \kappa>0\,,x\geq k,\ \alpha\neq 1\\[2ex] \left(1-\dfrac{k}{x}\right)^{\beta} & ;\ \beta,\alpha\ \kappa>0\,,x\geq k,\ \alpha=1\end{cases}$$

Properties of the model are derived and studied. The density function and its behaviour, survival function, hazard function, moments, moment generating function, mode, quantiles, entropies, mean residual life function, stochastic orders and order statistics are discussed. The maximum likelihood technique is implemented for determining the parameters of the model. The usefulness of the model is analyzed by using two real life data sets and the comparison is made based on the model selection techniques.

Alpha power transformed extended exponential distribution was proposed by Hassan [41] and studied the properties of the model with the application. Let *X* be a random variable following alpha power transformed extended exponential then its probability density function (pdf) and cumulative distribution function (cdf) is given by

$$f(x;\beta,\gamma,\alpha)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\dfrac{\gamma^{2}(1+\beta x)\exp(-\gamma x)}{\beta+\gamma}\,\alpha^{\frac{\gamma+\beta-(\gamma+\beta+\gamma\beta x)\exp(-\gamma x)}{\beta+\gamma}} & ;\ x>0,\ \alpha,\ \beta,\gamma>0,\alpha\neq 1\\[2ex] \dfrac{\gamma^{2}(1+\beta x)\exp(-\gamma x)}{\beta+\gamma} & ;\ x>0,\ \alpha,\ \beta,\gamma>0,\alpha=1\end{cases}$$

and

$$F(x;\beta,\gamma,\alpha)=\begin{cases}\dfrac{\alpha^{\frac{\gamma+\beta-(\gamma+\beta+\gamma\beta x)\exp(-\gamma x)}{\beta+\gamma}}-1}{\alpha-1} & ;\ x>0,\ \alpha,\ \beta,\gamma>0,\alpha\neq 1\\[2ex] \dfrac{\gamma+\beta-(\gamma+\beta+\gamma\beta x)e^{(-\gamma x)}}{\beta+\gamma} & ;x>0,\ \beta,\ \gamma>0,\alpha=1\end{cases}$$

Finally, an application of the model to a real data set is presented and compared with some other well-known distributions. The alpha power extended exponential distribution reduces to alpha power exponential, alpha power Lindley, exponential, Lindley and gamma distribution. The probability density function of APTEE distribution is reversed J-shaped, uni-model and right-skewed. Also, the hazard rate of APTEE distribution is increasing and decreasing.

Alpha power transformed Inverse Lomax distribution was suggested by Ramadan [42]. The probability density function (pdf) and cumulative distribution function (cdf) of the Alpha power inverse Lomax distribution having two shape parameters α, a and a scale parameter b is given as

$$f(x;a,b,\alpha)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\dfrac{ab}{x^2}\left(1+\dfrac{b}{x}\right)^{-a-1}\alpha^{1-(1+b/x^2)^{-a}} & ;x>0,\ a,b>0,\alpha\neq 1,\alpha>0\\[2ex] \dfrac{ab}{x^2}\left(1+\dfrac{b}{x}\right)^{-a-1} & ;x>0,a,b>0,\alpha=1\end{cases}$$

and

$$F(x;a,b,\alpha)=\begin{cases}\dfrac{\alpha^{1-\left(1+\frac{b}{x^2}\right)^{-a}}-1}{\alpha-1} & ;\ x>0,\alpha,a,b>0,\alpha\neq 1\\[2ex] 1-\left(1+\dfrac{b}{x^2}\right)^{-a} & ;\ x>0,\alpha,a,b>0,\alpha=1\end{cases}$$

Some statistical properties include moments, moment generating function, quantile function, mode, stress strength reliability, and order statistics. The model parameters are estimated by different methods. These methods include maximum likelihood estimator, least-squares, weighted least-squares, percentile, Cramer–von Mises, maximum product of spacing, Anderson–Darling, and right-tail Anderson–Darling. A simulation study was carried out to assess the performance of ML estimators. Analysis of a real data set is considered for illustrative purposes.

