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Alpha Power Modified Weibull Distribution: Actuarial Measures and Applications to Failure Data

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Abstract: In this study, a new modified Weibull distribution, called the alpha power modified Weibull distribution, is proposed and studied. The new distribution is a generalization of several well known distributions. The shapes of the density and hazard rate functions are obtained. The density function shows several shapes including skewed, approximately symmetric and decreasing shapes. The hazard rate function also shows shapes including decreasing, increasing, bathtub and modified bathtub shapes. Several properties of the distribution including moments, moment generating function, inequality measures, order statistics and stochastic ordering are derived. Also, several actuarial measures are derived. The numerical studies of the actuarial measures of the developed distribution are compared with other distributions. Various estimation methods are used to estimate the parameters of the distribution and a simulation study is conducted to ascertain the performance of the estimators. A bivariate extension of the distribution is also derived in the study. The distribution is used to model two real failure data sets to ascertain its usefulness. The results show that the new distribution can serve as an alternative to modeling failure data sets.

Keywords: Tail variance, value-at-risk, Weibull distribution, failure rate, bivariate distribution

1 Introduction

The need to properly model data is very crucial for successful practice in engineering, insurance and medicine among other fields. This usually involves finding a suitable distribution which can best explain the variations in the data or extract enough information from the data. In literature, there are several classical distributions. These include the Weibull, exponential, Pareto and gamma distributions, among others. These distributions are widely used to provide parametric fit to data sets. However, some data sets exhibit certain characteristics, such as non-monotonic failure rates, which render classical distributions inappropriate in providing a good parametric fit to the data sets. Also, due to the ever increasing complexity of data sets from different fields, no single distribution can provide a good fit to all of them. Due to this, researchers constantly develop new distributions to provide some flexibility in modeling data sets.

Due to the need for flexible distributions to model various data sets, several methods have been developed for the construction of new distributions. Some of these methods include transformed-transformer (T-X) method [1], exponentiated generalized class [2], quadratic transmutation map [3], cubic transmutation map [4,5] and alpha power transformation method [6]. These methods have been used extensively in literature to develop new distributions.

The Weibull distribution [7], though widely used, does not provide good fit to data sets which exhibit non-monotonic failure rates; such as bathtub, upside down bathtub, among other shapes. Thus, several extensions and modifications of the Weibull distribution have been developed by researches to make it more flexible and enhance its usefulness. Some of these include the exponentiated Weibull distribution [8], modified Weibull distribution [9], beta Weibull distribution [10], beta modified Weibull distribution [11], Kumaraswamy Weibull (KW) distribution [12], beta-exponentiated Weibull distribution [13], alpha power Weibull (APW) distribution [14], exponentiated power generalized Weibull (ExPGW) distribution [15], alpha power exponentiated Weibull (APExW) distribution [16], and modified beta modified Weibull distribution [17]. Others include exponent power Weibull (ExPW) distribution [18], heavy-tailed beta-power transformed Weibull

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distribution [19], modified beta inverse flexible Weibull extension distribution [20], and Marshall-Olkin alpha power inverse Weibull distribution [21], among others.

Though several distributions have been developed by researchers, including several extensions of the Weibull distribution, there is always the need for the development of new distributions. The Weibull distribution has gained popularity and usefulness among researchers and practitioners. Thus, an its extensions are expected to provide more flexible and useful distributions. Hence, in this article, a new extension of the modified Weibull distribution, proposed by Lai et al. [9], is constructed using the alpha power transformation. Several properties of the new distribution, actuarial measures and a bivariate extension of the distribution are studied.

The new distribution is a generalization of several known distributions including Weibull, modified Weibull, exponential, Rayleigh, among others. Thus, the new distribution is expected to be more flexible and useful with applications in different fields.

The rest of the article is organized as follows: section 2 presents the new distribution. Several statistical properties of the distribution, including the quantile function, moments, moment generating function, inequality measures, order statistics and stochastic ordering are studied in section 3. In section 4, some actuarial measures, including value-at-risk (VaR), tail VaR, tail variance, tail variance premium, mean excess function and limited expected value function, are derived and numerical results compared with other distributions. In section 5, the parameters of the distribution are derived using several estimation methods and simulation studies are performed in section 6. Applications to real data sets are given in section 7 to show the flexibility of the new distribution. A bivariate extension of the distribution is given in section 8 and the conclusion of the study is given in section 9. Some proposals for future work are given in section 10.

2 Alpha Power Modified Weibull Distribution (APMW)

In this section, the proposed modification of the modified Weibull distribution using the alpha power transformation is presented. Given a baseline distribution $F(x)$, the cumulative distribution function (CDF) and probability density function (PDF) of the alpha power family of distributions are given respectively as

$$G(x) = \begin{cases} \frac{\alpha^{F(x)}-1}{\alpha-1}, & \alpha > 0, \alpha \neq 0, x \in \mathbb{R} \\ F(x), & \alpha = 1, x \in \mathbb{R} \end{cases},$$

and

$$g(x) = \begin{cases} f(x)\alpha^{F(x)}\frac{\log\alpha}{\alpha-1}, & \alpha > 0, \alpha \neq 0, x \in \mathbb{R} \\ f(x), & \alpha = 1, x \in \mathbb{R} \end{cases}.$$

Using the modified Weibull distribution as the baseline distribution with CDF and PDF given respectively as $F(x) = 1 - e^{-ax^b e^{cx}}$, $x \geq 0$, $a > 0, b \geq 0, c > 0$ and $f(x) = a(b+cx)x^{b-1}e^{cx}e^{-ax^b e^{cx}}$, the new modified Weibull distribution, called the alpha power modified Weibull (APMW) distribution is obtained. The CDF and PDF of the new distribution are given respectively as

$$G(x) = \begin{cases} \frac{1-\alpha^{1-e^{-ax^b e^{cx}}}}{1-\alpha}, & \alpha > 0, \alpha \neq 0, x \geq 0 \\ 1-e^{-ax^b e^{cx}}, & \alpha = 1, x \geq 0 \end{cases}, \tag{1}$$

and

$$g(x) = \begin{cases} a(b+cx)x^{b-1}e^{cx-ax^b e^{cx}}\alpha^{1-e^{-ax^b e^{cx}}}\frac{\log\alpha}{\alpha-1}, & \alpha > 0, \alpha \neq 1, x \geq 0 \\ a(b+cx)x^{b-1}e^{cx-ax^b e^{cx}}, & \alpha = 1, x \geq 0 \end{cases}. \tag{2}$$

The survival function of the APMW distribution is defined as $S(x) = 1 - G(x)$ and given by

$$S(x) = \begin{cases} \frac{\alpha\left(1-\alpha^{-e^{-ax^b e^{cx}}}\right)}{\alpha-1}, & \alpha > 0, \alpha \neq 0, x \geq 0 \\ e^{-ax^b e^{cx}}, & \alpha = 1, x \geq 0 \end{cases}. \tag{3}$$

Also, the hazard rate function of the APMW distribution, defined as $\tau = g(x)/S(x)$, is given by

$$\tau(x) = \begin{cases} \frac{a(b+cx)x^{b-1}e^{cx-ax^b e^{cx}}\alpha^{-e^{-ax^b e^{cx}}}\log\alpha}{\left(1-\alpha^{-e^{-ax^b e^{cx}}}\right)}, & \alpha > 0, \alpha \neq 0, x \geq 0 \\ a(b+cx)x^{b-1}e^{cx}, & \alpha = 1, x \geq 0 \end{cases}.$$

The APMW distribution is a generalization of several distributions. These are shown in Table 1. This indicates that the APMW distribution is flexible.

Table 1: Special Distributions of the APMW distributions

No.	Distribution	a	b	α	c
1	Modified Weibull			1	
2	Alpha power Weibull				0
3	Weibull			1	0
4	Alpha power one parameter Weibull	1			0
5	One parameter Weibull	1		1	0
6	Alpha power exponential		1		0
7	Exponential		1	1	0
8	Alpha power Rayleigh		2		0
9	Rayleigh		2	1	0

When $\alpha = 1$ in Table 1, the modified Weibull distribution, whose properties are established in literature, is obtained (see Lai et al. [9]). Henceforth, only the case $\alpha \neq 0$ will be considered when referring to the APMW distribution.

For some parameter values, Fig. 1 shows the plots of the PDF of the APMW distribution. It can be observed that the APMW distribution can exhibit right skewed, left skewed, almost symmetric and decreasing shapes.

Also, Fig. 2 shows the plots of the hazard rate function of the APMW distribution for some parameter values. The plot shows that the hazard rate function can exhibit decreasing, increasing, bathtub, modified bathtub, J and reversed J shapes. The different shapes indicate the flexibility of the APMW distribution and its ability to model various data sets with different characteristics.

The expansion of the PDF of the APMW distribution is given below. The expansion of the PDF of a distribution is useful for the derivation of some properties of the distribution. Using the following series representation

$$\alpha^{-z} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} (\log \alpha)^i z^i,$$

the PDF of the APMW distribution can be written as

$$g(x) = \sum_{i=0}^{\infty} v_i f^*(x), \tag{4}$$

where $v_i = \frac{\alpha(-1)^i}{(i+1)!(\alpha-1)} (\log \alpha)^{i+1}$ and $f^*(x) = a(i+1)(b+cx)x^{b-1}e^{cx}e^{-a(i+1)x^b e^{cx}}$ is the PDF of the modified Weibull distribution with parameters $a(i+1)$, b and c.

3 Statistical Properties

In this section, various statistical properties of the APMW distribution are derived. These include the quantile function, moments, moment generating function, inequality measures, order statistics and stochastic ordering.

3.1 Quantile Function

The quantile function of a distribution is useful for the generation of random numbers from the distribution. The quantile function can also be used to derive other statistical measures, such as skewness and kurtosis, of a distribution. The quantile function of a distribution is obtained as the inverse function of the CDF. If $u \in (0,1)$, then the CDF of the APMW distribution in equation (1) can be equated to u and written as

$$\frac{1}{1-\alpha}\left(1 - \alpha^{1-e^{-ax_u^b e^{cx_u}}}\right) = u.$$

After some algebraic manipulations, we have

$$\frac{c}{b} x_u^b e^{\frac{c}{b} x_u} = \frac{c}{b}\left(-\frac{1}{a}\log\left\{\frac{\log(1-(1-\alpha)u) - \log\alpha}{\log\alpha}\right\}\right)^{\frac{1}{b}}.$$

Using the Lambert function defined as $W(xe^x) = x$, the quantile function is obtained as

$$Q(u) = \frac{b}{c} W\left[\frac{c}{b}\left(-\frac{1}{a}\log\left\{1 - \frac{\log(1-(1-\alpha)u)}{\log\alpha}\right\}\right)^{\frac{1}{b}}\right], \tag{5}$$

where $u \in (0,1)$. The quantile function obtained in equation (5) can be used to obtain the skewness and kurtosis of the distribution. This can be achieved via Bowley skewness (S) and Moors kurtosis (K) given respectively as

$$S = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{2}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}$$

and

$$K = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}.$$

Fig. 3 shows plots of skewness and kurtosis of the APMW distribution for some parameter values. It can be observed that the APMW distribution can exhibit various degree of skewness and kurtosis, including negative skewness. This illustrates the flexibility of the distribution and its ability to model data sets with varying degrees of skewness and kurtosis.

3.2 Moments

In this subsection, the non-central and incomplete moments of the APMW distribution are obtained. If the moments of a random variable exist, they can be used to obtain the measures of central tendency and dispersion, among other properties.

