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A New Family of Transmuted Distribution with Application to Lifetime Data

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Abstract: In this article, a new family of the perspectives group of transmuted distributions called the MH-Transmuted Family of Distributions (MH-TD) is developed. MH-TD is confirmed in the quadratic (MH-QTD) and cubic (MH-CTD) distribution families. The new proposed family increases the flexibility of the transmuted distributions, enabling the modeling of diverse data sets in different fields of sciences, including engineering, environment, and finance. Based on the MH-QTD and MH-CTD maps, some new distributions are developed, including MH-Quadratic and MH-Cubic Transmuted Pareto distributions, MH-Quadratic and MH-Cubic Transmuted Gambel distributions, and MH-Quadratic and MH-Cubic Transmuted Fréchet distributions, explaining the probability density and distribution function for each distribution. Moreover, based on the MH-QTD and MH-CTD maps, two new generalizations of the exponential distributions called MH-quadratic transmuted exponential (MH-QTED) and MH-cubic transmuted exponential (MH-CTED) are proposed and studied in detail to explain the utility of the proposed family MH-Transmuted. Several statistical characteristics of the MH-QTED and MH-CTED are discussed, including the distribution and density functions, the moment-generating function, and the use of the maximum likelihood method to estimate the distributions' parameters. Finally, the MH-QTED and MH-CTED models are fitted to a real-world, right-skewed dataset to determine their applicability.

Keywords: Quadratic and Cubic Transmuted, Pareto distribution, Gumbel distribution, Fréchet Distribution , Exponential distribution

1 Introduction

There are a lot of different ways to come up with new distributions by starting with some baseline distributions. For instance, the exponentiated family of distributions introduced by Gupta et al. [1] to model the data of failure time by $F(\tau) = K^{\theta}(\tau)$ where $F(\tau)$ and $G(\tau)$ are the proposed and baseline distribution functions, respectively, and θ is a positive real number. Eugene et al. [2] developed a general class of distributions generated from the logit of the beta random variable. The probability density function of the generalized class of distribution is

$$f(\tau) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} K^{\alpha-1}(\tau)[1 - K^{\beta-1}(\tau)K'(\tau)]$$

$$\alpha > 0 \ , \ \beta < \infty$$

If both α and β are whole numbers, then the probability distribution looks the same as the order statistics of the random variable τ.
Using the Kumaraswamy distribution's cumulative distribution function, Cordeiro and Castro [3] have introduced a new family of distributions called the Kw-G. The cdf of the Kw-G distribution is defined by

$$F(\tau) = 1 - \left[1 - K^{a}(\tau)\right]^{b}; \quad a, b > 0$$

A different idea for obtaining a new distribution involves transforming the baseline distribution; this method never contains any new parameters other than the parameter(s) included in the baseline distribution. For example, Kumar et al. [4]

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proposed the DUS transformation; its pdf is formulated as

$$f(\tau) = \frac{1}{1-e}\ g(\tau)\ e^{G(\tau)}$$

The SS transformation was introduced by Kumar et al. [5], using the sine function; the cdf of this method is

$$F(\tau) = \sin\left[\frac{\pi}{2}K(\tau)\right]$$

The MG transmutation map was presented by Kumar et al. [6]; the cdf of this form meets the following relationship

$$F(\tau) = \exp\left\{1 - \frac{1}{K(\tau)}\right\}$$

Shaw and Buckley [7] suggested another method of generating family of distributions; this method used a quadratic ranking transmutation map (QRT) to generate new distributions using any baseline distribution; the cdf of QRT is:

$$F(\tau) = \big(1+\nu\big)K(\tau) - \nu K^2(\tau)$$

where $\nu \in [-1,1]$ is the transmuted parameter.
General formula of the QRT was developed by Abed Al-Kadim [8]; the cdf of the cubic version is

$$F(\tau) = \big(1+\nu\big)K(\tau) - 2\nu K^2(\tau) + \nu K^3(\tau)$$

Two other classes of cubic transmuted distributions with two transmuted parameters were introduced based on the QRT, one by Granzottoa et al. [9] with the cdf of the form

$$F(\tau) = \nu_1 K(\tau) + \big(\nu_2 - \nu_1\big)K^2(\tau) - \big(1-\nu_2\big)K^3(\tau)$$

where $\nu_1 \in [0,1], \nu_2 \in [-1,1]$
while Rahman et al. [10] proposed the other class, with the cdf of the form

$$F(\tau) = \big(1+\nu_1\big)K(\tau) + \big(\nu_2 - \nu_1\big)K^2(\tau) - \nu_2 K^3(\tau)$$

where $\nu_1 \in [-1,1], \nu_2 \in [-1,1]$ and $-2 \le \nu_1 + \nu_2 \le 1$
Moreover, Rahman et al. ([11] and [12]) introduced the cubic of this transmuted for Weibull and Parito distributions.

This work introduces a new family of transmuted distributions, namely MH-transmuted map distributions. This new family of transmuted distributions provides tractable distributions and is qualified to fit complex data sets, such as ones with left- and right-skewed unimodal shapes. The quadratic and cubic transmutation map distributions of the proposed family are studied by applying well-known lifetime distributions, namely the Pareto, Gumbel, and Fréchet distributions. In order to evaluate the performance of the new transmuted family of distributions, the MH-quadratic and MH-cubic exponential distributions are studied in detail.

The rest of this paper is structured as follows: Section2 presents the proposed MH-family of transmuted distributions. Section 3 discusses some MH-quadratic transmuted distribution examples. Section 4 discusses some examples of MH-cubic transmuted distributions. Section 5 delves deeply into the MH-quadratic and MH-cubic exponential distributions. A real-world application of MH-quadratic and MH-cubic exponential distributions is presented in Section 6. Section 7 concludes with some final thoughts.

2 MH-Transmuted Family of Distributions

In this section, the proposed general transmuted family of distributions that can be used to generate new distributions is explained; this family is called the MH-Transmuted Family of Distributions (MH-TD). Furthermore, the quadratic and cubic versions of the proposed family are also discussed.
Let X be a random variable with a cdf $G(x)$ and a pdf $g(x)$, then a general cdf $F(x)$ of MH-TD is defined as

$$F(x) = G(x)\sum_{i=1}^{k} \lambda_i\ e^{\left[1-G^i(x)\right]} \tag{1}$$

where

- $|\lambda_i e| \leq 1$,for $i = 1,2,3,\ldots,k$
- $\sum_{i=1}^{k} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_k = 0$
- The general transmuted family reduces to the base distribution for $\lambda_i = 0; \forall i = 1,2,\ldots k$
- $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.

The density function (pdf) $f(x)$ corresponding to Eq. (1) is

$$f(x) = g(x)\left\{1 - \sum_{i=1}^{k} i\lambda_i G^{(i-1)}(x) e^{\left[1-G^i(x)\right]}\right\} \tag{2}$$

2.1 MH-Quadratic Transmuted Distributions (MH-QTD)

The MH-quadratic transmuted family distribution (MH-QTD) instance, which is addressed in this subsection, is formed by plugging $k = 2$ into the Eq. (1) and is represented as follows:

$$F(x) = G(x) + \lambda_1 G_1(x) + \lambda_2 G_2(x) \tag{3}$$

where

- $|\lambda_i e| \leq 1$, for $i = 1,2$
- $\lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_1 = -\lambda_2$
- $G_1(x) = e^{\left[1-G(x)\right]}$ and $G_2(x) = e^{\left[1-G^2(x)\right]}$

Now, if we take $\lambda = \lambda_1 = -\lambda_2$ and $|\lambda e| \leq 1$, we get

$$F(x) = G(x) + \lambda\left(G_1(x) - G_2(x)\right) \tag{4}$$

and the pdf is

$$f(x) = g(x)\left(1 - \lambda\left\{G_1(x) - 2G(x)G_2(x)\right\}\right) \tag{5}$$