Alpha power transformed Frechet distribution was developed and discussed by Nasiru et al. [43]. The APTF distribution is obtained by using the cdf and pdf of Frechet distribution. The probability density function (pdf) and distribution function is defined as follows

$$f(x;a,b,\alpha)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\alpha^{\exp\left(-\frac{a}{x}\right)^b} b a^b x^{-(b+1)}\exp\left(-\dfrac{a}{x}\right)^b & ;\quad a,b,x>0,\alpha>0,\alpha\neq 1\\[2ex] b a^b x^{-(b+1)}\exp\left(-\dfrac{a}{x}\right)^b & ;\ \alpha=1,a,b,\alpha>0,x>0\end{cases}$$

and

$$F(x;a,b,\alpha)=\begin{cases}\dfrac{\alpha^{\exp\left(-\frac{a}{x}\right)^b}-1}{\alpha-1} & ;\ x>0,\ a,\ b,\alpha>0,\alpha\neq 1\\[2ex] \exp\left(-\dfrac{a}{x}\right)^b & ;\ x>0,\ a,\ b,\ \alpha>0,\alpha=1\end{cases}$$

Some statistical properties of the model are obtained including quantile function, moments, moment generating function, incomplete moments mean residual life, mean inactivity time, entropy, stochastic ordering and order statistics.

The model parameters are investigated by the method of maximum likelihood estimation. The Monte Carlo simulation was to examine the finite properties of the maximum likelihood estimators for the parameters of the model. The applicability of the model is checked with a fit of two data sets to the model and make a comparison among the competing models.

Alpha power Gompertz distribution was proposed by Joseph Thomas Eghwerido et al. [44]. The probability density function (pdf) and cumulative distribution function (cdf) of the distribution is defined by

$$f(x;\theta,\eta,\alpha)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\alpha^{1-\exp\left(-\frac{\theta}{\eta}(\exp(\eta x)-1)\right)}\theta\exp\left(\eta x-\dfrac{\theta}{\eta}(\exp(\eta x)-1)\right) & ;\ x,\eta,\theta,\alpha>0,\ \alpha\neq 1\\[2ex] \theta\exp\left(\eta x-\dfrac{\theta}{\eta}(\exp(\eta x)-1)\right) & ;\ x,\eta,\theta,\alpha>0,\ \alpha=1\end{cases}$$

and

$$F(x;\theta,\eta,\alpha)=\begin{cases}\dfrac{\alpha^{1-\exp\left(-\frac{\theta}{\eta}(\exp(\eta x)-1)\right)}-1}{\alpha-1} & ;\ x,\eta,\theta,\alpha>0,\ \alpha\neq 1\\[2ex] 1-\exp\left(-\dfrac{\theta}{\eta}(\exp(\eta x)-1)\right) & ;\ x,\eta,\theta,\alpha>0,\alpha=1\end{cases}$$

Where, η , θ, α is the scale parameter and is the shape parameter.

Where, α is the shape parameter, controls the skewness and kurtosis of the distribution. The properties of the model are derived and discussed. These properties are survival function, hazard function, probability weighted moments, moments, entropy, quantile function, reversed hazard, moment generating function etc. The shape of the density function is left-skewed, decreasing, uni-modal. The shape of the hazard function is a bathtub curve. The ML estimates of the model are examined through simulation. The applicability of the model was validated by using two data sets.

Alpha power Lomax distribution suggested by Bhulat, Dogru and Arslan [45] obtained by using cdf and pdf of the Lomax Distribution. The density function and distribution function of the distribution is defined by

$$f(x;\alpha,\beta,\lambda)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\dfrac{\beta}{\lambda}\left(1+\dfrac{x}{\lambda}\right)^{-(\beta+1)}\alpha^{1-\left(1+\frac{x}{\lambda}\right)^{-\beta}} & ;\ x,\lambda,\beta,\alpha>0,\alpha\neq 1\\[2ex] \dfrac{\beta}{\lambda}\left(1+\dfrac{x}{\lambda}\right)^{-(\beta+1)} & ;\ x,\lambda,\beta,\alpha>0,\alpha=1\end{cases}$$