3.2.1 Non-central moment

The rth non-central moment of a distribution with PDF $g(x)$ is defined as

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r g(x) dx.$$

Using the expansion of the PDF of the APMW distribution in equation (4), we have

$$\mu_r' = \sum_{i=0}^{\infty} v_i a(i+1) \int_0^{\infty} x^r (b+cx) x^{b-1} e^{cx} e^{-a(i+1)x^b e^{cx}} dx. \tag{6}$$

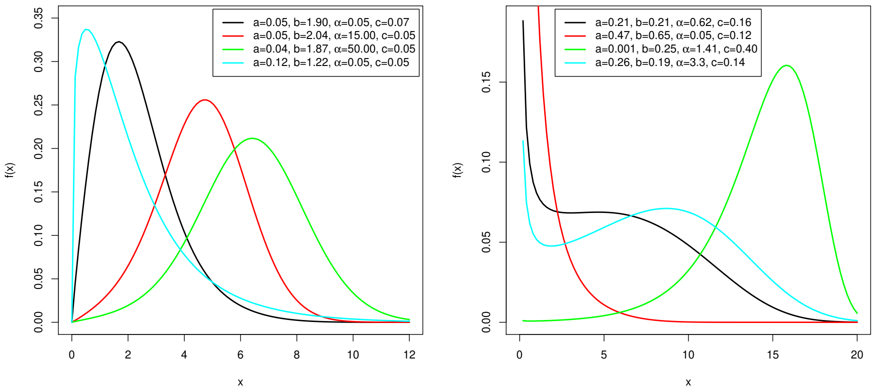

Fig. 1: Plot of the PDF of the APMW distribution

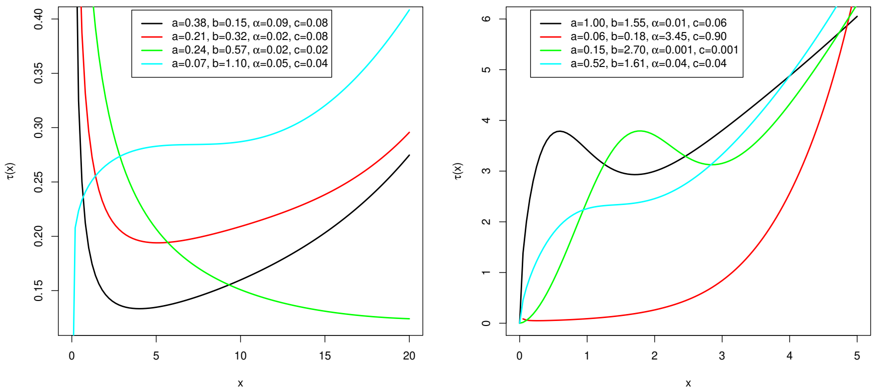

Fig. 2: Plot of hazard rate function of the APMW distribution

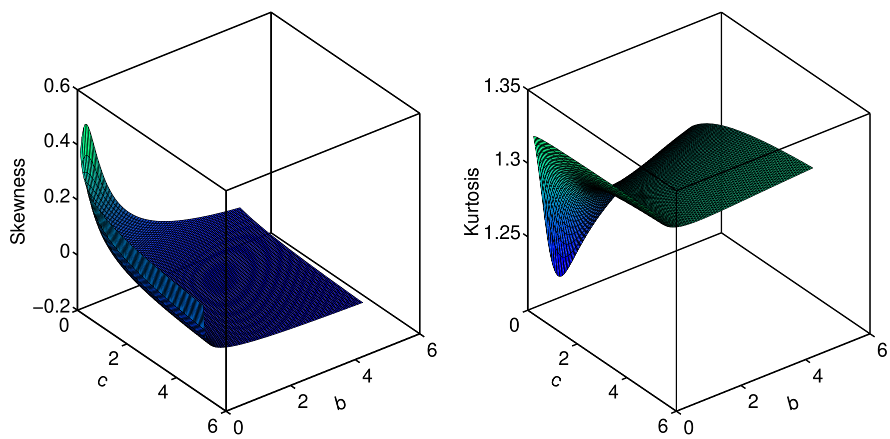

Fig. 3: Plot of skewness and kurtosis of the APMW distribution

Let $u = x^b e^{cx} \Rightarrow \frac{du}{dx} = (b+cx)x^{b-1}e^{cx}$ and $x = \frac{b}{c}W\left(\frac{c}{b}u^{\frac{1}{b}}\right)$, where $W(z) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(j-1)!} j^{j-2} z^j$. Also, as $x \to 0$, $u \to 0$ and as $x \to \infty$, $u \to \infty$. Substituting these into equation (6) gives

$$\mu_r' = \sum_{i=0}^{\infty} v_i a(i+1) \int_0^{\infty} \left(\frac{b}{c}W\left(\frac{c}{b}u^{\frac{1}{b}}\right)\right)^r e^{-a(i+1)u} du. \quad (7)$$

But

$$\frac{b}{c}W\left(\frac{c}{b}u^{\frac{1}{b}}\right) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(j-1)!} j^{j-2} \left(\frac{c}{b}\right)^j u^{\frac{j}{b}}, \\ = \sum_{j=1}^{\infty} e_j u^{\frac{j}{b}},$$

where $e_j = \frac{(-1)^{j+1}}{(j-1)!} j^{j-2} \left(\frac{c}{b}\right)^{j-1}$. Thus, equation (7) can be written as

$$\mu_r' = \sum_{i=0}^{\infty} v_i a(i+1) \int_0^{\infty} \left(\sum_{j=1}^{\infty} e_j u^{\frac{j}{b}}\right)^r e^{-a(i+1)u} du. \quad (8)$$

Again,

$$\left(\sum_{j=1}^{\infty} e_j u^{\frac{j}{b}}\right)^r = \sum_{j_1,\ldots,j_r=1}^{\infty} E_{j_1,\ldots,j_r} u^{\frac{s_r}{b}},$$

where $E_{j_1,\ldots,j_r} = e_{j_1} \ldots e_{j_r}$ and $s_r = j_1 + \cdots + j_r$. Substituting into equation (8) gives

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{j_1,\ldots,j_r=1}^{\infty} v_i a(i+1) E_{j_1,\ldots,j_r} \int_0^{\infty} u^{\frac{s_r}{b}} e^{-a(i+1)u} du. \quad (9)$$

Letting $t = a(i+1)u \Rightarrow u = \frac{t}{a(i+1)}$ and $\frac{dt}{du} = a(i+1)$. Also, as $u \to 0$, $t \to 0$ and as $u \to \infty$, $t \to \infty$. Substituting these into the equation (9) and with some algebraic manipulations, we obtain

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{j_1,\ldots,j_r=1}^{\infty} v_i \frac{E_{j_1,\ldots,j_r}}{(a(i+1))^{\frac{s_r}{b}}} \int_0^{\infty} t^{\frac{s_r}{b}} e^{-t} dt.$$

Using the gamma function defined as $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$, then the rth non-central moment of the APMW distribution can be written as

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{j_1,\ldots,j_r=1}^{\infty} v_i \frac{E_{j_1,\ldots,j_r}}{(a(i+1))^{\frac{s_r}{b}}} \Gamma\left(\frac{s_r}{b}+1\right). \quad (10)$$

If $r = 1$ in equation (10), then $E(X) = \mu$ is obtained and given as

$$\mu = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \frac{v_i e_j}{(a(i+1))^{\frac{j}{b}}} \Gamma\left(\frac{j}{b}+1\right). \quad (11)$$

3.2.2 Incomplete moment

The rth incomplete moment of a distribution is defined as $m_r(y) = \int_{-\infty}^{y} x^r g(x) dx$. Substituting the expansion of the PDF of the APMW in equation (4) into the definition, we have

$$m_r(y) = \sum_{i=0}^{\infty} v_i a(i+1) \int_0^{y} x^r (b+cx) x^{b-1} e^{cx - a(i+1)x^b e^{cx}} dx.$$

With similar algebraic manipulations as that of the rth non-central moment, we have the rth incomplete moment as

$$m_r(y) = \sum_{i=0}^{\infty} \sum_{j_1,\ldots,j_r=1}^{\infty} v_i \frac{E_{j_1,\ldots,j_r}}{(a(i+1))^{\frac{s_r}{b}}} \gamma\left(\frac{s_r}{b}+1, a(i+1)y^b e^{cy}\right), \quad (12)$$

where $\gamma(a,y) = \int_0^y x^{a-1} e^{-x} dx$ is the lower incomplete gamma function. Letting $r = 1$ gives the first incomplete moment of the APMW distribution as

$$m_1(y) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} v_i \frac{e_j}{(a(i+1))^{\frac{j}{b}}} \gamma\left(\frac{j}{b}+1, a(i+1)y^b e^{cy}\right). \quad (13)$$

3.3 Moment generating function

The moment generating function (MGF) of a distribution is useful in obtaining the moments of a distribution. The MGF of a distribution is defined as

$$M_X(t) = E\left[e^{tX}\right] = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'.$$

Substituting equation (10) into the definition gives the MGF of the APMW distribution as

$$M_X(t) = \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j_1,\ldots,j_r=1}^{\infty} v_i \frac{t^r E_{j_1,\ldots,j_r}}{r!(a(i+1))^{\frac{s_r}{b}}} \Gamma\left(\frac{s_r}{b}+1\right).$$

3.4 Inequality measures

Inequality measures are useful in studying disparities in income and poverty, and have applications in many fields including insurance, economics, reliability and medicine. The Lorenz and Bonferroni curves are studied for the APMW distribution.

The Lorenz curve of a distribution is defined as $L_G(y) = \frac{1}{\mu} \int_0^{y} x g(x) dx = \frac{1}{\mu} m_1(y)$, where $m_1(y)$ is the first incomplete moment given by equation (13). Thus, substituting equation (13) into the definition gives the Lorenz curve of the APMW distribution as

$$L_G(y) = \frac{1}{\mu}\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\frac{v_i e_j}{(a(i+1))^{\frac{j}{b}}}\gamma\left(\frac{j}{b}+1, a(i+1)y^b e^{cy}\right).$$

Alternatively, the Lorenz curve can be obtained in terms of the quantile function of a distribution [22] as

$$L_G(u) = \frac{1}{\mu}\int_0^u Q(t)dt,$$

where $u \in (0,1)$ and $Q(t)$ is the quantile function of the distribution. Lorenz curves of a distribution are of convex shapes and for $L_G(u) = u$, the curve becomes the bisector of the first quadrant, known as the point of minimal inequality. The area between $L_G(u) = u$ and the Lorenz curve is the area of concentration. An important inequality index that can be obtained using the Lorenz curve is the Gini index. It can be defined as $G = 1 - 2\int_0^1 L_G(u)du$.

Bonferroni curve of a distribution is defined as

$$B_G(y) = \frac{L_G(y)}{G(y)}.$$

Substituting the Lorenz curve of the APMW distribution and the CDF of the APMW distribution into the definition gives the Bonferroni curve of the APMW distribution. Similarly, Bonferroni curve can be defined in terms of the quantile function as

$$B_G(u) = \frac{L_G(u)}{u}, \quad u \in (0,1).$$

Bonferroni index is the area between the Bonferroni curve and $B_G(u) = 1$, and is defined as $B = 1 - \int_0^1 B_G(u)du$.

Fig. 4 shows the plots of the Lorenz and Bonferroni curves of the APMW distribution for some parameter values. It can be observed that the Lorenz curves of the APMW distribution are of convex shapes.