2.2 MH-Cubic Transmuted Distributions (MH-CTD)

The MH-cubic transmuted family distribution (MH-CTD) situation, which is studied in this subsection, is generated by setting $k = 3$ in Eq. (1) and is shown as follows:

$$F(x) = G(x) + \lambda_1\ G_1(x) + \lambda_2\ G_2(x) + \lambda_3\ G_3(x) \tag{6}$$

where

- ☐ $|\lambda_i e| \leq 1$ for $i = 1,2,3$
- ☐ $\lambda_1 + \lambda_2 + \lambda_3 = 0$ which means that $\lambda_3 = -[\lambda_1 + \lambda_2]$
- ☐ $G_3(x) = e^{\left[1-G^3(x)\right]}$

$$\begin{aligned} F(x) = &G(x) + \lambda_1\left[G_1(x) - G_3(x)\right] \\ &+ \lambda_2\left[G_2(x) - G_3(x)\right] \end{aligned} \tag{7}$$

and pdf is

$$f(x) = g(x)\ \ \zeta(x; \lambda_1, \lambda_2) \tag{8}$$

where

$$\begin{aligned} &|\lambda_i e| \leq 1 \text{ for } i = 1,2 \\ \zeta(x;\lambda_1,\lambda_2) &= 1 + \lambda_1\left[3G^2(x)G_3(x) - G_1(x)\right] \\ &+ \lambda_2\left[3G^2(x)G_3(x) - 2G(x)G_2(x)\right] \end{aligned} \tag{9}$$

3 Some Examples of (MH-QTD)

We explain some members of the MH-quadratic transmuted family of distributions given in Eqs.(4 , 5) for various baselines $G(x)$ and $g(x)$ in this section. Distributions such like Pareto, Gumbel, and Fréchet are examples.

3.1 MH-Quadratic Transmuted Pareto Distribution (MH-QTPD)

Assume that the random variable X follows the Pareto distribution. Hence, the pdf and cdf are respectively given as

$$\begin{aligned} G(x) =& 1-\left(\frac{m}{x}\right)^{\theta} = 1-p = G(p) \ ; \\ & \theta, m > 0, x \geq m \ \text{ and } \ p = \left(\frac{m}{x}\right)^{\theta} \end{aligned} \tag{10}$$

and

$$g(x) = \frac{\theta m^{\theta}}{x^{(\theta+1)}} = \left(\frac{\theta}{m}\right) \sqrt[\theta]{p^{(\theta+1)}} \tag{11}$$

where m is the minimum value of x, and must be positive, while θ is a shape parameter.

Remark 3.1 Consider the random variable $P = u(X) = \left(\frac{m}{X}\right)^{\theta}$ then for $(x = m) \Rightarrow (p = 1)$ and for $(x \to \infty) \Rightarrow (p \to 0)$ also $\frac{dx}{dp} = \left[\frac{-m}{\theta \sqrt[\theta]{p^{(\theta+1)}}}\right]$. Hence, $f(x)$ is expressed in term of the random variable P as follows

$$\begin{aligned} f(p) =& f[w(p)] \frac{dx}{dp} \\ =& \left(\frac{\theta}{m}\right) \sqrt[\theta]{p^{(\theta+1)}} \left(1-\lambda\left\{G_1(p)-2(1-p)G_2(p)\right\}\right) \\ & \left[\frac{-m}{\theta \sqrt[\theta]{p^{(\theta+1)}}}\right] \\ =& (-1)\left(1-\lambda\left\{G_1(p)-2(1-p)G_2(p)\right\}\right); \text{if} p \in [0,1] \end{aligned}$$

Theorem 3.1 Consider the random variable X with the MH-QTPD, then

(A_{311}) The corresponding cdf and pdf are respectively given as

$$F(x) = (1-p) + \lambda \ \left[G_1(p) - G_2(p)\right] \tag{12}$$

and

$$\begin{aligned} f(x) =& \left(\frac{\theta}{m}\right) \ \sqrt[\theta]{p^{(\theta+1)}} \\ & \left\{1-\lambda\left[G_1(p)-2(1-p) \ G_2(p)\right]\right\} \end{aligned} \tag{13}$$

where $|\lambda \ \ e| \leq 1$ is a transmuted parameter.

(A_{312}) $f(x)$ of Eq.(12), is a pdf.

Proof.

(A_{311})The proof is easy to see. Eq.(12) is achieved by substituting Eq.(10) into Eq.(4) and Eq.(13) is obtained by substituting Eq.(10) and Eq.(11) into Eq.(5) or differentiating Eq.(12) with respect to x.

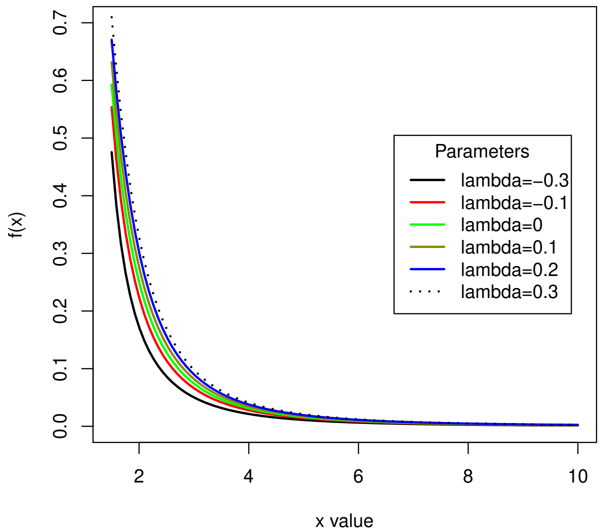

Fig. 1: The pdf of MH-QTPD for various values of parameter λ with $\theta = 2$ and $m = 1$ where $\mu = 2$ and $\beta = 2$.

(A_{312})To show that $f(x)$ is pdf , we must prove that $f(x) \geq 0$ and $\int_m^\infty f(x)dx = 1$. the proof of $f(x) \geq 0$ is deduced from the following two limits

$$\lim_{x\to\infty} f(x) = \lim_{p\to 0} f(p) = \left(\frac{\theta}{m}\right) \lim_{p\to 0} \sqrt[\theta]{p^{(\theta+1)}} \left\{1-\lambda\left[G_1(p) - 2(1-p)\ G_2(p)\right]\right\} = 0$$

$$\lim_{x\to m} f(x) = \lim_{p\to 1} f(p) = \left(\frac{\theta}{m}\right) \lim_{p\to 1} \sqrt[\theta]{p^{(\theta+1)}} \left\{1-\lambda\left[G_1(p) - 2(1-p)\ G_2(p)\right]\right\} = \frac{\theta(1-\lambda e)}{m}$$

since $\theta, m > 0,\ \ |\lambda e| \leq 1,$ then $\theta(1-\lambda e) > 0$. while the proof of $\int_m^\infty f(x)dx = 1$ is given below

$$\begin{aligned}\int_m^\infty f(x)dx &= \int_1^0 f(p)\ dp \\ &= \int_0^1 \left\{1-\lambda\left[G_1(p) - 2(1-p)\ G_2(p)\right]\right\}dp \\ &= \int_0^1 \left(1-\lambda\left\{e^p - 2(1-p)e^{\left[1-(1-p)^2\right]}\right\}\right)dp \\ &= 1-\lambda(e-1)+\lambda(e-1) = 1\end{aligned}$$

Therefore, the theorem is proved.
Some shapes of the pdf and cdf of MH-QTPD for selected values of λ at $\theta = 2$ and $m = 1$ are illustrated in Figure (1) and Figure (2) respectively.