And

$$F(x;\alpha,\beta,\lambda)=\begin{cases}\dfrac{\alpha^{1-\left(1+\frac{x}{\lambda}\right)^{-\beta}}-1}{\alpha-1} & ;\ x,\lambda,\ \beta,\ \alpha>0,\alpha\neq 1\\ 1-\left(1+\dfrac{x}{\lambda}\right)^{-\beta} & ;\ x,\ \lambda,\ \beta,\ \alpha>0,\alpha=1\end{cases}$$

Properties of the model such as quantile function, survival function, hazard function, reversed hazard rate, moments, moment generating function, order statistics. The maximum likelihood estimation of the unknown parameters is discussed. A simulation study was performed to investigate the performance of the ML estimates. The flexibility of the new distribution is illustrated using data sets.

Alpha Power Log-Logistic distribution was presented by Aldahlan [46] for modeling the data set of carbon fibres. The probability density function (pdf) and cumulative distribution function (cdf) of the distribution is given by

$$f(x;a,b,\alpha)=\begin{cases}\dfrac{b\log\alpha}{a^{b}(\alpha-1)}x^{b-1}\left(1+\left(\dfrac{x}{a}\right)^{b}\right)^{-2}\alpha^{1-\left(1+\left(\frac{x}{a}\right)^{b}\right)^{-1}} & ;\ \alpha>0,a,b>0,x>0,\alpha\neq 1\\ x^{b-1}\dfrac{b}{a^{b}}\left(1+\left(\dfrac{x}{a}\right)^{b}\right)^{-2} & ;\ a,b>0,x,\ \alpha>0,\alpha=1\end{cases}$$

and

$$F(x;a,b,\alpha)=\begin{cases}\dfrac{\alpha^{1-\left(1+\left(\frac{x}{a}\right)^{b}\right)^{-1}}-1}{\alpha-1} & ;\quad x>0,\ \alpha\neq 1,\alpha>0,a,b>0,\\ 1-\left(1+\left(\dfrac{x}{a}\right)^{b}\right)^{-1} & ;\quad x>0,\alpha=1,\alpha>0,a,b>0,\end{cases}$$

These properties include quantile function, moments, probability weighted moments, and renyi entropy. The parameters of the model are obtained by maximum likelihood estimation method. The simulation study was also carried out to know the behaviour of the parameters of the model. The goodness of fit criteria were evaluated and make a comparison among other models. It was revealed that the model performs better for the considered data as compared to BXII, Zografos-Balakrishnan BXII, Marshal-Olkin BXII, five parameter Kumaraswamy BXII, Topp Leone BXII, BXIII, five parameter beta BXII, beta exponentiated BXII distributions.

Alpha power Weibull distribution was developed by Nassar et al. [47]. The probability density function (pdf) and cumulative distribution function (cdf) of the distribution is defined by

$$f(x;\alpha,\beta,\lambda)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}\alpha^{1-e^{-\lambda x^{\beta}}} & ;\quad x>0,\alpha>0,\lambda,\beta>0,\alpha\neq 1\\[2ex] \lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}} & ;\ x>0,\ \alpha,\lambda,\beta>0,\alpha=1\end{cases}$$

and

$$F(x;\alpha,\beta,\lambda)=\begin{cases}\dfrac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1} & ;\quad x>0,\alpha>0,\lambda,\beta>0,\alpha\neq 1\\[2ex] 1-e^{-\lambda x^{\beta}} & ;\quad x>0,\alpha,\lambda,\beta>0,\alpha=1\end{cases}$$

The properties of the model are derived and discussed such as reliability function, hazard rate function, mean residual function, quantiles, moments, moment generating function, entropy, order statistics, stress-strength parameter. The simulation study was performed by using Mathcad to illustrate the behaviour of model parameters. The maximum likelihood estimation is established for calculating the parameters of the model. Two real data sets are used to illustrate the importance of the proposed model.