3.5 Order Statistics

In many areas of statistical theory and practice, order statistics are of significance. Given that $X_1, X_2, \ldots, X_n$ are random samples from the APMW distribution. Let $X_{i:n}$ be the ith order statistics from the sample. The PDF of the ith order statistic is defined as

$$g_{i:n}(x) = \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{p=0}^{n-i}\binom{n-i}{p}(-1)^p \times [G(x)]^{p+i-1}. \tag{14}$$

From equation (1) and using generalized binomial expansion, we have

$$\begin{aligned}[G(x)]^{p+i-1} &= \left(\frac{1}{1-\alpha}\right)^{p+i-1}\left(1-\alpha^{1-e^{-ax_u^b e^{cx_u}}}\right)^{p+i-1} \\ &= \left(\frac{1}{1-\alpha}\right)^{p+i-1}\sum_{k=0}^{p+i-1}(-1)^k\binom{p+i-1}{k} \\ &\quad \times \left(\alpha^{1-e^{-ax_u^b e^{cx_u}}}\right)^k. \end{aligned} \tag{15}$$

Substituting equations (2) and (15) into equation (14), and after some algebraic manipulations, we obtain the PDF of the ith order statistic as

$$g_{i:n}(x) = \sum_{p=0}^{n-i}\sum_{k=0}^{p+i-1}\sum_{l=0}^{\infty} v_{pkl}^{*} f^{**}(x), \tag{16}$$

where

$$v_{pkl}^{*} = \frac{\alpha^{k+1}(1-\alpha)^{-(p+i)}(\log\alpha)^{l+1}}{B(i,n-i+1)}\binom{n-i}{p}\binom{p+i-1}{k} \times \frac{(-1)^{p+k+l}}{(l+1)!}(k+1)^l$$

and $f^{**}(x) = a(l+1)(b+cx)x^{b-1}e^{cx}e^{-(l+1)x^b e^{cx}}$ is the PDF of the modified Weibull distribution with parameters $a(l+1), b$ and c, and $B(a,b)$ is the beta function.

The first and nth order statistics of the APMW distribution are also obtained respectively as:

$$g_{1:n}(x) = ng(x)[1-G(x)]^{n-1} = \sum_{k=0}^{n-1}\sum_{l=0}^{\infty}(\alpha-1)^{n-1}v_{kl}^{**}f^{**}(x),$$

and

$$g_{n:n}(x) = ng(x)[G(x)]^{n-1} = \sum_{k=0}^{n-1}\sum_{l=0}^{\infty}\alpha v_{kl}^{**}f^{**}(x),$$

where $v^{**} = \frac{n(\log\alpha)^{l+1}(-1)^{k+1}}{l!(1-\alpha)^n}\binom{n-i}{k}$ and $f^{**}(x)$ is as defined in equation (16).

3.6 Stochastic Ordering

In reliability theory and other fields, stochastic ordering is used for the assessment of comparative behaviour. Given the two random variables X_1 and X_2, with density functions $g_{X_1}(x;a,b,\alpha_1,c)$ and $g_{X_2}(x;a,b,\alpha_2,c)$, respectively. If the following conditions hold, the random variable X_2 is stochastically greater than X_1 in the following ordering:

i. Likelihood ratio order ($X_1 \le_{\mathrm{lr}} X_2$): if $\frac{g_{X_1}(x)}{g_{X_2}(x)}$ is an decreasing function of x.

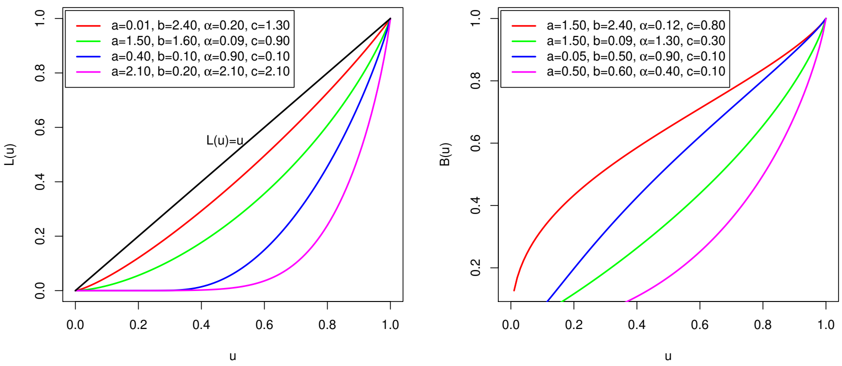

Fig. 4: Plot of Lorenz and Bonferroni curves of the APMW distribution

ii. Stochastic order: $(X_1 \leq_{\text{st}} X_2)$ if $F_{X_1}(x) \leq F_{X_2}(x)$ for all x.

iii. Hazard rate order $(X_1 \leq_{\text{hr}} X_2)$: if $\frac{\tau_{X_1}(x)}{\tau_{X_2}(x)}$ is decreasing for all x.

iv. Mean residual life order $(X_1 \leq_{\text{mrl}} X_2)$: if $E[X_1 - t \mid X_1 < t] \leq E[X_2 - t \mid X_2 < t]$.

The stochastic orders above are related to each other [23] as follows:

$$(X_1 \leq_{\text{lr}} X_2) \Rightarrow (X_1 \leq_{\text{hr}} X_2) \Rightarrow (X_1 \leq_{\text{mrl}} X_2) \Rightarrow (X_1 \leq_{\text{st}} X_2).$$

Given the ratio of the PDFs of X_1 and X_2 as

$$\frac{g_{X_1}(x)}{g_{X_2}(x)} = \left(\frac{\alpha_2 - 1}{\alpha_1 - 1}\right)\left(\frac{\log \alpha_1}{\log \alpha_2}\right)\left(\frac{\alpha_1}{\alpha_2}\right)^{1-e^{-ax^b e^{cx}}},$$

the differential of the logarithm of the ratio is given as

$$\frac{d}{dx}\log\frac{g_{X_1}(x)}{g_{X_2}(x)} = a(b+cx)x^{b-1}e^{cx-ax^b e^{cx}}\log\left(\frac{\alpha_1}{\alpha_2}\right). \quad (17)$$

If $\alpha_1 < \alpha_2$ in equation (17), then $\frac{d}{dx}\log\frac{g_{X_1}(x)}{g_{X_2}(x)} < 0$ for all x. Thus, for $\alpha_1 < \alpha_2$, X_2 is greater than X_1 in stochastic order with respect to likelihood ratio order, $(X_1 \leq_{lr} X_2)$. By implication $(X_1 \leq_{hr} X_2)$, $(X_1 \leq_{mrl} X_2)$ and $(X_1 \leq_{st} X_2)$.

4 Actuarial Measures

In actuarial practice, the assessment of risk exposures is very critical to firms. Risk measures have been developed for this purpose. Given a probability distribution, risk measures are used to assess the degree of risk exposure to a firm. In this section, various actuarial measures are derived for the APMW distribution. These measures include the value-at-risk (VaR), tail VaR, tail variance, tail variance premium, limited expected value function and mean excess function.

4.1 Value-at-risk

Value-at-risk (VaR), also known as quantile risk measure, is a widely used risk measure. VaR is usually specified at a given confidence level p. Thus, VaR_p represents the loss that will not be exceeded with probability p. VaR is defined as $VaR_p = \inf\{x \in \mathbb{R} : P(X \leq x) \geq p\}$. For a continuous distribution, the definition of VaR simplifies to the quantile of the distribution, that is $VaR_p = Q(p)$. Hence, the VaR for APMW distribution is given as

$$VaR_p = \frac{b}{c}W\left[\frac{c}{b}\left(-\frac{1}{a}\log\left\{1 - \frac{\log(1-(1-\alpha)p)}{\log \alpha}\right\}\right)^{\frac{1}{b}}\right],$$

where $p \in (0,1)$. For simplicity, let $VaR_p = Q_p$.

4.2 Tail value-at-risk

Tail value-at-risk (TVaR), also known as conditional tail expectation, defines the expected value of the worst case of a loss. That is, the loss with $1-p$ probability. Given a loss random variable X and $P(X \leq Q_p) = p$, then TVaR is defined as $E\left[X \mid X > Q_p\right]$. Thus, TVaR is given as

$$TVaR_p(x) = \frac{1}{1-p}\int_{Q_p}^{\infty} xg(x)dx. \quad (18)$$

From the PDF of the APMW in equation (4), we have

$$\int_{Q_p}^{\infty} xg(x)dx = \sum_{i=0}^{\infty} v_i a(i+1)\int_{Q_p}^{\infty} x(b+cx)x^{b-1}e^{cx}e^{-a(i+1)x^b e^{cx}}dx.$$

After some algebraic manipulations, we have

$$\int_{Q_p}^{\infty} xg(x)dx = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\frac{v_i e_j}{(a(i+1))^{\frac{j}{b}}}\Gamma\left(\frac{j}{b}+1, a(i+1)Q_p^b e^{cQ_p}\right), \quad (19)$$

where $\Gamma(a,y)=\int_y^{\infty}x^{a-1}e^{-x}dx$ is the upper incomplete gamma function.

Substituting equation (19) into (18) gives the TVaR of the APMW distribution as

$$TVaR_p(x)=\frac{1}{1-p}\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\frac{v_ie_j}{(a(i+1))^{\frac{j}{b}}}\times\Gamma\left(\frac{j}{b}+1,a(i+1)Q_p^be^{cQ_p}\right).$$

4.3 Tail variance

The tail variance (TV) measure computes the variance beyond VaR. The TV of a distribution is defined as

$$TV_p(x)=E\left[X^2\left|X>Q_p\right.\right]-(TVaR_p)^2.$$

Using the expansion of the PDF of the APMW distribution given by equation (4) and after some algebraic manipulations, we have

$$E\left[X^2\left|X>Q_p\right.\right]=\frac{1}{1-p}\sum_{i=0}^{\infty}\sum_{j_1,j_2=1}^{\infty}\frac{v_iE\,j_1,j_2}{(a(i+1))^{\frac{s_2}{b}}}\times\Gamma\left(\frac{s_2}{b}+1,a(i+1)Q_p^be^{cQ_p}\right),$$

where $s_2=j_1+j_2$ and $E_{j_1,j_2}=e_{j_1}\cdot e_{j_2}$. Substituting this into the definition gives the TV of the APMW distribution as

$$TV_p(x)=\frac{1}{1-p}\sum_{i=0}^{\infty}\sum_{j_1,j_2=1}^{\infty}\frac{v_iE\,j_1,j_2}{(a(i+1))^{\frac{s_2}{b}}}\times\Gamma\left(\frac{s_2}{b}+1,a(i+1)Q_p^be^{cQ_p}\right)-(TVaR_p)^2.$$

4.4 Tail variance premium

Another widely used measure is the tail variance premium (TVP). TVP of a distribution is defined as

$$TVP_p(x)=TVaR_p+\delta TV_p,\quad \delta\in(0,1).$$

For the APMW distribution, TVP is given as

$$TVP_p(x)=\frac{1}{1-p}\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\frac{v_ie_j}{(a(i+1))^{\frac{j}{b}}}\times\Gamma\left(\frac{j}{b}+1,a(i+1)Q_p^be^{cQ_p}\right)+\delta TV_p,\quad \delta\in(0,1).$$

4.5 Limited expected value function

Limited expected value function (LEVF) of a distribution is defined as

$$E\left[L\wedge Q_p\right]=E\left[\min\left(L,Q_p\right)\right]=\int_0^{Q_p}xg(x)dx+Q_p\left(1-F\left(Q_p\right)\right).$$

Using the first incomplete moment of the APMW distribution given in equation (13), we have the limited expected value function of the APMW distribution given as

$$E\left[L\wedge Q_p\right]=\frac{1}{1-p}\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\frac{v_ie_j}{(a(i+1))^{\frac{j}{b}}}\times\Gamma\left(\frac{j}{b}+1,a(i+1)Q_p^be^{cQ_p}\right)+\frac{\alpha Q_p}{\alpha-1}\left(1-\alpha^{-e^{-aQ_p{}^be^{cQ_p}}}\right).$$