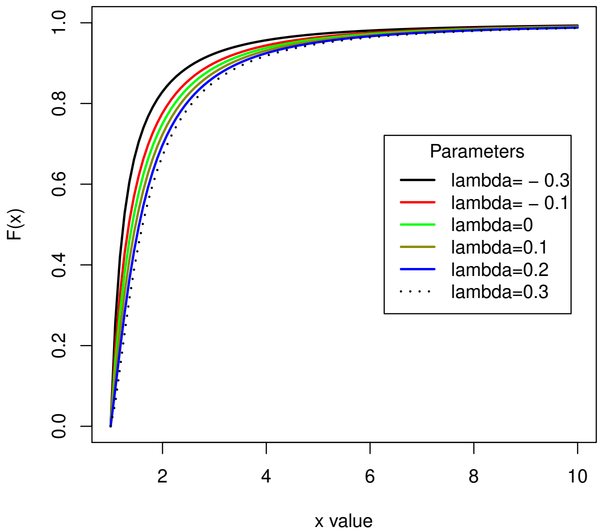

Fig. 2: The cdf of MH-QTPD for various values of parameter λ with $\theta = 2$ and $m = 1$ where $\mu = 2$ and $\beta = 2$.

3.2 MH- Quadratic Transmuted Gumbel Distribution (MH-QTGD)

Assume that X is a random variable with the Gumbel distribution. As a result, the pdf and cdf are respectively defined as

$$G(x) = e^{-e^{-\left(\dfrac{x-\mu}{\sigma}\right)}} = e^{-z} = G(z); \qquad (14)$$
$$\sigma \geq 0; x, \mu \in \Re \text{ and } z = e^{-\left(\dfrac{x-\mu}{\sigma}\right)}$$

and

$$g(x) = \left(\frac{1}{\sigma}\right) e^{-\left[\left(\dfrac{x-\mu}{\sigma}\right) + e^{-\left(\dfrac{x-\mu}{\sigma}\right)}\right]} = \left(\frac{z}{\sigma}\right) e^{-z} \qquad (15)$$

where σ and μ are the scale and location parameters respectively.

Remark 3.2 Consider the random variable $Z = u(X) = e^{-\left(\dfrac{X-\mu}{\sigma}\right)}$ then for $(x \to -\infty) \Rightarrow (z \to \infty)$ and for $(x \to \infty) \Rightarrow (z \to 0)$ also $\frac{dx}{dz} = \left(\dfrac{-\sigma}{z}\right)$. Then, in terms of the random variable Z, $f(x)$ may be rewritten as follows.

$$\begin{aligned} f(z) =& f[w(z)]\frac{dx}{dz} \\ =& \left(\frac{z}{\sigma}\right)e^{-z}\left\{1-\lambda\left[G_1(z)-2e^{-z}G_2(z)\right]\right\}\left(\frac{-\sigma}{z}\right) \\ =& (-e^{-z})\left\{1-\lambda\left[G_1(z)-2e^{-z}G_2(z)\right]\right\} \quad \text{if} z \in [0,\infty) \end{aligned}$$

Theorem 3.2 Suppose the random variable X follows the MH-QTGD. Then

(A_{321}) The cdf and pdf respectively are defined as

$$F(x) = e^{-z} + \lambda\left[G_1(z) - G_2(z)\right] \qquad (16)$$

and

$$f(x) = \left(\frac{z}{\sigma}\right)\left\{e^{-z} - \lambda\left[e^{-z}G_1(z) - 2e^{-2z}G_2(z)\right]\right\} \tag{17}$$

where $|\lambda| \leq \left(\frac{1}{e}\right)$ is a transmuted parameter.

(A_{322}) $f(x)$ of Eq.(16), is a pdf.

Proof.

(A_{321})The proof is easy to see . Eq.(16) is achieved by substituting Eq.(14) into Eq.(4) and Eq.(17) is obtained by substituting Eq.(14) and Eq.(15) into Eq.(5).

(A_{322})To show that $f(x)$ is pdf , we must prove that

$$f(x) \geq 0 \text{and} \int_{-\infty}^{\infty} f(x)dx = 1$$

The proof of $f(x) \geq 0$ is derived from the two limits below,

$$\lim_{x\to\infty} f(x) = \lim_{z\to 0} f(z) = \left(\frac{1}{\sigma}\right)\lim_{z\to 0} z$$
$$\left\{e^{-z} - \lambda\left[e^{-z}G_1(z) - 2e^{-2z}G_2(z)\right]\right\} = 0$$

whilst from L'Hopital's rule we get

$$\lim_{x\to -\infty} f(x) = \lim_{z\to\infty} f(z) = \left(\frac{1}{\sigma}\right)\lim_{z\to\infty} z$$
$$\left\{e^{-z} - \lambda\left[e^{-z}G_1(z) - 2e^{-2z}G_2(z)\right]\right\} = 0$$

Moreover, the Proof of $\int_{-\infty}^{\infty} f(x)dx = 1$ is as follows

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_{\infty}^{0} f(z)dz = \\ &= \int_{0}^{\infty}\left\{e^{-z} - \lambda\left[e^{-z}G_1(z) - 2e^{-2z}G_2(z)\right]\right\}dz \\ &= \int_{0}^{\infty}\left\{e^{-z} - \lambda\left[e^{-z}e^{1-e^{-z}} - 2e^{-2z}e^{\left(1-e^{-2z}\right)}\right]\right\}dz \\ &= 1 - \lambda(e-1) + \lambda(e-1) = 1\end{aligned}$$

With this, the theorem is proved.
Some shapes of the pdf and cdf of MH-QTGD for selected values of λ at $\mu = 2$ and $\sigma = 2$ are illustrated in Figure (3) and Figure (4) respectively.

3.3 MH-Quadratic Transmuted Fréchet Distribution (MH-QTFD)

Assume that X is a random variable with the Fréchet distribution. As a result, the pdf and cdf are respectively represented as

$$\begin{gathered}G(x) = e^{-\left(\frac{s}{x}\right)^{\alpha}} = e^{-y} = G(y); \\ x > 0 \quad \text{and} \quad y = \left(\frac{s}{x}\right)^{\alpha}\end{gathered} \tag{18}$$

and

$$g(x) = \left(\frac{\alpha}{s}\right)\left(\frac{s}{x}\right)^{\alpha+1} e^{-\left(\frac{s}{x}\right)^{\alpha}} = \left(\frac{\alpha}{s}\right)\sqrt[\alpha]{y^{(\alpha+1)}}e^{-y} \tag{19}$$

where $\alpha, s \in (0,\infty)$ are the scale and shape parameters respectively.

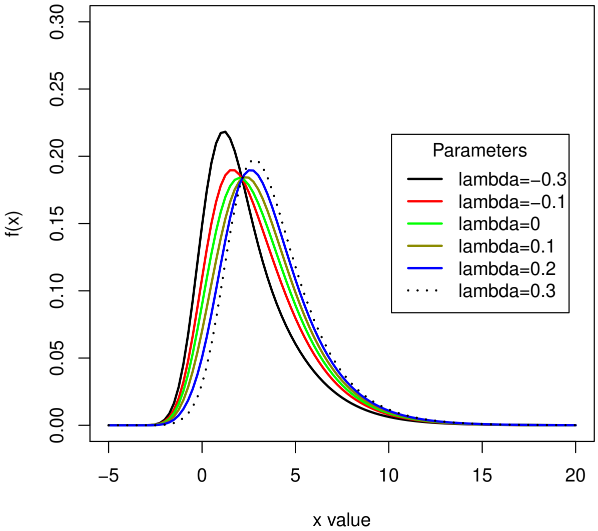

Fig. 3: The pdf of MH-QTGD for various values of parameter λ setting $\mu = 2$ and $\sigma = 2$.