Alpha Power Weibull Frechet distribution was obtained by Thomas et al. [48]. They derived mathematical properties of the model such as survival function, hazard function, reverse hazard rate function, cumulative hazard function, quantile function, probability weighted moments, random number generation, odds function and order statistics. The maximum likelihood method is established for estimating the parameters of the model. A simulation study is conducted to know the behaviour of parameters. It is revealed that the (mean square error) MSE and Bias decrease on increasing the sample sizes. The variance also decreases with the increase in sample sizes. The performance of the model is examined with other competing distributions by using the gas fibre and carbon data. The Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), Hannan-Quinn Information Criteria (HQIC), The Anderson Darling statistic, Cramer-von Mises statistic, Kolmogorov Smirnov (KS) statistic, Log-likelihood and the p-value were considered for the comparison between the alpha power Weibull Frechet distribution and Kumaraswamy Lomax, Gompertz, Beta Lomax and the Alpha power Inverted Exponential distributions

Alpha power Exponentiated Inverse Rayleigh distribution was developed by Noor et al. [49]. The probability density function (pdf) and distribution function of the distribution is given by

$$f(x;\alpha,\beta,\theta)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\dfrac{2\beta\theta}{x^{2}}e^{-\frac{\beta\theta}{x^{2}}}\alpha^{e^{\frac{\beta\theta}{x^{2}}}} & ;\ x>0,\ \alpha\neq 1,\alpha>0,\theta,\beta>0\\[2ex] \dfrac{2\beta\theta}{x^{2}}e^{-\frac{\beta\theta}{x^{2}}} & ;\ x>0,\ \alpha=1,\alpha>0,\theta,\beta>0\end{cases}$$

and

$$F(x;\alpha,\beta,\theta)=\begin{cases}\dfrac{\alpha^{e^{-\frac{\beta\theta}{x^2}}}-1}{\alpha-1} & ; \quad x>0,\ \alpha\neq 1,\alpha>0,\ \theta,\ \beta>0\\[2ex] e^{-\frac{\beta\theta}{x^2}} & ; \quad x>0,\ \alpha=1,\alpha>0,\ \theta,\ \beta>0\end{cases}$$

The properties of the model were studied which includes survival function, hazard function, quantile function, stress-strength parameter, moments, order statistics, residual life time, mean waiting time, entropy, moment generating function. The parameters of the model were investigated by using maximum likelihood estimation. The simulation study was performed for the performance of the parameters in the model.

Alpha power inverse Weibull distribution was suggested by Basheer [50]. The density function and distribution function of the model is defined by

$$f(x;\alpha,\beta,\lambda)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\lambda\beta x^{-\beta-1}e^{-\lambda x^{\beta}}\alpha^{1-e^{-\lambda x^{\beta}}} & ; \quad x\geq 0,\ \alpha>0,\lambda,\beta>0,\alpha\neq 1\\[2ex] \lambda\beta x^{-\beta-1}e^{-\lambda x^{\beta}} & ; \quad x\geq 0,\ \lambda,\alpha,\ \beta>0,\alpha=1\end{cases}$$

and

$$F(x;\alpha,\beta,\lambda)=\begin{cases}\dfrac{\alpha^{1-e^{-\lambda x^{\beta}}}-1}{\alpha-1} & ; \quad \alpha>0,\alpha,\ \lambda>0,\ \beta>0,\ \alpha\neq 1,x\geq 0\\[2ex] 1-e^{-\lambda x^{\beta}} & ; \quad \alpha>0,\ \lambda>0,\ \beta>0,\ \alpha=1,\ x\geq 0\end{cases}$$

The reliability properties of the proposed model were investigated. These include reliability analysis, hazard rate function, reversed hazard rate function, mean residual life, and mean inactivity time, stress-strength reliability. The parameters of the model were estimated by using maximum likelihood estimation for reliability analysis. The statistical properties of the model were also obtained such as quantile function, moments, moment generating function, renyi entropy, Shannon entropy, stochastic ordering and order statistics. The simulation study was also conducted for the model parameters. The real data was used to illustrate Alpha power inverse distribution for comparing with many known distributions such as exponentiated (generalized) inverse Weibull (GIW), Kumaraswamy inverse Weibull (KIW) and inverse Weibull (IW) distributions.