4.6 Mean excess function

The mean excess function (MEF), also known as the mean residual function or complete expectation of life, is another useful measure. In an insurance context, the mean excess function is the expected payment per claim on a policy with a fixed amount deductible of t, with payments less than t not honoured. In a mortality context, it can be defined as the remaining lifetime of a life, given that the life reached age t. The mean excess function is defined as

$$e(t)=E\left[X-t\left|X>t\right.\right]=\frac{1}{S(t)}\int_t^{\infty}xg(x)dx-t.$$

Using equations (3) and (19), with $Q_p=t$, the mean excess function of the APMW distribution is given as

$$e(t)=\frac{\alpha-1}{\alpha}\left(1-\alpha^{-e^{-ax^be^{cx}}}\right)^{-1}\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\frac{v_ie_j}{(a(i+1))^{\frac{j}{b}}}\times\Gamma\left(\frac{j}{b}+1,a(i+1)t^be^{ct}\right)-t.$$

4.7 Numerical simulations of some actuarial measures

The numerical study of the actuarial measures defined are studied in this subsection. The study is done for the APMW, APExE, APW and exponential distributions. The study is carried as follows:

i. Generate a random sample of 150 from the distributions using their quantile functions.
ii. The parameters of the sample is estimated using maximum likelihood method.
iii. Steps i and ii are repeated for 1500 times and the VaR, TVaR, TV and TVP computed.
iv. Steps i to iii are repeated for $p=(0.60,0.65,0.70,0.75,0.80,0.85,0.90,0.975,0.999)$.

Tables 2 and 3 show the simulation results for the actuarial measures of the four distributions for different sets of parameter values. It can be observed that the APMW distribution has higher values of the actuarial measures as compared to the other competing distributions. Fig. 5 and Fig. 6 illustrate the actuarial measures in Table 2. Also, Fig. 7 and Fig. 8 illustrate the actuarial measures in Table 3. The results show that the APMW distribution have higher values of VaR, TVaR and TVP, and decreasing values of TV. This suggests that the APMW distribution can be used as an alternative distribution to model insurance loss data.

5 Parameter Estimation

Different parameter estimation methods are employed to estimate the parameters of the APMW distribution. These include the maximum likelihood estimation (MLE), maximum product spacing (MPS), ordinary least squares (OLS), Cramér-von Mises (CVM), Anderson-Darling (AD) and percentile (PC) methods of estimation.

5.1 Maximum Likelihood Estimation Method

Suppose that $x_1, x_2, \ldots, x_n$ are random samples from the APMW distribution with density function given by equation (2) and let $\psi = (a, b, \alpha, c)'$ be the set of parameters of the distribution. Then the likelihood function for the density is defined as

$$\ell = n\log(a) + \sum_{i=1}^{n}\log(b+cx_i) + (b-1)\sum_{i=1}^{n}\log(x_i) + c\sum_{i=1}^{n}x_i - a\sum_{i=1}^{n}\left(x_i^b e^{cx_i}\right) + \sum_{i=1}^{n}\left(1-e^{ax_i^b e^{cx_i}}\right)\log(\alpha). \quad (20)$$

The differential of equation (20) with respect to each parameter gives the score functions as follows:

$$\frac{\partial \ell}{\partial a} = \sum_{i=1}^{n}\log(\alpha)x_i^b e^{cx_i - ax_i^b e^{cx_i}} - \sum_{i=1}^{n}x_i^b e^{cx_i} + \frac{n}{a},$$

$$\frac{\partial \ell}{\partial b} = \sum_{i=1}^{n}a\log(\alpha)x_i^b\log(x_i)e^{cx_i - ax_i^b e^{cx_i}} - a\sum_{i=1}^{n}x_i^b e^{cx_i}\log(x_i) + \sum_{i=1}^{n}\frac{1}{b+cx_i} + \sum_{i=1}^{n}\log(x_i),$$

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{n}\frac{1-e^{-ax_i^b e^{cx_i}}}{\alpha},$$

$$\frac{\partial \ell}{\partial c} = \sum_{i=1}^{n}a\log(\alpha)x_i^{b+1}e^{cx_i - ax_i^b e^{cx_i}} - a\sum_{i=1}^{n}x_i^{b+1}e^{cx_i} + \sum_{i=1}^{n}\frac{x_i}{b+cx_i} + \sum_{i=1}^{n}x_i.$$

The parameter estimates $\hat{\psi} = (\hat{a}, \hat{b}, \hat{\alpha}, \hat{c})'$ are obtained by equating the score function to zero and solving for the parameters. The solution to the equations will not yield closed-forms for the estimates as the score functions are non-linear. Thus, numerical methods are employed in this study to solve the non-linear equations.

5.2 Maximum Product Spacing

An alternative method to maximum likelihood method is the maximum product spacing method (MPS). Given the uniform spacing

$$S_i = G(x_{(i)} \mid \psi) - G(x_{(i-1)} \mid \psi),$$

where $G(x_{(0)} \mid \psi) = 0$, $G(x_{(n+1)} \mid \psi) = 1$ and $\sum_{i=0}^{n+1} S_i(\psi) = 1$. The MPS parameter estimates of the distribution can be obtained by maximizing the following function

$$D(\psi) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log S_i(\psi)$$

with respect to $\psi = (a, b, \alpha, c)$.

5.3 Ordinary Least Squares Estimation Method

Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be ordered samples from a random sample from the APMW distribution. Then the following function is minimized to obtain the OLS estimates of the parameters of the APMW distributions:

$$L(\psi) = \sum_{i=1}^{n}\left[G(x_{(i)}|\psi) - \frac{i}{n+1}\right]^2.$$

5.4 Cramér-von Mises Minimum Distance Estimation Method

The CVM estimation method obtains the parameter estimates by minimizing the difference between the empirical and cumulative distribution functions. That is, the estimators are obtained by minimizing the function

$$C(\psi) = \frac{1}{12n} + \sum_{i=1}^{n}\left[G(x_{(i)}|\psi) - \frac{2i-1}{2n}\right]^2.$$

5.5 Anderson-Darling Estimation Method

The Anderson-Darling (AD) is a minimum distance estimator. The AD parameter estimates of the APMW distribution are obtained by minimizing the function

$$A(\psi) = -n - \frac{1}{n}\sum_{i=1}^{n}\left[\log G(x_{(i)} \mid \psi) + \log\left(1 - G(x_{(i)} \mid \psi)\right)\right],$$

with respect to $\psi = (a, b, \alpha, c)$.

Table 2: Numerical Estimates of Actuarial Measures

Distribution	Parameters	p	VaR	TVaR	TV $(\delta = 0.2)$	TVP
APMW	$a = 0.08$, $b = 1.30$, $\alpha = 1.80$, $c = 0.10$	0.600	5.0792	6.9940	2.1597	7.4259
		0.650	5.4393	7.2420	1.9746	7.6369
		0.700	5.8188	7.5111	1.7946	7.8701
		0.750	6.2275	7.8093	1.6173	8.1328
		0.800	6.6805	8.1493	1.4395	8.4372
		0.850	7.2047	8.5540	1.2565	8.8053
		0.900	7.8570	9.0729	1.0593	9.2848
		0.975	9.6102	10.5361	0.6656	10.6693
		0.999	12.3269	12.9421	0.3267	13.0074
APExE	$a = 0.80$, $\alpha = 1.20$, $c = 1.80$	0.600	1.8255	3.1634	1.6668	3.4967
		0.650	2.0165	3.3411	1.6518	3.6715
		0.700	2.2322	3.5443	1.6375	3.8718
		0.750	2.4820	3.7824	1.6237	4.1072
		0.800	2.7820	4.0713	1.6105	4.3934
		0.850	3.1618	4.4406	1.5979	4.7602
		0.900	3.6882	4.9569	1.5857	5.2740
		0.975	5.4490	6.7036	1.5681	7.0172
		0.999	9.4813	10.7315	1.5627	11.0440
Ex	$a = 0.95$	0.600	0.9645	2.0171	1.1080	2.2388
		0.650	1.1051	2.1577	1.1080	2.3793
		0.700	1.2673	2.3200	1.1080	2.5416
		0.750	1.4593	2.5119	1.1080	2.7335
		0.800	1.6941	2.7468	1.1080	2.9684
		0.850	1.9970	3.0496	1.1080	3.2712
		0.900	2.4238	3.4764	1.1080	3.6980
		0.975	3.8830	4.9357	1.1080	5.1573
		0.999	7.2713	8.3240	1.1080	8.5456
APW	$\alpha = 2.30$, $\beta = 1.20$, $\lambda = 0.50$	0.600	2.0433	3.4508	1.6439	3.7796
		0.650	2.2589	3.6368	1.6016	3.9571
		0.700	2.4984	3.8468	1.5588	4.1586
		0.750	2.7711	4.0899	1.5148	4.3929
		0.800	3.0923	4.3807	1.4686	4.6744
		0.850	3.4904	4.7466	1.4183	5.0303
		0.900	4.0282	5.2483	1.3602	5.5204
		0.975	5.7386	6.8791	1.2248	7.1240
		0.999	9.3337	10.3788	1.0534	10.5895