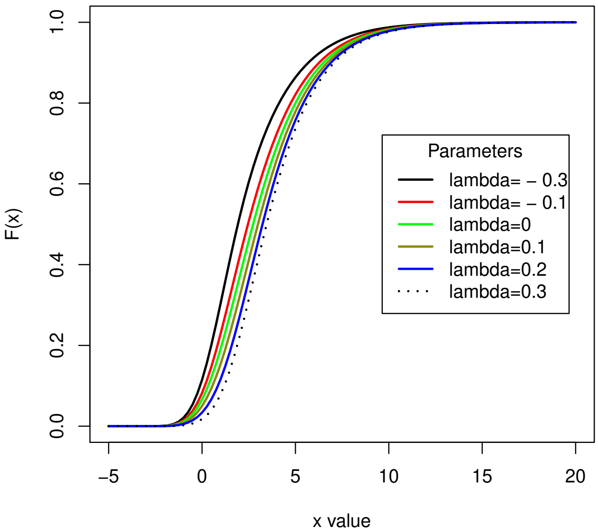

Fig. 4: The cdf of MH-QTGD for various values of parameter λ setting $\mu = 2$ and $\sigma = 2$.

Remark 3.3 Consider the random variable $Y = u(X) = (\frac{s}{X})^{\alpha}$ then for $(x \to 0) \Rightarrow (y \to \infty)$ and for $(x \to \infty) \Rightarrow (y \to 0)$ also $\frac{dx}{dy} = \left(\frac{-s}{\alpha \sqrt[\alpha]{y^{(\alpha+1)}}}\right)$.

Then, in term of the random variable Y, $f(x)$ may be rewritten as follows

$$\begin{aligned} f(y) =& f[w(y)]\frac{dx}{dy} \\ =& (\frac{\alpha}{s})\sqrt[\alpha]{y^{(\alpha+1)}}e^{-y}\left\{1-\lambda\left[G_1(y)-2e^{-y}G_2(y)\right]\right\} \\ & \left(\frac{-s}{\alpha\sqrt[\alpha]{y^{(\alpha+1)}}}\right) \\ =& -e^{-y}\left\{1-\lambda\left[G_1(y)-2e^{-y}G_2(y)\right]\right\} \quad \text{if} y\in[0,\infty) \end{aligned}$$

Theorem 3.3 Assume that X is a random variable with the (MH-QTFD), then

(A_{331}) The corresponding cdf and pdf respectively are given as

$$F(x) = e^{-y} + \lambda\left[G_1(y) - G_2(y)\right]; |\lambda e| \leq 1 \tag{20}$$

and

$$\begin{aligned} f(x) =& (\frac{\alpha}{s})\ \sqrt[\alpha]{y^{(\alpha+1)}}\ e^{-y} \\ & \left\{1-\lambda\left[G_1(y)-2e^{-y}G_2(y)\right]\right\} \end{aligned} \tag{21}$$

(A_{332}) $f(x)$ of Eq.(21), is a pdf.

Proof.

(A_{331})The proof is easy to see. Eq.(20) is achieved by substituting Eq.(18) into Eq.(4) and Eq.(21) is achieved by substituting Eq.(18) and Eq.(19) into Eq.(5) .

(A_{332})To show that $f(x)$ is pdf , we must prove that $f(x)\geq 0$ and $\int_0^\infty f(x)dx = 1$. The proof of $f(x)\geq 0$ comes from the limits below

$$\begin{aligned} \lim_{x\to\infty} f(x) =& \lim_{y\to 0}(\frac{\alpha}{s})\sqrt[\alpha]{y^{(\alpha+1)}}e^{-y} \\ & \left\{1-\lambda\left[G_1(y)-2e^{-y}G_2(y)\right]\right\} = 0 \end{aligned}$$

whilst from L'Hopital's rule

$$\begin{aligned} \lim_{x\to 0} f(x) =& \lim_{y\to\infty}(\frac{\alpha}{s})\sqrt[\alpha]{y^{(\alpha+1)}}e^{-y} \\ & \left\{1-\lambda\left[G_1(y)-2e^{-y}G_2(y)\right]\right\} = 0 \end{aligned}$$

Moreover, the following is the Proof of $\int_0^\infty f(x)dx = 1$

$$\begin{aligned} \int_0^\infty f(x)dx =& \int_\infty^0 f(y)dy \\ =& \int_0^\infty e^{-y}\left\{1-\lambda\left[G_1(y)-2e^{-y}G_2(y)\right]\right\}dy \\ =& \int_0^\infty e^{-y}\left\{1-\lambda\left[e^{1-e^{-y}}-2e^{-y}e^{\left(1-e^{-2y}\right)}\right]\right\}dy \\ =& 1-\lambda(e-1)+\lambda(e-1) = 1 \end{aligned}$$

This concludes the theorem's proof.
Some shapes of the pdf and cdf of MH-QTFD for selected values of λ at $\alpha = 2$ and $s = 2$ are illustrated in Figure (5) and Figure (6) respectively.

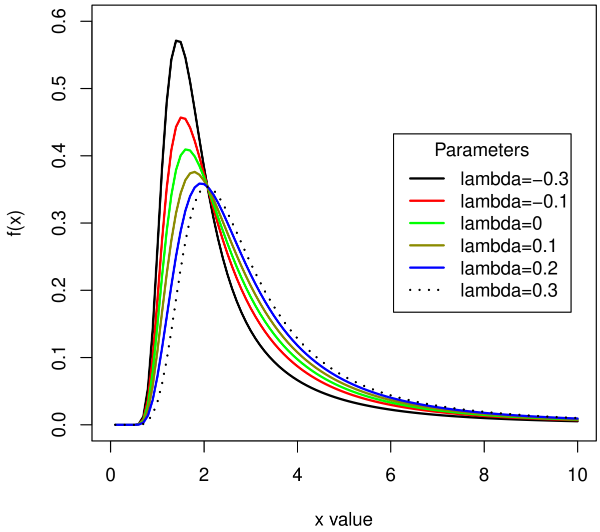

Fig. 5: The pdf of MH-QTFD for various values of parameter λ with $\alpha = 2$ and $s = 2$.

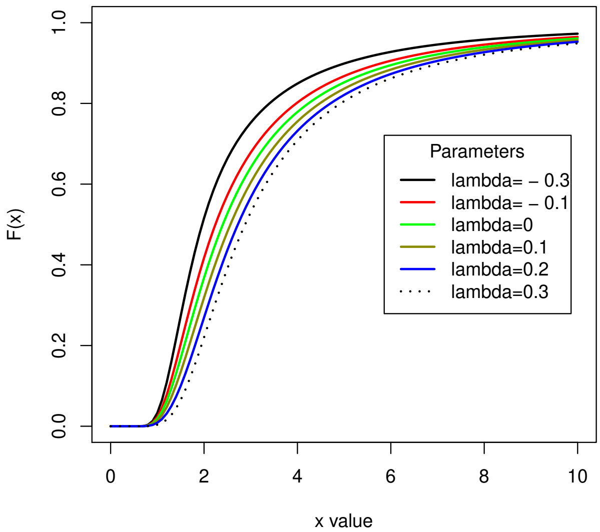

Fig. 6: The cdf of MH-QTFD for various values of parameter λ with $\alpha = 2$ and $s = 2$.

4 Some Examples of MH-CTD

In this section, we show some members of the MH-cubic transmuted family of distributions given in Eqs. (7 , 8) for various choices of baseline $G(x)$ and $g(x)$. Specifically, Pareto, Gumbel, and Fréchet distributions

4.1 MH-Cubic Transmuted Pareto Distribution (MH-CTPD)

The cdf and pdf of the MH-CTPD are given in the following theorem.
Theorem 4.1 Assume X is a random variable with the (MH-CTPD). Then

(A_{411})The cdf and pdf are defined, respectively, as

$$F(x) = (1-p) + \lambda_1\left[G_1(p) - G_3(p)\right] + \lambda_2\left[G_2(p) - G_3(p)\right] \tag{22}$$

and

$$f(x) = (\frac{\theta}{m}) \sqrt[\theta]{p^{(\theta+1)}} \;\; \zeta(p;\lambda_1,\lambda_2) \tag{23}$$

where $|\lambda_i| \leq (\frac{1}{e})$; $i = 1,2$ are transmuted parameters and $p = (\frac{m}{x})^{\theta}$.