Alpha power inverted Topp Leone distribution was obtained by Amal et al [51]. The density function and distribution function of the distribution is given by

$$f(x;\alpha,\lambda)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)2\lambda x(1+x)^{-2\lambda-1}(1+2x)^{\lambda-1}\alpha^{1-\left(\frac{(1+2x)^{\lambda}}{(1+x)^{2\lambda}}\right)} & ; \quad x>0,\ \alpha>0,\ \lambda>0,\ \alpha\neq 1\\[2ex] 2\lambda x(1+x)^{-2\lambda-1}(1+2x)^{\lambda-1} & ; \quad x>0,\ \lambda>0,\alpha>0,\alpha=1\end{cases}$$

and

$$F(x;\alpha,\lambda)=\begin{cases}\dfrac{\alpha^{1-\left(\frac{(1+2x)^{\lambda}}{(1+x)^{2\lambda}}\right)}-1}{\alpha-1} & ;\ x,\lambda,\alpha>0,\ \alpha\neq 1\\[2ex] 1-\left(\dfrac{(1+2x)^{\lambda}}{(1+x)^{2\lambda}}\right) & ;\ x,\lambda>0,\ \alpha>0,\ \alpha=1\end{cases}$$

Some structural properties of the model are obtained such as reliability function, residual and reversed residual life, moments, incomplete moments, and quantile function and renyi entropy measure. The parameters of the model were estimated by using Bayesian and nom-Bayesian methods. The maximum likelihood, weighted least square, maximum product of spacing estimators of the parameters have been obtained. A simulation study was also performed to illustrate the restricted sample properties of the proposed model. It has been observed that the Bayesian estimates provide more accurate results for the model parameters. The performance of the model were demonstrated by using the data sets related to reliability, engineering and medicine.

Alpha power Kumaraswamy distribution was obtained by Ali [52]. The probability density function (pdf) and distribution function of the model is as follows

$$f(x;\theta,\beta,\alpha)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\beta\theta\ 1-(1-x^{\beta})^{\theta}\,x^{\beta-1}\alpha^{1-(1-x^{\beta})^{\theta}} & ;\ 0<x<1,\ \alpha,\ \beta,\ \theta>0,\ \alpha\neq 1\\[2ex] \beta\theta\ 1-(1-x^{\beta})^{\theta}\,x^{\beta-1} & ;\ 0<x<1,\ \beta,\ \theta>0,\ \alpha=1\end{cases}$$

and

$$F(x;\theta,\beta,\alpha)=\begin{cases}\dfrac{\alpha^{1-(1-x^{\beta})^{\theta}}-1}{\alpha-1} & ;\ 0<x<1,\ \alpha,\beta,\theta>0,\alpha\neq 1\\[2ex] 1-(1-x^{\beta})^{\theta} & ;0<x<1,\ \beta,\theta>0,\alpha=1\end{cases}$$

The properties of the Alpha power Kumaraswamy were obtained which include mean deviation, mode, moments, moment generating function, survival function, hazard function, stress-strength reliability, order statistics and renyi entropy. The model parameters were obtained by using the method of maximum likelihood estimation. The simulation study was also performed on different sets of parametric values.

Alpha power Inverted Exponential distribution was introduced by Ceren, Ozel et al. [53] for modeling the data of patients suffering from head and neck cancer. The density and distribution function of the model is defined by

$$f(x;\alpha,\lambda)=\begin{cases}\left(\dfrac{\log\alpha}{\alpha-1}\right)\dfrac{\lambda}{x^{2}}\exp\left(-\dfrac{\lambda}{x}\right)\alpha^{e^{\left(-\frac{\lambda}{x}\right)}} & ;\ x>0,\alpha,\lambda>0,\alpha\neq 1\\[2ex] \dfrac{\lambda}{x^{2}}\exp\left(-\dfrac{\lambda}{x}\right) & ;\ x>0,\ \alpha,\ \lambda>0,\alpha=1\end{cases}$$

and

$$F(x;\alpha,\lambda)=\begin{cases}\dfrac{\alpha^{e^{-(\lambda/x)}}-1}{\alpha-1} & ;\ x>0,\ \lambda>0,\alpha>0,\alpha\neq 1\\ \\ e^{-\left(\frac{\lambda}{x}\right)} & ;\ x>0,\alpha>0,\lambda>0,\ \alpha=1\end{cases}$$