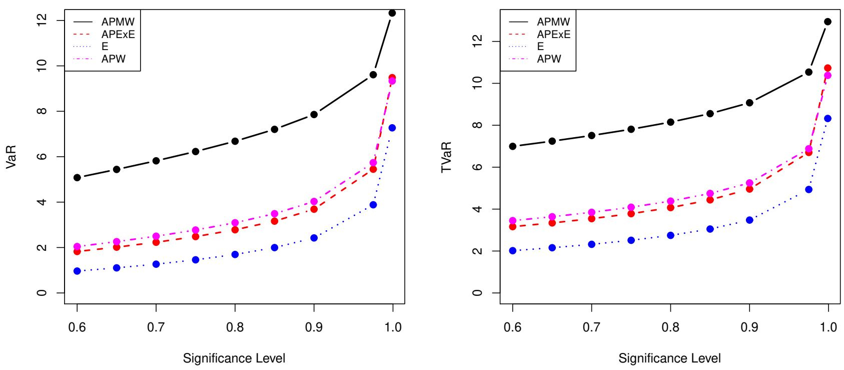

Fig. 5: Plot of VaR and TVaR using parameter values in Table 2

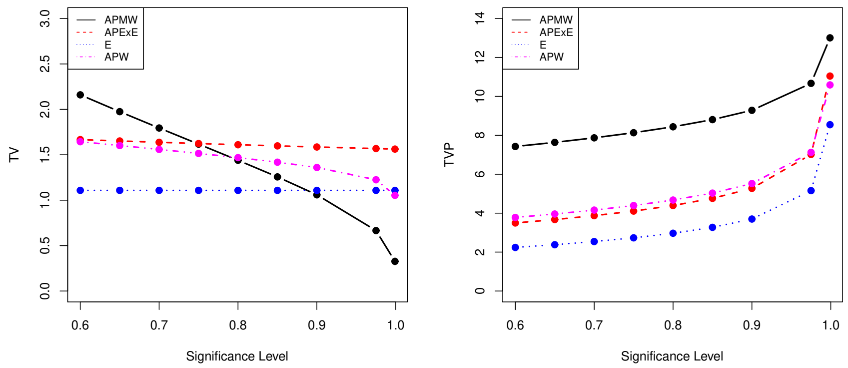

Fig. 6: Plot of TV and TVP using parameter values in Table 2

Table 3: Numerical Estimates of Actuarial Measures

Distribution	Parameters	p	VaR	TVaR	TV ($\delta = 0.6$)	TVP
		0.600	2.8409	5.6169	6.7338	9.6572
		0.650	3.2318	5.9862	6.6032	9.9481
	$a = 0.20$	0.700	3.6805	6.4088	6.4509	10.2793
APMW	$b = 1.10$	0.750	4.2076	6.9032	6.2695	10.6649
	$\alpha = 0.40$	0.800	4.8467	7.5001	6.0472	11.1284
	$c = 0.02$	0.850	5.6594	8.2552	5.7639	11.7135
		0.900	6.7797	9.2906	5.3778	12.5173
		0.975	10.3453	12.5694	4.2483	15.1184
		0.999	17.1378	18.8712	2.6686	20.4723
		0.600	0.5307	1.1042	0.3187	1.2954
		0.650	0.6094	1.1806	0.3174	1.3711
		0.700	0.6995	1.2685	0.3162	1.4582
	$a = 1.80$	0.750	0.8052	1.3720	0.3149	1.5610
APExE	$\alpha = 2.00$	0.800	0.9337	1.4982	0.3136	1.6864
	$c = 0.80$	0.850	1.0980	1.6603	0.3124	1.8477
		0.900	1.3278	1.8878	0.3111	2.0745
		0.975	2.1046	2.6613	0.3093	2.8469
		0.999	3.8950	4.4506	0.3087	4.6358
		0.600	0.6109	1.2775	0.4444	1.5442
		0.650	0.6999	1.3665	0.4444	1.6332
		0.700	0.8026	1.4693	0.4444	1.7360
		0.750	0.9242	1.5909	0.4444	1.8575
Ex	$a = 1.50$	0.800	1.0730	1.7396	0.4444	2.0063
		0.850	1.2647	1.9314	0.4444	2.1981
		0.900	1.5351	2.2017	0.4444	2.4684
		0.975	2.4593	3.1259	0.4444	3.3926
		0.999	4.6052	5.2718	0.4444	5.5385
		0.600	0.9473	2.2077	1.8214	3.3005
		0.650	1.1035	2.3769	1.8523	3.4883
		0.700	1.2864	2.5743	1.8877	3.7070
	$\alpha = 2.10$	0.750	1.5060	2.8107	1.9293	3.9683
APW	$\beta = 0.80$	0.800	1.7793	3.1040	1.9797	4.2919
	$\lambda = 1.20$	0.850	2.1389	3.4888	2.0440	4.7152
		0.900	2.6590	4.0431	2.1330	5.3229
		0.975	4.5469	6.0377	2.4222	7.4910
		0.999	9.4813	11.1665	3.0038	12.9687

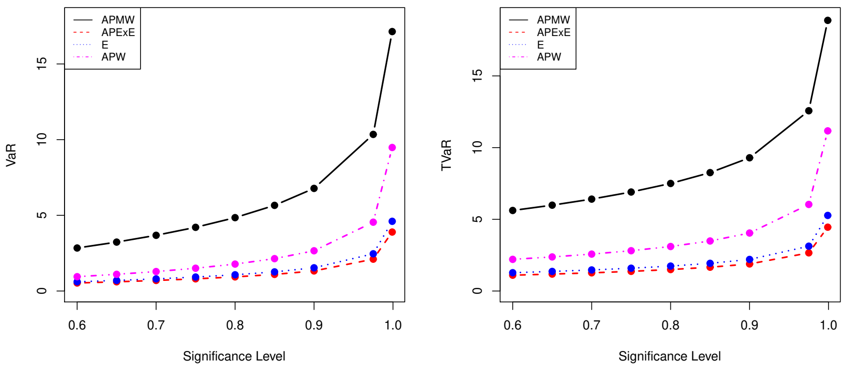

Fig. 7: Plot of VaR and TVaR using parameter values in Table 3

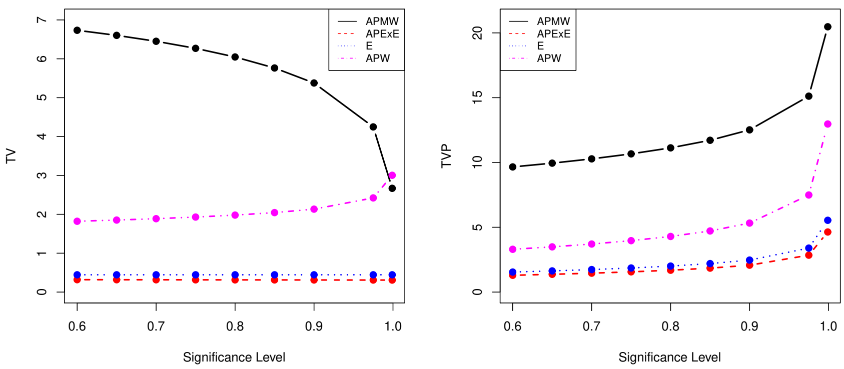

Fig. 8: Plot of TV and TVP using parameter values in Table 3

5.6 Percentile Method

The percentile (PC) method can be used to estimate the parameters of a distribution. Given that $q_i = \frac{i}{n+1}$ is an unbiased estimator of $G(x_{(i)} \mid \psi)$, then the PC estimates of the parameters of APMW can be obtained by minimizing

$$P(\psi) = \sum_{i=1}^{n} \left[x_{(i)} - Q(q_i)\right]^2,$$

with respect to $\psi = (a, b, \alpha, c)$, where $Q(q_i)$ is the quantile function of the APMW distribution given by equation (5).

6 Simulation Study

In this section, Monte Carlo simulation study is carried out to assess the performance of the MLE, MPS, OLS, CVM, AD and PC estimators of the parameters of the APMW distribution. In comparing the various estimation methods, the average estimate (AE), average bias (AB) and root mean square errors (RMSE) of the estimators are computed and then examined. The simulation study was carried out using the following parameter sets: **I**: $(a, b, \alpha, c) = (0.9, 0.6, 2.5, 0.8)$ and **II**: $(a, b, \alpha, c) = (0.3, 3.2, 0.5, 6.8)$. The study was carried out using the following process:

i. Generate random observations of size $n = 30, 80, 150, 250$ and 500 using the given parameter values and quantile function of the APMW distribution defined by equation (5).
ii. Compute the estimates of the parameters using the MLE, MPS, OLS, CVM, AD and PC estimators.
iii. Repeat steps i and ii for $N = 1,500$ times.
iv. Compute the average estimate (AE), average bias (AB) and root mean square errors (RMSE) of the estimators of the parameters. The AE, AB and RMSE are computed using the following

$$\mathrm{AE} = \frac{1}{N}\sum_{i=1}^{N}\widehat{\psi}_i,\ \mathrm{AB} = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\psi}_i - \psi)$$

and

$$\mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\widehat{\psi}_i - \psi)^2},$$

where $\psi = (a,b,\alpha,c)$.

The simulation results consisting of the AE, AB and RMSE of the estimators for parameter sets I and II are presented in Tables 4 and 5 respectively. It can be observed from the results that as the sample size increases, AE, AB and RMSE of all the estimators decrease. This implies that all the estimators are consistent as the sample size increases. However, the results reveal that the MLE estimators generally have least values of AE, AB and RMSE than the other estimators.

7 Applications

In this section, the APMW distribution is applied to two real data sets to ascertain its usefulness and flexibility. The performance of the APMW distribution is compared to several competing distributions including KW distribution, ExPGW distribution, APExW distribution, APW distribution, alpha power exponentiated exponential (APExE) distribution [24], Weibull Burr XII (WBXII) distribution [25], ExPW distribution and Weibull distribution [7]. The CDF of the distributions are given as follows:

- KW:

$$F(x) = 1 - \left[1 - \left(1 - e^{-(\alpha x)^c}\right)^a\right]^b, \quad a,b,\alpha,c > 0$$

- ExPGW:

$$F(x) = \left[1 - e^{1-(1+ax^c)^\alpha}\right]^b, \quad a,b,\alpha,c > 0$$

- APExW:

$$F(x) = \frac{1}{1-\alpha}\left[1 - \alpha^{\left(1-e^{-ax^b}\right)^c}\right], \quad a,b,\alpha,c > 0$$

- APW:

$$F(x) = \frac{1}{1-\alpha}\left[1 - \alpha^{1-e^{-ax^b}}\right], \quad a,b,\alpha > 0$$

- APExE:

$$F(x) = \frac{1}{1-\alpha}\left[1 - \alpha^{\left(1-e^{-ax}\right)^b}\right], \quad a,b,\alpha > 0$$

- WBXII:

$$F(x) = 1 - e^{-a\left[\left(1+x^b\right)^c - 1\right]^\alpha}, \quad a,b,\alpha,c > 0$$

- ExPW:

$$F(x) = \frac{1}{1-e}\left(1 - e^{\left(\frac{ax^b}{c+ax^b}\right)}\right), \quad a,b,c > 0$$

- W:

$$F(x) = 1 - e^{-\left(\frac{x}{a}\right)^b}, \quad a,b > 0$$

The performance of the distributions are compared using the following goodness-of-fit measures: Akaike information criterion (AIC), Bayesian information criterion (BIC), Anderson-Darling (AD) and Kolmogrove-Smirnov (KS). The distribution with the largest value of the measures, but with highest KS measure p-value is the best distribution which describes the data sets.

The shapes of the hazard rate functions of the data sets are obtained by using a graphical method based on the total time on test (TTT) transformation [26]. Given the order statistics of a sample $x_{i:n}$ $(i = 1,\ldots,n)$, the TTT plot is obtained by plotting the scaled empirical TTT given by

$$G\left(\frac{t}{n}\right) = \frac{\sum_{i=1}^{t} x_{i:n} + (n-t)x_{t:n}}{\sum_{i=1}^{n} x_{i:n}},$$

where $t = 1,\ldots,n$, against $\frac{t}{n}$. The corresponding hazard rate function is decreasing if the TTT transform is convex below the 45^o line and it is increasing if the TTT is concave above the 45^o line. Also, the hazard rate function is bathtub if it is convex below and then concave above the 45^o line. Finally, it is unimodal or upside down bathtub if it is concave above and then convex below the 45^o line [26,27].

7.1 Data Set 1: Turbochargers Failure Data

The first data set considered is the time-to-failure (10^3h) rate of turbocharger of a type of engine. The data set consists of 40 observations and can be found in Xu et al. [28]. The data is given as follows: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.

The characteristics of the first data set is ascertained by obtaining the shape of the failure data set and the shape of the hazard rate function of the data set. A histogram and TTT plot of the data are obtained for this purpose and shown in Fig. 9. Fig. 9 (a) shows the histogram of the data. It can be observed that the data is negatively skewed. Also, Fig. 9 (b) shows the TTT plot of the data set. The failure rate is an increasing function as it can be observed that the TTT plot is concave in shape. Hence, the shapes of the histogram and TTT plot of the data set indicates that the APMW distribution is a suitable candidate for modeling it.

Table 6 shows the parameter estimates of all the distributions with their corresponding standard errors in parenthesis.