$(A_{412})f(x)$ of Eq.(23), is a pdf.

Proof.

(A_{411})The proof is easy to see. Eq.(22) is achieved by substituting Eq.(10) into Eq.(7) and Eq.(23) is achieved by substituting Eq.(10) and Eq.(11) into Eq.(8) or differentiating Eq.(22) with respect to x.

(A_{412})To show that $f(x)$ is pdf , we must prove that $f(x) \geq 0$ and $\int_m^\infty f(x)dx = 1$. The proof of $f(x) \geq 0$ is drawn from Remark 3.1 and the following two limits

$$\lim_{x\to\infty} f(x) = (\frac{\theta}{m}) \lim_{p\to 0} \sqrt[\theta]{p^{(\theta+1)}} \;\; \zeta(p;\lambda_1,\lambda_2) = 0$$

$$\lim_{x\to m} f(x) = (\frac{\theta}{m}) \lim_{p\to 1} \sqrt[\theta]{p^{(\theta+1)}} \;\; \zeta(p;\lambda_1,\lambda_2) = \frac{\theta\left(1-\lambda_1 e\right)}{m}$$

and since $\theta, m > 0, |\lambda_1 e| \leq 1$ which tends to $\theta(1-\lambda_1 e) > 0$. In addition, the Proof of $\int_m^\infty f(x)dx = 1$, using Remark 2.1 we find that

$$\begin{aligned}\int_m^\infty f(x)dx &= \int_1^0 f(p)dp \\ &= \int_0^1 \zeta(p;\lambda_1,\lambda_2)dp \\ &= 1 + \lambda_1\left[(e-1)-(e-1)\right] + \\ &\lambda_2\left[(e-1)-(e-1)\right] = 1\end{aligned}$$

Therefore, the theorem is proved.
Some shapes of the pdf and cdf of MH-CTPD for selected values of λ_1 and λ_2 at $\theta = 2$ and $m = 1$ are illustrated in Figure (7) and Figure (8) respectively.

4.2 MH- Cubic Transmuted Gumbel Distribution (MH-CTGD)

The cdf and pdf of the MH-CTGD are illustrated in the theorem below.
Theorem 4.2 Assume that X is a random variable with the (MH-CTGD). Then

(A_{421}) The cdf and pdf are defined, respectively, as

$$F(x) = e^{-z} + \lambda_1\left[G_1(z) - G_3(z)\right] + \lambda_2\left[G_2(z) - G_3(z)\right] \tag{24}$$

and

$$f(x) = \left(\frac{z}{\sigma}\right) \; e^{-z} \;\; \zeta(z;\lambda_1,\lambda_2) \tag{25}$$

where $|\lambda_i e| \leq 1$; $i = 1,2$ are transmuted parameters and $z = e^{-(\frac{x-\mu}{\sigma})}$.

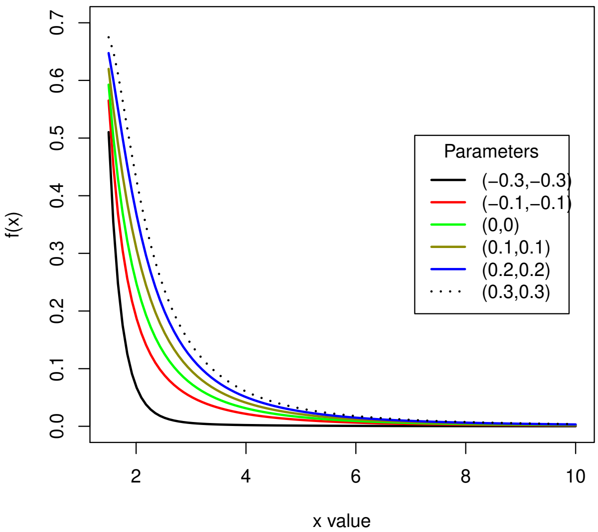

Fig. 7: The pdf of MH-CTPD for various values of parameter λ_1 and λ_2 setting $\theta = 2$ and $m = 1$.

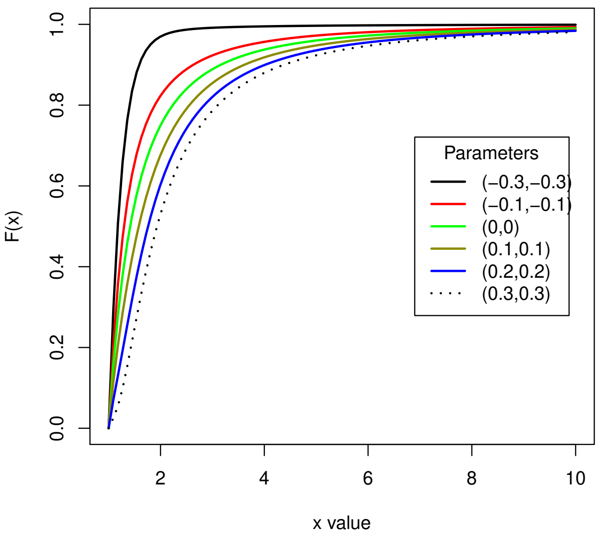

Fig. 8: The cdf of MH-CTPD for various values of parameter λ_1 and λ_2 setting $\theta = 2$ and $m = 1$.

(A_{422}) $f(x)$ of Eq.(25) is a pdf.

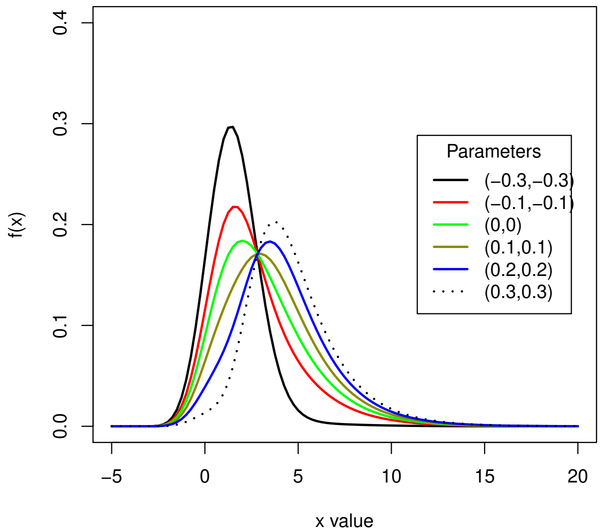

Fig. 9: The pdf of MH-CTGD for various values of parameter λ_1 and λ_2 setting $\mu = 2$ and $\sigma = 2$.

Proof.

(A_{421})The proof is easy to see. Eq.(24) is achieved by swappint Eq.(14) with Eq.(7) and Eq.(25) is achieved by swapping Eq.(14) and Eq.(15) with Eq.(8) .

(A_{422})To show that $f(x)$ is pdf , we must prove that

$$f(x) \geq 0 \text{ and } \int_{-\infty}^{\infty} f(x)dx = 1.$$

Moreover, the proof of $f(x) \geq 0$ is deduced from Remark 3.2, Eq.(9) and the two limits below

$$\lim_{x \to \infty} f(x) = \lim_{z \to 0} f(z) = \lim_{z \to 0} \left(\frac{z}{\sigma}\right) \; e^{-z} \;\; \zeta(z; \lambda_1, \lambda_2) = 0$$

whilst from L'Hopital's rule we get

$$\lim_{x \to -\infty} f(x) = \lim_{z \to \infty} f(z) = \lim_{z \to \infty} \left(\frac{z}{\sigma}\right) \; e^{-z} \;\; \zeta(z; \lambda_1, \lambda_2) = 0$$

Furthermore, the proof of $\int_{-\infty}^{\infty} f(x)dx = 1$ is satisfied from the definition of the random variable Z in Remark 3.2 and Eq.(9) that is

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \int_{\infty}^{0} f(z)dz \\ &= -\int_{0}^{\infty} e^{-z} \;\; \zeta(z; \lambda_1, \lambda_2)dz \\ &= 1 + \lambda_1 \left[(e-1) - (e-1)\right] + \\ &\lambda_2 \left[(e-1) - (e-1)\right] = 1 \end{aligned}$$

This concludes the theorem's proof.
Some shapes of the pdf and cdf of MH-CTGD for selected values of λ_1 and λ_2 at $\mu = 2$ and $\sigma = 2$ are illustrated in Figure (9) and Figure (10) respectively.