The properties of the model were investigated including survival function, hazard rate function, quantile function, skewness and kurtosis, order statistics. The validation of the model were investigated by fitting a data set by using Akaike Information Criteria (AIC), Corrected Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), Hannan-Quinn (HQIC), and log-likelihood (LL). It was observed that the alpha power inverted exponential model provides a better fit for the considered data set as compared to weighted Lindley introduced by Shanker, Lindley introduced by Shanker, inverted exponential, generalized inverted exponential introduced by Sharma, inverse Rayleigh introduced by Sharma and Singh, three parameter weighted Lindley, inverse Lindley distribution introduced by Sharma.

Maryam and Kannan [54] introduced the Alpha power transformed Aradhana distribution. They studied its moments, moment generating function, characteristic function and order statistics properties. They also established a maximum likelihood technique for the estimation of parameters. Maryam and Kannan [55] obtained the Alpha power transformed Garima distribution. They derived moments, moment generating, characteristic function, stress-strength reliability, mean waiting time, mean residual life, order statistics, entropies, Bonferroni, and Lorenz curves. They have used a method of least square estimation and a maximum likelihood technique for estimation of parameters. Maryam and Kannan [56] developed the Alpha power transformed Rama distribution. Some properties are discussed including moments, moment generating, characteristic function, order statistics, entropies, Bonferroni, and Lorenz curves. The method of maximum likelihood technique is used for the estimation of parameters. All the author(s) has derived and discussed numerous characteristics of the distributions and their applicability in different areas are demonstrated by using real-life data from different disciplines.

4 Conclusion

In this paper, we have reviewed different methods for generating distributions based on different families of distributions. The model and expressions for probability density function, cumulative distribution function for each of distribution is provided. We have studied the behavior of probability density function and a hazard rate function of different distributions.

Acknowledgement

The author(s) are very grateful to reviewers for their useful comments for the improvement of this paper. The reviews were helpful to finalize the manuscript.

References

[1] K. Pearson, Contributions to the mathematical theory of evolution, skew variation in homogeneous material, Philosophical Transactions of the Royal Society, **186**, 343–414 (1895).
[2] N. L. Johnson, Systems of frequency curves generated by methods of translation, Biometrika, **36**, 274-282 (1949).
[3] J.W. Tukey, Technical Report 36, In the practical relationship between the common transformations of percentages of counts and amounts, Princeton University, Princeton, N. J, Statistical Techniques Research Group, *(*1960).
[4] L. Amoroso, Ricerche intorno. alla curva dei redditi, Annali di Matematica. Pura ed Application, Series B, **2(1)**, 123–159 (1925).
[5] N. L. Johnson, S. Kotz, and N. Balakrishnan, Continuous univariate distributions, I. Wiley, New York., (1994).
[6] E.W. Stacy, A generalization of the gamma distribution, Annals of Mathematical Statistics, **33 (3),** 1187-1192 (1962).
[7] N. Balakrishnan and Y. Peng, Generalized gamma frailty model, Statistics in Medicine, **25(16),** 2797-2816 (2006).

[8] G. M., Cordeiro, E. M. M. Ortega, and G. O Silva, The exponentiated generalized gamma distribution with application to lifetime data, Journal of Statistical Computation and Simulation, **81(7),** 827–842 (2012).

[9] S. K. Agarwal, and J. A. Al-Saleh, Generalized gamma type distribution and its hazard rate function, Communications in Statistics - Theory and Methods, **30(2),** 309–318 (2001).

[10] M. C. Donald, and J. B. Xu, A generalization of the beta distribution with application, Journal of Econometrics, **66**, 133–152 (1984).

[11] A. Azzalini, A class of distributions which includes the normal ones, Scandinavian Journal of Statistics, **12**, 171–178 (1985).

[12] G. S. Mudholkar, and D. K. Srivastava, Exponentiated Weibull family for analyzing bathtub failure-rate data, IEEE Transactions on Reliability, **42(2)**, 99–302 (1993).