Table 4: Simulation Results for parameter values set I

n	Estimate	Parameter	MLE	MPS	OLS	CVM	AD	PC
30	AE	a	0.8715	0.7846	0.9775	0.9377	1.0018	0.9408
		b	0.6232	0.7051	0.5312	0.5615	0.5718	0.4978
		α	2.6120	2.0432	3.4886	3.1835	3.4723	3.5045
		c	0.9461	1.0621	0.8901	0.953	0.8591	0.9129
	AB	a	0.3525	0.3346	0.428	0.4107	0.3972	0.4286
		b	0.1696	0.1899	0.2276	0.2110	0.1807	0.238
		α	1.4523	1.2573	1.7446	1.6134	1.7791	1.8004
		c	0.3383	0.3850	0.4168	0.4152	0.3457	0.402
	MSE	a	0.2122	0.1790	0.3338	0.3038	0.2977	0.2792
		b	0.0498	0.0691	0.0763	0.0697	0.0517	0.0818
		α	2.5932	1.8715	3.7197	3.2507	3.7941	3.8734
		c	0.1880	0.2445	0.2763	0.2874	0.1908	0.2444
80	AE	a	0.9307	0.8219	0.9582	0.9461	0.9823	1.1052
		b	0.6144	0.6521	0.5633	0.5852	0.5940	0.5920
		α	2.9679	2.1970	3.3801	3.1640	3.3615	3.9346
		c	0.8323	0.9206	0.8295	0.8444	0.7981	0.7353
	AB	a	0.2651	0.2428	0.3024	0.3026	0.2840	0.3979
		b	0.1111	0.1177	0.1396	0.1381	0.1207	0.1665
		α	1.5570	1.2859	1.7319	1.6420	1.7303	1.9922
		c	0.2103	0.2263	0.2627	0.2631	0.2175	0.2983
	MSE	a	0.1129	0.0913	0.1591	0.1645	0.1380	0.2521
		b	0.0207	0.0256	0.0299	0.0299	0.0229	0.0425
		α	3.0090	2.0059	3.6309	3.3175	3.6139	4.5162
		c	0.0701	0.0814	0.1091	0.1101	0.0750	0.0765
150	AE	a	0.9210	0.8290	0.9424	0.9331	0.9607	1.0521
		b	0.6034	0.6299	0.5763	0.5866	0.5888	0.5822
		α	3.0112	2.3319	3.2785	3.1762	3.2957	3.8570
		c	0.8194	0.8865	0.8146	0.8316	0.7968	0.7512
	AB	a	0.2122	0.2080	0.2503	0.2565	0.2294	0.3151
		b	0.0825	0.0849	0.1026	0.1030	0.0881	0.1268
		α	1.5403	1.3216	1.6979	1.6488	1.6685	1.9379
		c	0.1587	0.1703	0.2024	0.2100	0.1700	0.2241
	MSE	a	0.0681	0.0628	0.1012	0.1063	0.0838	0.1598
		b	0.0112	0.0121	0.0159	0.0167	0.0121	0.0249
		α	2.9600	2.1398	3.5166	3.3354	3.4149	4.3328
		c	0.0402	0.0458	0.0646	0.0684	0.0460	0.0765
250	AE	a	0.9350	0.8493	0.9456	0.9129	0.9346	1.0273
		b	0.5992	0.6235	0.5798	0.5914	0.5886	0.5826
		α	3.1111	2.4106	3.3212	3.0184	3.1868	3.8389
		c	0.8028	0.8579	0.8079	0.8226	0.8034	0.7563
	AB	a	0.1923	0.1892	0.2315	0.2184	0.2060	0.2597
		b	0.0663	0.0711	0.0796	0.0802	0.0695	0.0957
		α	1.5438	1.2770	1.7052	1.5707	1.6064	1.8887
		c	0.1299	0.1440	0.1672	0.1665	0.1410	0.1763
	MSE	a	0.0529	0.0498	0.0798	0.0723	0.0431	0.1024
		b	0.0071	0.0086	0.0098	0.0100	0.0044	0.0145
		α	3.0038	2.0690	3.5304	3.0613	3.2072	4.1991
		c	0.0260	0.0324	0.0437	0.0436	0.0310	0.0471
500	AE	a	0.9158	0.8564	0.9334	0.9193	0.9238	0.9968
		b	0.5962	0.6120	0.5849	0.5892	0.5910	0.5833
		α	2.9696	2.5179	3.1792	3.0764	3.1111	3.6484
		c	0.8068	0.8424	0.8020	0.8119	0.8045	0.7654
	AB	a	0.1666	0.1692	0.2007	0.199	0.1771	0.2105
		b	0.0501	0.0509	0.0607	0.0605	0.0529	0.0691
		α	1.3842	1.2577	1.5970	1.5465	1.5043	1.7310
		c	0.1023	0.1097	0.1299	0.1334	0.1108	0.1301
	MSE	a	0.0385	0.0379	0.0562	0.0547	0.0431	0.0636
		b	0.0039	0.0042	0.0056	0.0057	0.0044	0.0074
		α	2.5432	2.0646	3.1722	3.0238	1.5043	3.7246
		c	0.0164	0.0183	0.0258	0.0272	0.0185	0.0260

Table 5: Simulation Results for parameter values set II

n	Estimate	Parameter	MLE	MPS	OLS	CVM	AD	PC
30	AE	a	0.4607	0.5251	0.6107	0.6491	0.5065	0.5706
		b	3.0448	3.4419	3.4086	3.7558	3.0790	3.4835
		α	0.4435	0.4916	0.2966	0.2879	0.3683	0.3078
		c	8.2470	8.2361	6.3840	6.4262	7.2910	6.2508
	AB	a	0.3800	0.4005	0.4311	0.4387	0.3934	0.4045
		b	1.4483	1.4632	1.4207	1.4236	1.5305	1.1909
		α	0.2968	0.2642	0.3678	0.3524	0.3421	0.3564
		c	2.5818	2.2297	2.1643	1.7766	2.3816	1.8172
	MSE	a	0.1635	0.1785	0.2004	0.2073	0.1735	0.1836
		b	3.0491	2.9853	3.0986	3.0267	3.4313	2.0336
		α	0.1021	0.0825	0.1487	0.1404	0.1324	0.1435
		c	13.6139	11.7628	6.9114	5.2455	9.6950	4.6029
80	AE	a	0.4051	0.5019	0.6116	0.6473	0.5257	0.4668
		b	2.8783	3.3474	3.4920	3.7082	3.2706	3.1157
		α	0.4479	0.4812	0.3359	0.3215	0.3999	0.3998
		c	8.0804	7.4663	6.0312	5.9772	6.7797	6.8915
	AB	a	0.3498	0.3738	0.4072	0.4182	0.3814	0.3576
		b	1.1853	1.0253	0.9881	0.9861	1.0925	1.0545
		α	0.2753	0.2535	0.3259	0.3249	0.3060	0.3148
		c	2.3639	1.7242	1.6447	1.4698	1.7694	1.7961
	MSE	a	0.1430	0.1620	0.1883	0.1966	0.1678	0.1514
		b	2.0914	1.5003	1.5256	1.4364	1.8503	1.6782
		α	0.0913	0.0785	0.1234	0.1228	0.1100	0.1156
		c	13.6139	5.9712	3.8560	3.2162	5.1087	4.8328
150	AE	a	0.3823	0.4872	0.6112	0.6142	0.5109	0.4374
		b	2.8455	3.2291	3.5903	3.5910	3.2616	3.0132
		α	0.4528	0.5100	0.3719	0.3783	0.4381	0.4351
		c	7.9573	7.3319	6.0081	6.0403	6.6925	7.1484
	AB	a	0.3320	0.3607	0.3940	0.3954	0.3612	0.3488
		b	1.0212	0.8870	0.8331	0.8194	0.8818	0.9493
		α	0.2655	0.2343	0.3007	0.2964	0.2831	0.2864
		c	2.1559	1.6339	1.3528	1.3265	1.5276	1.8169
	MSE	a	0.1321	0.1541	0.1813	0.1826	0.1561	0.1445
		b	1.5908	1.1524	1.0041	0.9746	1.2114	1.4087
		α	0.0857	0.0674	0.1082	0.1042	0.0963	0.0987
		c	8.3814	5.0869	2.3544	2.3343	3.5647	5.0249
250	AE	a	0.3688	0.4650	0.6069	0.5994	0.5019	0.4073
		b	2.8847	3.1943	3.5429	3.5607	3.2764	3.0382
		α	0.4440	0.4970	0.4163	0.4182	0.4624	0.4327
		c	7.7989	7.1965	6.0057	6.0764	6.6651	7.1874
	AB	a	0.3160	0.3377	0.3856	0.3807	0.3497	0.3227
		b	0.8774	0.7491	0.7053	0.7370	0.7594	0.8230
		α	0.2573	0.2214	0.2683	0.2785	0.2590	0.2745
		c	1.9875	1.5262	1.2748	1.2361	1.3695	1.6012
	MSE	a	0.1230	0.1409	0.1761	0.1826	0.1493	0.1287
		b	1.1825	0.8215	0.6872	0.7464	0.8490	1.0453
		α	0.0823	0.0623	0.0894	0.0936	0.0820	0.0913
		c	6.7715	3.9973	1.9075	1.8415	2.6011	3.9121
500	AE	a	0.3656	0.4470	0.5856	0.5737	0.4803	0.3974
		b	2.9018	3.1554	3.4539	3.4354	3.2335	3.0395
		α	0.4505	0.4915	0.4734	0.5000	0.5178	0.4581
		c	7.7085	7.1876	6.1380	6.1943	6.6642	7.2074
	AB	a	0.3059	0.3233	0.3673	0.3552	0.3186	0.3095
		b	0.7555	0.6522	0.6125	0.6093	0.5964	0.6598
		α	0.2333	0.2068	0.2355	0.2367	0.2260	0.2352
		c	1.8411	1.4714	1.1646	1.1125	1.1895	1.4926
	MSE	a	0.1171	0.1321	0.1652	0.1589	0.1317	0.1207
		b	0.8806	0.6275	0.5134	0.5116	0.5363	0.644
		α	0.0691	0.0554	0.0698	0.0699	0.0636	0.0707
		c	5.7051	3.6576	1.6177	1.5060	1.916	3.2514

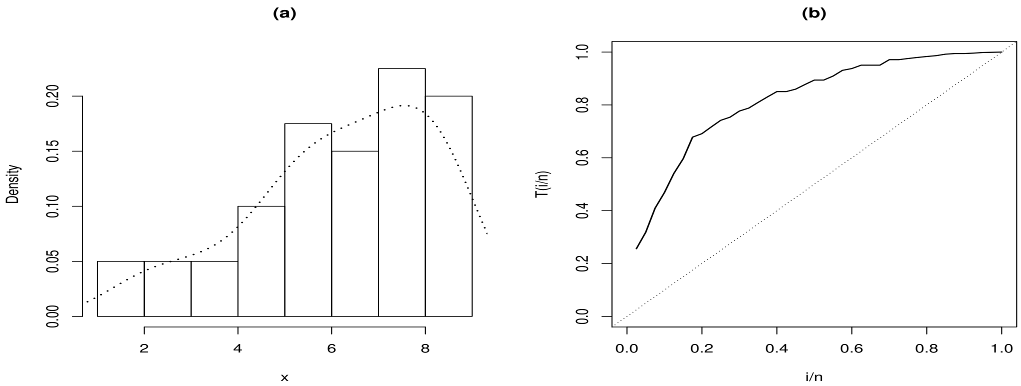

Fig. 9: (a) Histogram and (b) TTT plot for first data set

Table 6: Parameter Estimates of Distributions for First Data Set

Distribution	a	b	α	c
APMW	0.8907	38.7527	4.3115	0.0558
	(0.0727)	(0.0000)	(0.0680)	(0.0000)
KW	0.8901	38.7527	0.0558	4.3115
	(0.4963)	(0.0249)	(0.0069)	(2.3621)
ExPGW	0.0048	1.6922	3.7637	2.0854
	(0.0102)	(1.0860)	(2.5158)	(0.9262)
APExW	0.0039	3.0603	3.9313	1.0497
	(0.0185)	(2.0472)	(1.1832)	(1.2671)
APW	0.0078	2.7960	11.4946	
	(0.0073)	(0.4566)	(0.0197)	
APExE	0.5795	7.2001	34.1927	
	(0.0712)	(3.6640)	(1.4477)	
WBXII	0.0031	3.0248	1.6987	0.5929
	(0.0051)	(1.1277)	(0.6237)	(0.2312)
ExPW	0.0024	4.2626		3.5438
	(0.0028)	(0.5949)		(0.0035)
W	6.9202	3.8728		
	(0.2947)	(0.5176)		

Table 7 shows the goodness-of-fit measures for the distributions. Table 7 also shows the p-values of the AD and KS test statistics. The results from the Table 7 show that the APMW distribution has the least of all the goodness-of-fit measures, except the Weibull distribution in terms of the BIC measure, and the highest of the AD and KS p-values. Hence, the results indicate that the APMW distribution best describes the data set.