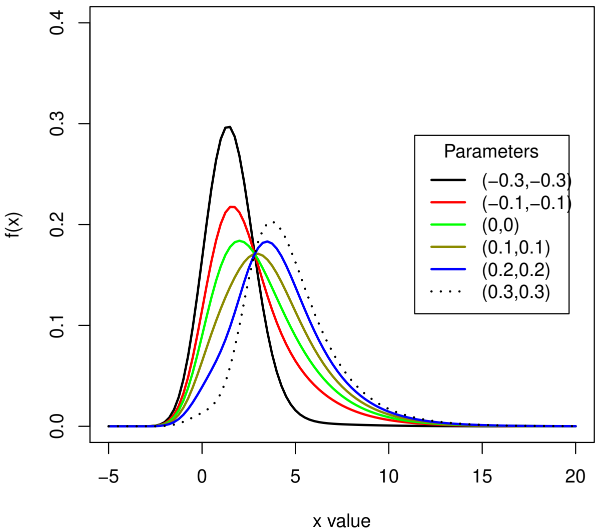

Fig. 10: The cdf of MH-CTGD for various values of parameter λ_1 and λ_2 setting $\mu = 2$ and $\sigma = 2$.

4.3 MH-Cubic Transmuted Fréchet Distribution (MH-CTFD)

The cdf and pdf of the MH-CTFD are given in the following theorem.

Theorem 4.3 Let X be a random variable with the (MH-CTFD). Then

(A_{431}) The cdf and pdf are defined, respectively, as

$$F(x) = e^{-y} + \lambda_1 \big[G_1(y) - G_3(y)\big] + \lambda_2 \big[G_2(y) - G_3(y)\big] \tag{26}$$

and

$$f(x) = (\frac{\alpha}{s}) \sqrt[\alpha]{y^(1+\alpha)} \; e^{-y} \;\; \zeta(y;\lambda_1,\lambda_2) \tag{27}$$

where $\mid \lambda_i e \mid \leq 1; \;\; i = 1,2$ are transmuted parameters and $y = (\frac{s}{x})^{\alpha}$.

(A_{432}) $f(x)$ of Eq.(27), is a probability density function.

Proof.

(A_{431})The proof is easy to see. Eq.(26) is achieved by swapping Eq.(18) with Eq.(7) and Eq.(27) is achieved by swapping Eq.(18) and Eq.(19) with Eq.(8).

(A_{432})To show that $f(x)$ is a pdf , we must establish that

$$f(x) \geq 0 \;\text{ and } \int_{-\infty}^{\infty} f(x)dx = 1.$$

And the proof of $f(x) \geq 0$ is drown from Remark 3.3, Eq.(9) and the two limits below

$$\lim_{x\to\infty} f(x) = (\frac{\alpha}{s}).\lim_{y\to 0} \sqrt[\alpha]{y^(1+\alpha)} \;\; e^{-y} \;\; \zeta(y;\lambda_1,\lambda_2) = 0$$

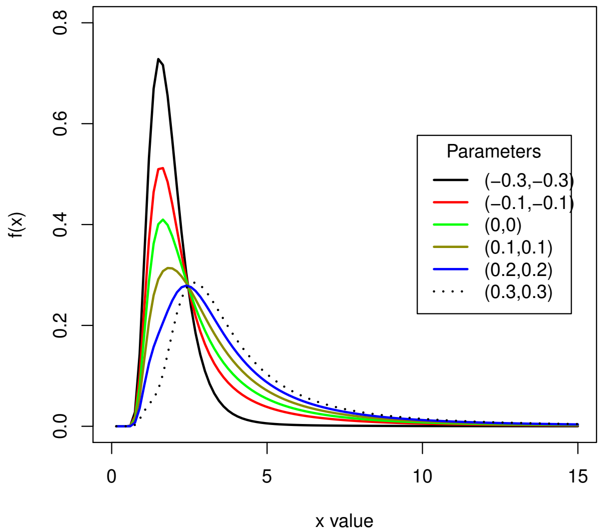

Fig. 11: The pdf of MH-CTFD for various values of parameter λ_1 and λ_2 setting $\alpha = 2$ and $s = 2$.

$$\lim_{x\to 0} f(x) = \left(\frac{\alpha}{s}\right) . \lim_{y\to\infty} \sqrt[\alpha]{y^{(}1+\alpha)} \;\; e^{-y} \;\; \zeta(y;\lambda_1,\lambda_2) = 0$$

Moreover, the Proof of $\int_0^\infty f(x)dx = 1$ is satisfied from the definition of the random variable Y given in Remark 2.3 that is

$$\begin{aligned}\int_0^\infty f(x)dx &= \int_\infty^0 f(y)dy \\ &= -\int_0^\infty \;\; e^{-y} \;\; \zeta(y;\lambda_1,\lambda_2)dy \\ &= 1+\lambda_1\big[(e-1)-(e-1)\big]+ \\ &\lambda_2\big[(e-1)-(e-1)\big] = 1\end{aligned}$$

Therefore, the theorem is proved.
Some shapes of the pdf and cdf of MH-CTFD for selected values of λ_1 and λ_2 at $\alpha = 2$ and $s = 2$ are illustrated in Figure (11) and Figure (12) respectively.

5 MH-Transmuted of Exponential Distributions (MH-TED)

This section pertains to the study of the MH-quadratic and MH-cubic transmuted exponential distributions (MH-QTED and MH-CTED) in detail, including distribution functions, density functions, and some statistical properties. In addition, the estimate of distribution parameters using the maximum likelihood technique is investigated.
The exponential distribution (ED) is one of the widely used lifetime continuous distributions. It is often used to model the time elapsed between events. The cdf and pdf of the exponential random variable X are defined as follows:

$$G(x) = 1-e^{-ax} = 1-\omega = G(\omega) \tag{28}$$

and

$$g(x) = ae^{-ax} = a \;\; \omega = g(\omega) \tag{29}$$

where $a > 0$ is a rate or inverse scale parameter and $\omega = e^{-ax}$; $x > 0$.

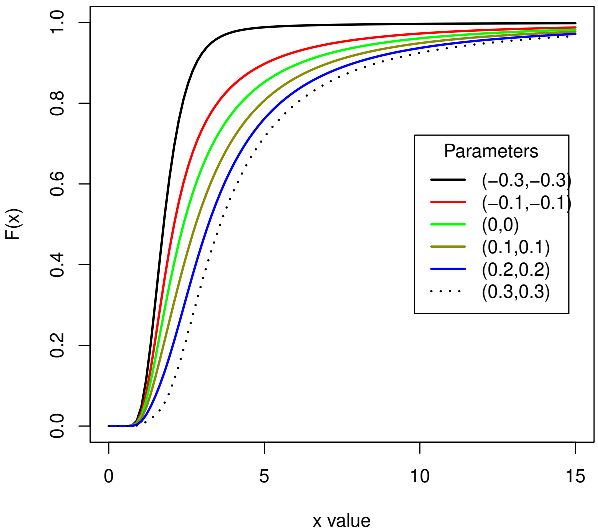

Fig. 12: The cdf of MH-CTFD for various values of parameter λ_1 and λ_2 setting $\alpha = 2$ and $s = 2$.

5.1 MH-Quadratic Transmuted Exponential Distribution (MH-QTED)

In this subsection, the distribution and denisty functions of MH-QTED are introduced.