[13] A. W. Marshall, and I. Olkin, A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families, Biometrika, **84**, 641–652 (1997).

[14] N. Eugene, C. Lee, and F. Famoye, Beta-normal distribution and its applications, Communications in Statistics—Theory and Methods, **31(4),** 497–512 (2002).

[15] J.T.A.S. Ferreira, and M.F.J. Steel, A constructive representation of univariate skewed distributions, Journal of American Statistical Association, **101**, 823–829 (2006).

[16] M. C. Jones, Families of distributions arising from distributions of order statistics, Test, Springer, **13**, 1–43 (2004).

[17] S. Nadarajah, and S. Kotz, The beta Gumbel distribution, Mathematical Problems in Engineering, **10**, 323–332 (2004).

[18] A. Akinsete, F. Famoye, and C. Lee, The beta-pareto distribution, Statistics, **42**, 547–563 (2008).

[19] K. Zografos, and N. Balakrishnan, On families of beta- and generalized gamma-generated distributions and associated inference, Statistical Methodology, **6**, 344–362 (2009).

[20] W. Barreto-Souza, A.H.S. Santos, and G.M. Cordeiro, The beta generalized exponential distribution, Journal of Statistical Computation and Simulation, **80**, 159–172 (2010).

[21] Gupta, R. D. and D. Kundu, Generalized exponential distributions, Australian and New Zealand Journal of Statistics, **41**, 173 – 188 (1999).

[22] M.C. Jones, and P.V. Larsen, Multivariate distributions with support above the diagonal, Biometrika, **91**, 975–986 (2004).

[23] B.C. Arnold, E. Castillo, and J.M. Sarabia, Families of multivariate distributions involving the Rosenblatt construction, Journal of the American Statistical Association, **101**, 1652–1662 (2006).

[24] M. Razzaghi, Beta-normal distribution in dose-response modeling and risk assessment for quantitative responses, Environmental and Ecological Science, **16**, 21–36 (2009).

[25] M. C. Jones, Kumaraswamy's distribution: a beta-type distribution with some tractability advantages, Statistical Methodology, **6**, 70–81 (2009).

[26] P. Kumaraswamy, A generalized probability density functions for double-bounded random processes, Journal of Hydrology, **46**, 79–88 (1980).

[27] C. Alexander, G. M. Cordeiro, E. M. M. Ortega, and J. M. Sarabia, Generalized beta-generated distributions, Computational statistics and Data Analysis, **56(6),** 1880–1896 (2012).

[28] A. Alzaatreh, C. Lee, and F. Famoye, A new method for generating families of continuous distributions, Metron, **71 (1),** 63-79 (2013a).

[29] A. Alzaatreh, C. Lee, and F. Famoye, Gamma-Pareto distribution and its applications, Journal of Modern Applied Statistical Methods, **11(1),** 78–94 (2012a).

[30] S. Nadarajah, and S. Kotz, The beta exponential distribution, Reliability Engineering and System Safety, **91**, 689–697 (2006).

[31] W.T., Shaw, I. R.C. Buckley, The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map, (2009).

[32] A. Mahdavi, and D. Kundu, A new method for generating distributions with an application to exponential distribution, Communications in Statistics—Theory and Methods, **46(13),** 6543-6557 (2017).

[33] Mohamed E. Mead, Gauss M. Cordeiro, Ahmed Z. Afify and Hazem Al-Mofleh, The Alpha power transformation family: properties and applications, Pakistan Journal of Statistics and Operation research, **15(3),** 525-545 (2019).

[34] A. S Hassan, M. Elgarhy, Alpha power transformed power Lindley distribution, Journal of Probability and Statistics, (2019).

[35] S. Dey, I. Ghosh, and D. Kumar, Alpha-power transformed Lindley distribution: properties and associated inference with application to earthquake data, Annals of Data Science, 1–28 (2018).

[36] S. Dey, M. Nassar, and D. Kumar, Alpha power transformed inverse Lindley distribution: A distribution with an upside-down bathtub-shaped hazard function, Journal of Computational and Applied Mathematics., **348**, 130–145 (2018).