The histogram of the data set with the estimated densities of the distributions are shown in Fig. 10. It can be observed that the APMW distribution best estimates the data set.

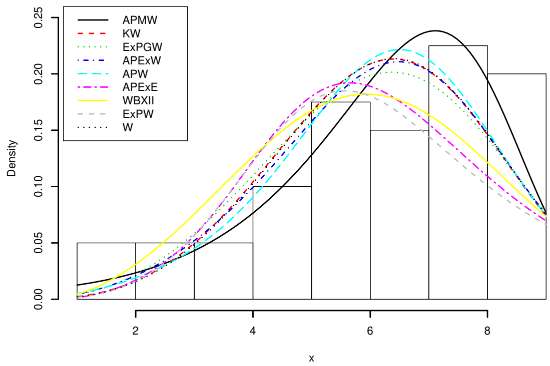

Fig. 10: Plots of empirical and fitted densities for first data set

Fig. 11 shows the Probability-Probability (P-P) plots of the distributions. Fig. 10 and Fig. 11 show that the APMW distribution best describes the data as the expected and observed probabilities cluster more along the diagonal.

Fig. 12 shows the profile log-likelihood plots of the estimated parameters of the APMW distribution. The intersection of the vertical and horizontal lines indicate the parameter estimates on Fig. 12. It can be observed that the estimated parameters are the maxima for the first data set.

7.2 Data Set 2: Remission Times of Bladder Cancer Patients

The second data set consist of remission times (in months) of a random sample of 128 bladder cancer patients reported [29]. The data is given as follows: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22,

Table 7: Goodness-of-fit Measures for first data

Distribution	AIC	BIC	AD		KS	
			Statistic	p-value	Statistic	p-value
APMW	168.2466	175.0021	0.2780	0.9535	0.0881	0.9155
KW	172.8478	179.6033	0.6768	0.5775	0.1110	0.7083
ExPGW	172.2541	179.0096	0.6562	0.5955	0.1113	0.7051
APExW	172.2289	178.9844	0.5330	0.7124	0.1017	0.8020
APW	170.3263	175.3930	0.5194	0.7260	0.0989	0.8288
APExE	179.9017	184.9683	1.0723	0.3210	0.1308	0.5003
WBXII	175.5777	182.3332	1.0752	0.3197	0.1220	0.5914
ExPW	184.0918	189.1585	1.2824	0.2380	0.1665	0.2172
W	168.9510	172.3288	0.6583	0.5937	0.1077	0.7426

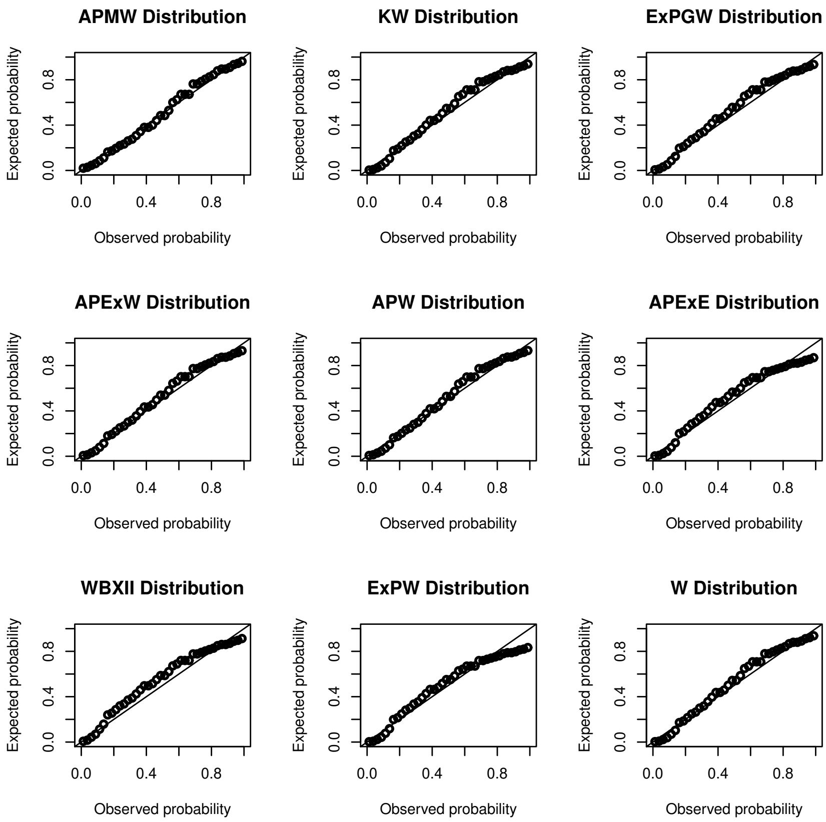

Fig. 11: P-P plots of fitted distributions for first data set

13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 1.46, 18.10, 11.79, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 13.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 12.07, 6.76, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

The histogram and the TTT plot of the second data set are shown in Fig. 13. The shape of the histogram, shown in Fig. 13 (a), indicates that the data set is positively skewed. The TTT plot in Fig. 13 (b) shows that its shape is concave and then convex, which indicates that the shape of the failure rate of the data set is upside down bathtub. Again, these characteristics show that the APMW distribution is suitable for modelling the data.

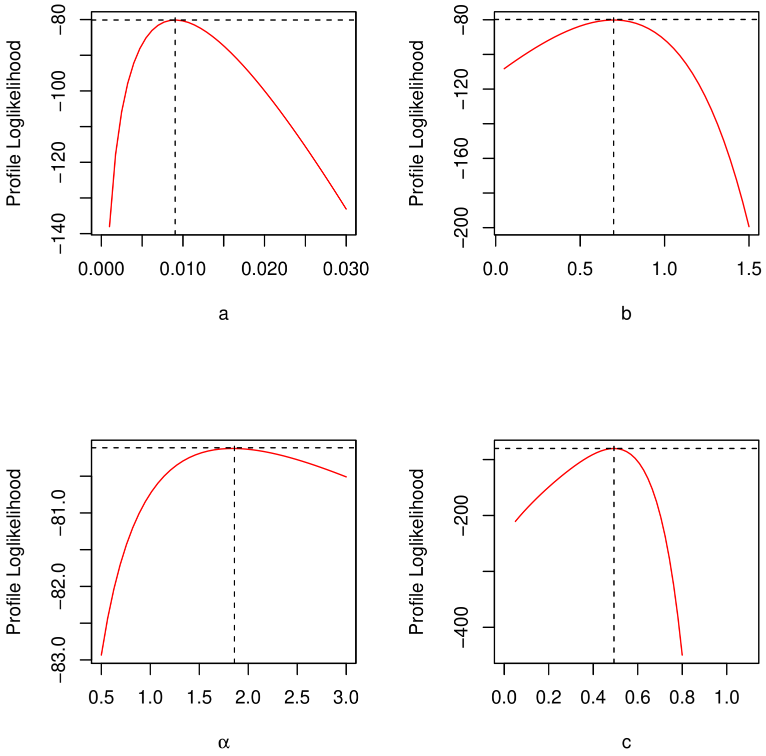

Fig. 12: Profile log-likelihood plots for APMW estimated parameters for first data set

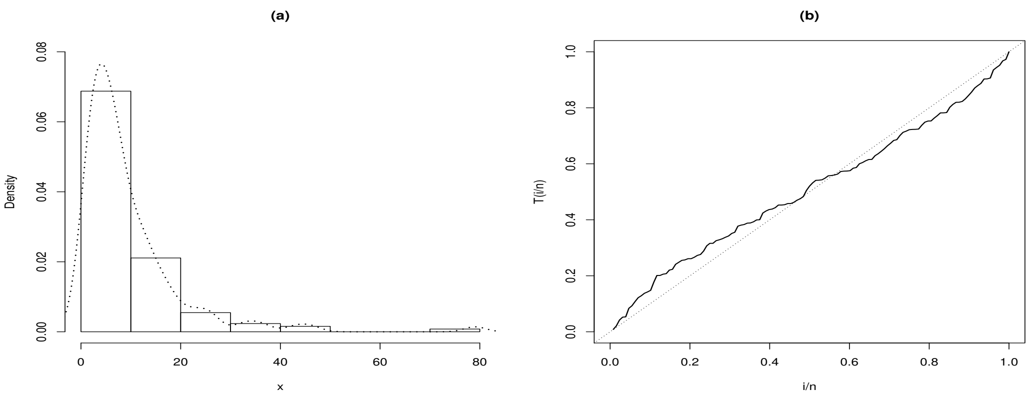

Fig. 13: (a) Histogram and (b) TTT plot for second data set

Table 8 gives the parameter estimates of all the distributions with their corresponding standard errors in parenthesis.

Table 9 presents the goodness-of-fit measures for the APMW and the competing distributions. Table 9 shows that, APMW distribution has the least in terms of AD and KS measures and the highest of their p-values. However, the APW distribution has the least statistic in terms of the AIC and BIC measures. The results show that the APMW distribution competitively describes the data set.

Fig. 14 shows the histogram and estimated densities of the estimated distributions. It can be observed that the

Table 8: Parameter Estimates of Distributions for Second Data Set

Distribution	a	b	α	c
APMW	0.0178 (0.0126)	1.3623 (0.1452)	0.0329 (0.0893)	0.0069 (0.0088)
KW	3.5790 (3.9282)	2.4525 (5.6202)	0.1973 (0.2138)	0.5077 (0.4428)
ExPGW	0.0799 (0.0211)	6.8417 (3.1827)	9.5559 (0.2808)	0.2806 (0.0588)
APExW	0.0632 (0.0861)	0.9532 (0.3283)	0.0195 (0.0496)	1.5088 (0.7355)
APW	0.0167 (0.0086)	1.2726 (0.0864)	0.0176 (0.0343)	
APExE	0.0534 (0.0298)	1.4154 (0.1624)	0.0210 (0.0482)	
WBXII	1.7374 (3.2358)	0.8455 (0.3128)	2.1844 (0.6472)	0.2869 (0.1493)
ExPW	0.0297 (0.0082)	1.6988 (0.1267)		0.4011 (0.0007)
W	9.6582 (0.8574)	1.0528 (0.0680)		

APMW distribution competes favourably with the other distributions.