Remark 5.1 Consider the random variable $\Omega = u(X) = e^{-aX}$ then for $(x \to 0) \Rightarrow (\omega \to 1)$ and for $(x \to \infty) \Rightarrow (\omega \to 0)$ also $\frac{dx}{d\omega} = \left(\frac{-1}{a\omega}\right)$.

Then, in terms of the random variable Ω, $f(x)$ may be expressed as follows:

$$\begin{aligned} f(\omega) =& f[w(\omega)]\frac{dx}{d\omega} \\ =& a\omega\left\{1-\lambda\left[G_1(\omega)-2(1-\omega)G_2(\omega)\right]\right\}\left(\frac{-1}{a\omega}\right) \\ =& \left\{\lambda\left[G_1(\omega)-2(1-\omega)G_2(\omega)\right]-1\right\} \text{if } \ \omega \in [0,1] \end{aligned}$$

Theorem 5.1 Assume that X is a random variable with the (MH-QTED). Then

(A_{511}) The cdf and pdf are defined, respectively, as

$$F(x) = \left(1-\omega\right) + \lambda\left\{e^{\omega} - e^{\left[1-(1-\omega)^2\right]}\right\} \quad (30)$$

and

$$f(x) = a\omega\left\{1-\lambda\left[e^{\omega} - 2(1-\omega)e^{\left[1-(1-\omega)^2\right]}\right]\right\} \quad (31)$$

where $|\lambda| \leq (\frac{1}{e})$ is a transmuted parameter.

(A_{512}) $f(x)$ of Eq.(31), is a pdf.

Proof.

(A_{511})The proof is easy to see.

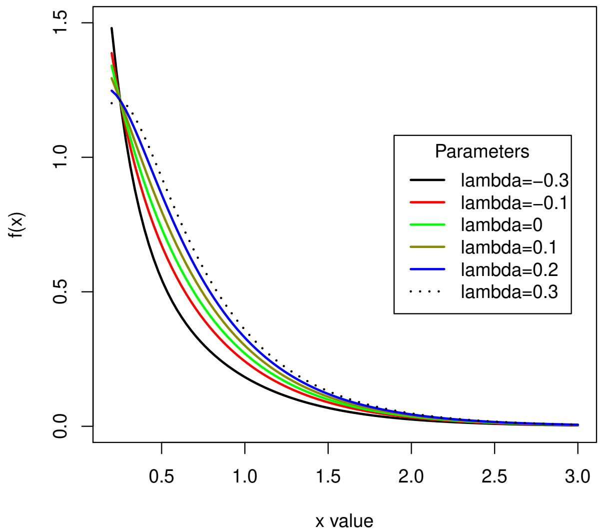

Fig. 13: The pdf of MH-QTED for various values of parameter λ setting $a = 2$.

(A_{512})To show that $f(x)$ is a pdf , we must establish that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$. The proof of $f(x) \geq 0$ is deduced from the following two limits

$$\lim_{x\to\infty} f(x) = \lim_{\omega\to 0} f(\omega) = a \lim_{\omega\to 0} \omega \left\{1 - \lambda\left[G_1(\omega) - 2(1-\omega)G_2(\omega)\right]\right\} = 0$$

$$\lim_{x\to 0} f(x) = \lim_{\omega\to 1} f(x) = a \lim_{\omega\to 1} \omega \left\{1 - \lambda\left[G_1(\omega) - 2(1-\omega)G_2(\omega)\right]\right\} = a\{1-\lambda e\}$$

Since $a > 0$ and $|\lambda e| \leq 1$ then $a\{1-\lambda e\} \geq 0$. Moreover, the Proof of $\int_0^{\infty} f(x)dx = 1$ is

$$\begin{aligned}\int_0^{\infty} f(x)dx &= \int_1^0 f(\omega)d\omega \\ &= \int_0^1 \left\{1 - \lambda\left[G_1(\omega) - 2(1-\omega)G_2(\omega)\right]\right\}d\omega \\ &= \int_0^1 \left\{1 - \lambda\left[e^{\omega} - 2(1-\omega)e^{\left[1-(1-\omega)^2\right]}\right]\right\}d\omega \\ &= 1 + \lambda\left[(e-1) - (e-1)\right] = 1\end{aligned}$$

This concludes the theorem's proof.
Some shapes of the pdf and cdf of MH-QTED for selected values of λ at $a = 2$ are illustrated in Figure (13) and Figure (14) respectively.

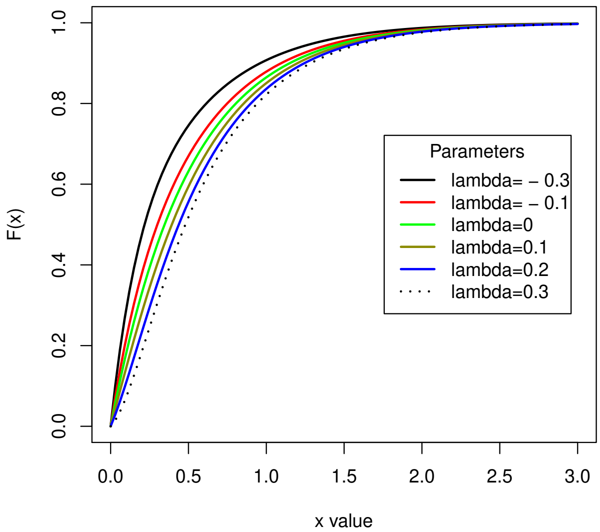

Fig. 14: The cdf of MH-QTED for various values of parameter λ setting $a = 2$.

5.1.1 The Moment Generating Function (MGF) of MH-QTED

The moment generating function of MH-QTED is demonstrated in the following theorem.
Theorem 5.2 Let X be a random variable with the MH-QTED. The MGF is defined, respectively, as

$$M_X(t) = \left(\frac{a}{a-t}\right) - \lambda \left[(-1)^{\left(\frac{t}{a}\right)} \left(\Gamma(1-\frac{t}{a}, -1) - \Gamma(1-\frac{t}{a}) \right) - \left(\frac{2ea^2}{2a^2 - 3at + t^2} \right) H(t) \right] \tag{32}$$

Where

- $\Gamma(.)$ and $\Gamma(.,.)$ are Gamma and Incomplete Gamma functions ([13]) respectively
- $H(t) = {}_pF_q\left[\{1, \frac{3}{2}\}, \{\frac{3}{2} - \frac{t}{2a}, 2 - \frac{t}{2a}\}, -1\right]$ is the generalized hypergeometric function.

Proof. From the definition of $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} f(x) e^{tx} dx = \int_0^{\infty} f(x) e^{tx} dx$. and Using Remark 5.1 to obtain

$$\begin{aligned} M_X(t) &= \int_0^1 \omega^{-\left(\frac{t}{a}\right)} \\ &\left\{ 1 - \lambda \left[G_1(\omega) - 2(1-\omega) G_2(\omega) \right] \right\} d\omega \\ &= I_1 - \lambda \left(I_2 - I_3 \right) \end{aligned} \tag{33}$$

where

$$I_1 = \int_0^1 \omega^{-\left(\frac{t}{a}\right)} d\omega$$

$$I_2 = \int_0^1 \omega^{-\left(\frac{t}{a}\right)} G_1(\omega) d\omega$$

$$I_3 = \int_0^1 \omega^{-\left(\frac{t}{a}\right)} \{2(1-\omega)G_2(\omega)\} d\omega$$

Using, Wolfram Mathematica software, to get the integrals I_1, I_1 and I_3 as follows:

$$\begin{aligned} I_1 &= \frac{a}{a-t}; \qquad t < a \\ I_2 &= (-1)^{\left(\frac{t}{a}\right)} \left(\Gamma(1-\frac{t}{a};-1) - \Gamma(1-\frac{t}{a}) \right) \\ I_3 &= \left(\frac{2ea^2}{2a^2 - 3at + t^2} \right) H(t) \end{aligned} \tag{34}$$

By substituting Eq.(34) in Eq.(33) we obtain the proved of theorem.
Now, we can get the r^{th} moments of MH-QTED by this derivatives

$$E(x^r) = \left[\frac{d^r}{dt^r} M_x(t) \right]_{t=0}; \quad r = 1, 2, \ldots\ldots \tag{35}$$

Hence, putting $r = 1, 2$ to get the arithmetic mean and variance of MH-QTED as follows

$$\begin{aligned} E(X) &= \frac{1}{a}[1 + (0.7862)\lambda] \\ E(X^2) &= \frac{1}{a^2}[2 - (0.4)\lambda] \\ Var[X] &= \frac{1}{a^2}[1 - (1.9724)\lambda - (0.6)\lambda^2] \end{aligned} \tag{36}$$

The arithmetic mean and variance of MH-QTED for various combinations of model parameters are given in Table(1).