[37] M. Eltehiwy, On the Alpha power transformed power Inverse Lindley Distribution, Journal of the Indian Probability and Statistics, **21**, 201-224 (2020).

[38] U. Patrick, and E. Harrison, Alpha power transformed Quasi Lindley distribution, International Journal of Advanced Statistics and Probability, **9(1),** 6-7 (2021).

[39] I. Elbatal., M. Elgarhy, and B. M. G. Kibria, Alpha power transformed Weibull-G family of distributions, Journal of Statistical Theory and Applications, 130-145 (2021).

[40] M. H. Sakthivel, and A. Arul chezhian, Alpha power transformed Pareto distribution with application, Journal of Mathematik, 1-13 (2020).

[41] A. S. Hassan, R. E Mohamed, M. Elgarhy, and F. Fayomi, Alpha power transformed extended exponential distribution, properties and applications, Journal of Nonlinear Sciences and Applications, **12(4),** 62–67 (2018).

[42] A. Ramadan, Zein Eldin, Muhammad Ahsan ul Haq, Sharqa Hashmi, and Mahmoud Elsehety, Alpha power transformed inverse Lomax distribution with different methods of estimation and applications, Journal of Probability and Statistics, (2020).

[43] S. Nasiru, and N. Peter, Alpha power transformed Frechet Distribution, International Journal Applied Mathematics & Information Sciences, **13(1),** 129-141 (2019).

[44] J. T. Eghwerido, L. C. Nzei, and F. I. Agu, The Alpha power Gompertz distribution: Characterization Properties and Applications, Sankhya, **83**, 449–475 (2021).

[45] Murat Y. Bulut, Fatma Z. Dogru, Olcay Arslan, Alpha power Lomax distribution, properties and applications, Journal of Reliability and Statistical Studies, **14(1)** (2021).

[46] M. Aldahlan, Alpha power transformed Log-logistic distribution with application to breaking stress data, Advances in Mathematical Physics**, 22**, 1-25 (2020).

[47] T. E. Joseph, T. U. Ovokaefe, E. E., Eferhonore, The Alpha power Weibull Frechet distribution, properties and applications, Turkish Journal of Science, **5(3),**170-185 (2020).

[48] M. Nasaar, A. Alzaatreh, M. Mead, and O. Abo-Kasem, Alpha power Weibull distribution: properties and applications, Communication in Statistics- Theory and Methods, **46,** 10236-10252 (2017).

[49] S. Noor, M. Ali, and A. Khalil, Alpha power exponentiated inverse Rayleigh distribution and its Applications to Real and Stimulated data, PLOS ONE, **14** (2021).

[50] Abdul Kareem and M. Basheer, Alpha power inverse Weibull distribution with reliability application, Journal of Taibah University for Science, **13(1),** 423-432 (2019).

[51] M.G. Ibrahim, A. S. Hassan, E. M. Almetwally, and M. H. Algomy, Parameter estimation of Alpha power inverted Topp-Leone distribution with Applications, Intelligent Automation and Soft Computing**, 29(2)**, (2021).

[52] A. A. Mohamed, On the Alpha power Kumaraswamy distribution: properties, simulation and application, Revista Colombiana de Estadística - Applied Statistics, **43(2)**, 285-313 (2020).

[53] U. Ceren, G. O. Kadiler, and Selen, Alpha power inverted Exponential distribution, its properties and applications, Gazi University Journal of Science, **3(13),** 954-965 (2018),

[54] M. Mohiuddin and R. Kannan, Alpha power transformed Aradhana distribution, its properties and applications, Indian Journal of Science and technology, **14(30),** 2483-2493 (2021).

[55] M. Mohiuddin and R. Kannan, A new Generalization of Garima distribution with applications to real life data, Applied Mathematics and Information Sciences, **15(5),** 577-592 (2021).

[56] M. Mohiuddin and R. Kannan, A new Generalization of Rama distribution with applications to machinery data, International Journal of Emerging Technologies in Engineering Research, **9(9),** 1-13 (2021).