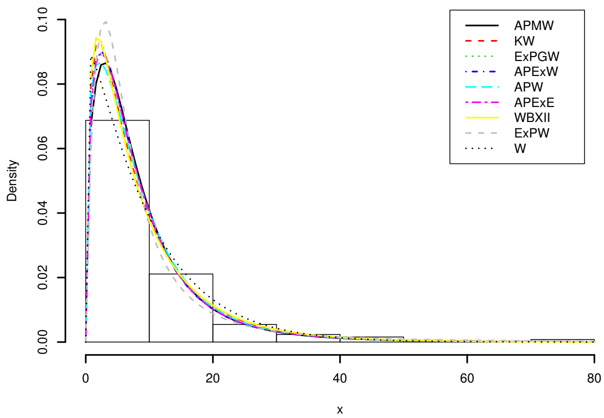

Fig. 14: Plots of empirical and fitted densities for second data set

Fig. 15 shows the P-P plots for the second data set for all the fitted distributions. The plots confirm that all the distributions best describe the data set.

Fig. 16 shows the profile log-likelihood plots of the estimated parameters of the APMW distribution for the second data set. The intersection of the vertical and horizontal lines on Fig. 16 indicate the parameter estimates. Again, tt can be observed that the estimated parameters are the maxima for the second data set.

8 Bivariate Extension

In this section, a bivariate extension of the APMW distribution is presented. Let $G_{X_1}(x_1)$ and $G_{X_2}(x_2)$ be marginal distribution functions of X_1 and X_2, respectively. Let (X_1,X_2) be a pair of random variables, a joint distribution function can be defined using a copula associated with the pair. If K is a copula associated with the pair (X_1,X_2), the joint distribution function is given by

$$G_{X_1,X_2}(x_1,x_2) = K(G_{X_1}(x_1), G_{X_2}(x_2)).$$

In this article, we use Clayton copula [30] which is defined as

$$K(m,n) = \left[m^{-\theta}+n^{-\theta}-1\right]^{-\frac{1}{\theta}}, \qquad \theta \geq 0.$$

Thus, the joint distribution of the random pair (X_1,X_2) is given as

$$G_{X_1,X_2}(x_1,x_2) = \left[(G_{X_1}(x_1))^{-\theta}+(G_{X_1}(x_1))^{-\theta}-1\right]^{-\frac{1}{\theta}}, \theta \geq 0.$$

This gives the joint bivariate distribution as

$$G_{X_1,X_2}(x_1,x_2) = \begin{cases} \left[\sum_{i=1}^{2}\frac{1}{1-\alpha_i}\left(1-\alpha_i^{1-e^{-a_i x_i^{b_i} e^{c_i x_i}}}\right)^{-\theta}-1\right]^{-\frac{1}{\theta}}, \\ \qquad \alpha_i > 0, \alpha_i \neq 0, i=1,2 \\ \left[\sum_{i=1}^{2}\left(1-e^{-a_i x_i^{b_i} e^{c_i x_i}}\right)^{-\theta}-1\right]^{-\frac{1}{\theta}}, \\ \qquad \alpha_i = 1, i=1,2 \end{cases}, \tag{21}$$

where $a_i > 0, b_i \geq 0, c_i > 0, i=1,2$ are marginal parameters. Also, the joint density function is given by

$$g_{X_1,X_2}(x_1,x_2) = \begin{cases} \left[\sum_{i=1}^{2}\left(\frac{a_i(b_i+c_i x_i)x_i^{b_i-1}\log\alpha_i}{(\alpha_i-1)e^{a_i x_i^{b_i} e^{c_i x_i}-c_i x_i}}\alpha_i^{1-e^{-a_i x_i^{b_i} e^{c_i x_i}}}\right) \right. \\ \quad \left. \times\left(\frac{1-\alpha_i}{1-\alpha_i^{1-e^{-a_i x_i^{b_i} e^{c_i x_i}}}}\right)^{\theta+1}\right] \\ \quad \times\left[\sum_{i=1}^{2}\frac{\left(1-\alpha_i^{1-e^{-a_i x_i^{b_i} e^{c_i x_i}}}\right)^{-\theta}}{1-\alpha_i}-1\right]^{-\frac{1}{\theta}-1}, \\ \qquad \alpha > 0, \alpha \neq 1, \\ \left[\sum_{i=1}^{2}\left(a_i(b_i+c_i x_i)x_i^{b_i-1}e^{c_i x_i-a_i x_i^{b_i} e^{c_i x_i}}\right)\right. \\ \quad \left.\times\left(1-e^{c_i x_i-a_i x_i^{b_i} e^{c_i x_i}}\right)^{-\theta-1}\right] \\ \quad \times\left[\sum_{i=1}^{2}\left(1-1-e^{-a_i x_i^{b_i} e^{c_i x_i}}\right)^{-\theta}\right]^{-1\frac{1}{\theta}-1}, \\ \qquad \alpha = 1. \end{cases} \tag{22}$$

Table 9: Goodness-of-fit Measures for Second Data Set

Distribution	AIC	BIC	AD		KS	
			Statistic	p-value	Statistic	p-value
APMW	828.0655	839.0736	0.1182	0.9998	0.0329	0.9991
KW	831.6745	843.0826	0.2364	0.9771	0.0424	0.9757
ExPGW	832.2547	843.6628	0.2801	0.9522	0.0440	0.9651
APExW	830.0074	841.4155	0.1253	0.9997	0.0353	0.9972
APW	828.7486	837.3047	0.2102	0.9874	0.0435	0.9690
APExE	828.2043	836.5804	0.1289	0.9996	0.0356	0.9969
WBXII	832.5345	843.9426	0.3127	0.9282	0.0472	0.9381
ExPW	831.9942	840.5503	0.2795	0.9526	0.0417	0.9791
W	834.1968	839.9009	0.8743	0.4302	0.0663	0.6272

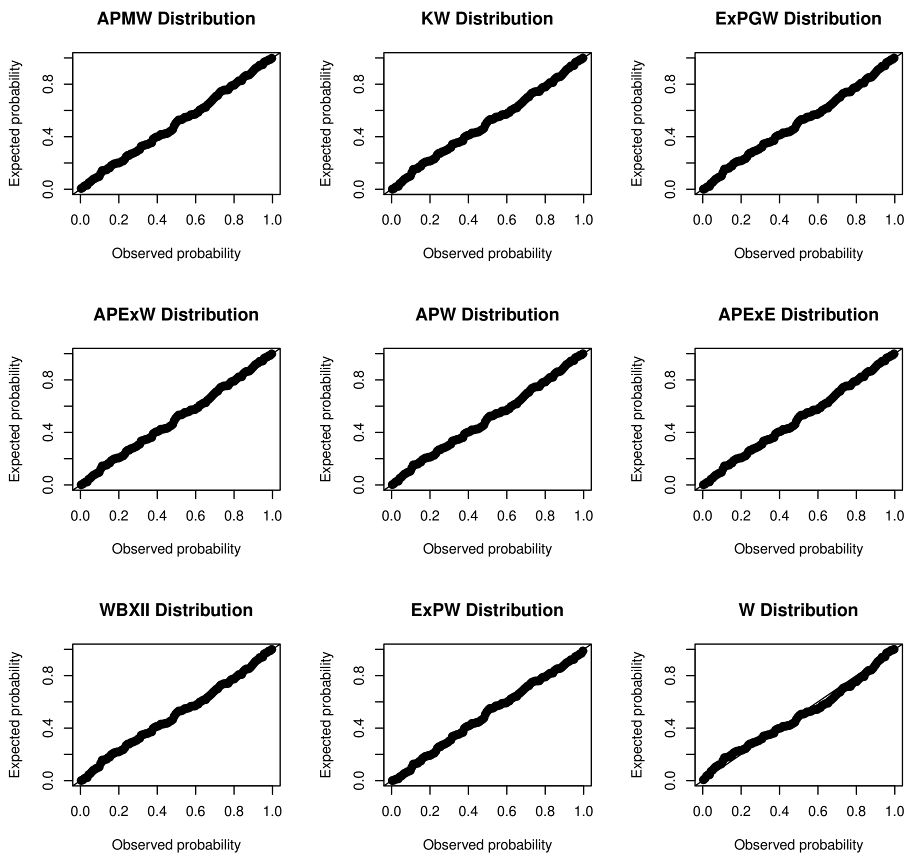

Fig. 15: P-P plots of fitted distributions for second data set

The conditional density functions can be obtained as

$$g_{X_1,X_2}(x_1 \mid X_2 = x_2) = \frac{g_{X_1,X_2}(x_1,x_2)}{g_{X_2}(x_2)}$$

and

$$g_{X_1,X_2}(x_2 \mid X_1 = x_1) = \frac{g_{X_1,X_2}(x_1,x_2)}{g_{X_1}(x_1)}$$

.

The main focus of this article was on the univariate case of APMW distribution. Hence, we did not treat into details the bivariate case. However, the bivariate case will be explored in a future article.

9 Conclusion

In this paper, alpha power modified Weibull distribution is proposed and studied. The new distribution generalizes

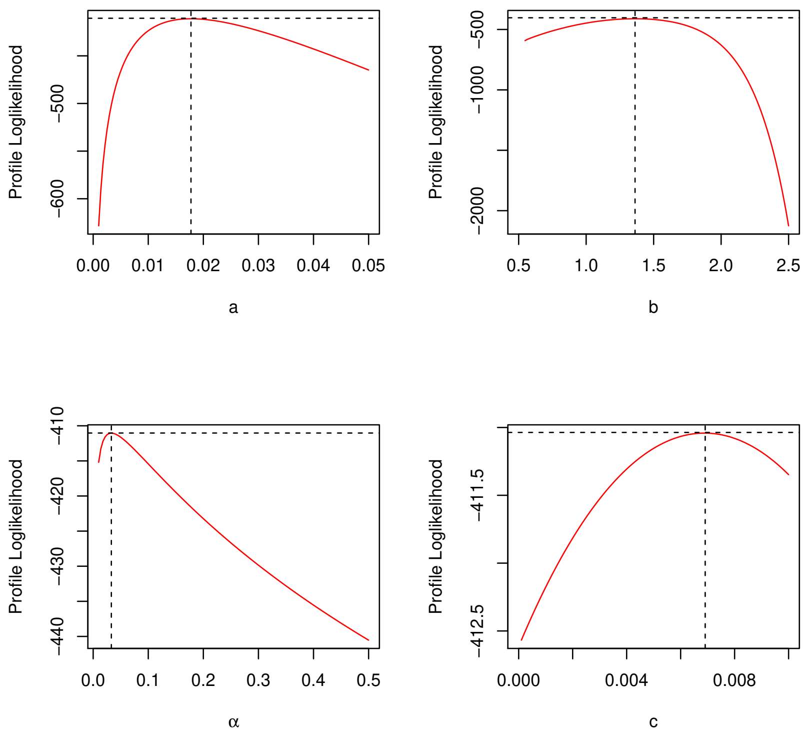

Fig. 16: Profile log-likelihood plots for APMW estimated parameters for second data set

several well known distributions. From the shapes of the hazard rate function, the new distribution can model data sets with decreasing, increasing, bathtub, modified bathtub shapes. Several properties of the distribution are studied and various estimation methods used to estimate the parameters of the distribution. It was shown that the estimators are consistent via a simulation study. Also, some actuarial measures were derived and a simulation study conducted on them. A bivariate extension of the distribution is also derived in the study. The usefulness of the distribution is ascertained by using it to model two real lifetime data sets. The results show that the new distribution can serve as an alternative to modeling lifetime data.

10 Future Work

The article extensively considered the univariate case of the APMW distribution with applications to complete data sets. Future work would consider the application of the distribution to censored data. It would also consider the development of cure rate models using APMW distribution to describe the susceptible group of a population.

Again, a bivariate extension of the APMW is introduced in the current study. In future work, a general multivariate extension would be considered. The properties of the distribution would be studied and applications to real multivariate data sets considered.

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Conflict of interest

The authors declare no conflicts of interest.

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