Table 1: Mean and Variance of MH-QTED

Mean					
	$\lambda = -0.3$	$\lambda = -0.1$	$\lambda = 0$	$\lambda = 0.1$	$\lambda = 0.3$
$a = 2$	0.3821	0.4607	0.5000	0.5393	0.6179
$a = 4$	0.1910	0.2303	0.2500	0.2697	0.3090
$a = 6$	0.1274	0.1536	0.1667	0.1798	0.2060
Variance					
$a = 2$	0.3240	0.2778	0.2500	0.2191	0.1482
$a = 4$	0.0810	0.0694	0.0625	0.0548	0.0370
$a = 6$	0.0360	0.0309	0.0278	0.0243	0.0165

Table(1) demonstrates that increasing the inverse scale parameter a decreases the mean and variance while maintaining the transformed parameter *lambda* unchanged. In contrast, when the inverse scale parameter a is held constant and the transmuted parameter λ is increased, the mean rises while the variance falls.

5.1.2 The Parameter Estimation of MH-QTED

The maximum likelihood estimate (MLE) for MH-QTED parameters is discussed in this part.

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from MH-QTED. Then the likelihood function is given by

$$L = \prod_{i=0}^{n} f(x_i; a, \lambda) = a^n \prod_{i=0}^{n} (\omega_i \ \ \Lambda_i); \quad \omega_i = e^{-ax_i}$$

where

$$\Lambda_i = 1 - \lambda \Big[G_1(\omega_i) - 2(1-\omega_i) G_2(\omega_i) \Big]$$

Therefore, the log-likelihood function is

$$l = \ln L = n \ln a + \sum_{i=0}^{n} \big(\ln \omega_i + \ln \Lambda_i \big) \tag{37}$$

by differentiating the log-likelihood function in Eq.(37) with respect to the unknown parameters a, λ and equating them to zero. We obtain the following likelihood equations:

$$\begin{aligned} &\sum_{i=0}^{n} \left(\frac{\lambda \omega_i \big[G_1(\omega_i) + 2\{1 - 2(1-\omega_i)^2\} G_2(\omega_i) \big]}{\Lambda_i} - 1 \right) \\ &\ln \omega_i = n \\ &\sum_{i=0}^{n} \left(\frac{G_1(\omega_i) + 2(1-\omega_i) G_2(\omega_i)}{\Lambda_i} \right) = 0 \end{aligned} \tag{38}$$

5.2 MH-Cubic Transmuted Exponential Distribution (MH-CTED)

In this subsection the distribution and density functions of MH-CTED is demonstrated.
Theorem 5.3 Let X be a random variable with the (MH-CTED). Then

(A_{521}) The cdf and pdf are defined, respectively, as

$$\begin{aligned} F(x) = \big(1-\omega\big) &+ \lambda_1 \big[G_1(\omega) - G_3(\omega) \big] \\ &+ \lambda_2 \big[G_2(\omega) - G_3(\omega) \big] \end{aligned} \tag{39}$$

and

$$f(x) = a\omega \ \ \zeta(\omega; \lambda_1, \lambda_2) \tag{40}$$

where $|\lambda_i \ \ e| \leq 1$ for $i = 1, 2$ are transmuted parameters and $\omega = e^{-ax}$.

(A_{522}) $f(x)$ of Eq.(40), is a pdf.

Proof.

(A_{521})The proof is easy to see.
(A_{522}). To show that $f(x)$ is a pdf , we must establish that

$$f(x) \geq 0 \ \text{ and } \int_{-\infty}^{\infty} f(x) dx = 1.$$

The proof of $f(x) \geq 0$ is drawn from Remark 5.1, Eq.(9) and the two limits below

$$\lim_{x \to \infty} f(x) = a \lim_{\omega \to 0} \omega \ \ \zeta(\omega; \lambda_1, \lambda_2) = 0$$

$$\lim_{x \to 0} f(x) = a \lim_{\omega \to 1} \omega \ \ \zeta(\omega; \lambda_1, \lambda_2) = a\big\{1 - \lambda_1 e\big\}$$

Moreover, using Remark 5.1 the Proof of $\int_0^\infty f(x)dx = 1$ is as follows

$$\begin{aligned} \int_0^\infty f(x) dx = &\int_0^1 f(\omega) d\omega \\ = &\int_0^1 \zeta(\omega; \lambda_1, \lambda_2) d\omega \\ = &1 + \lambda_1 \big[(e-1) - (e-1) \big] \\ &+ \lambda_2 \big[(e-1) - (e-1) \big] \\ = &1 \end{aligned}$$

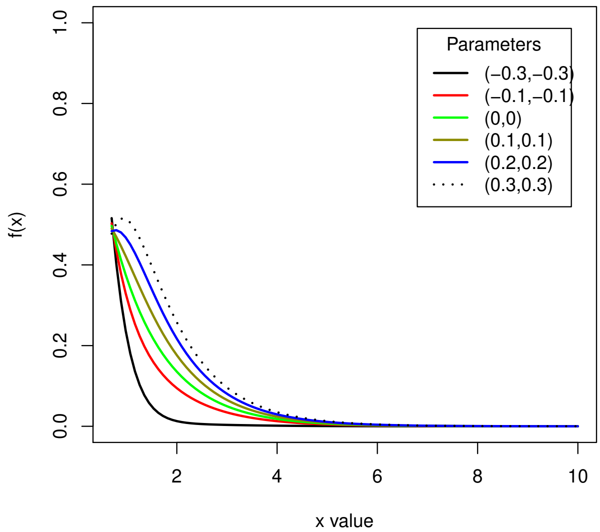

Fig. 15: The pdf of MH-CTED for various values of parameter λ_1 and λ_2 setting $a = 2$.

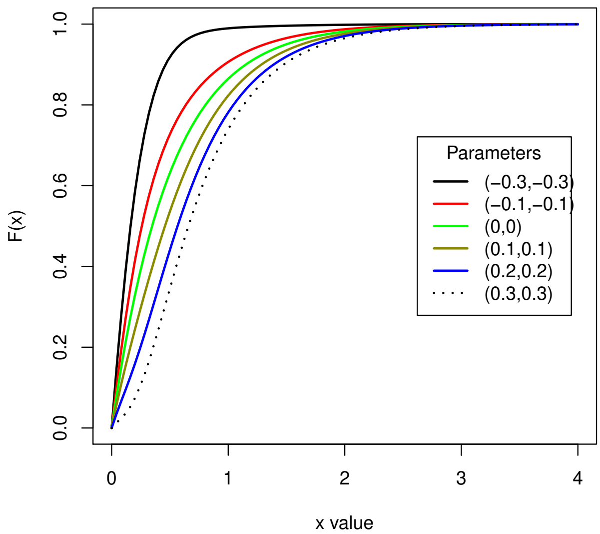

Fig. 16: The cdf of MH-CTED for various values of parameter λ_1 and λ_2 setting $a = 2$.

Therefore, the theorem is proved.
Some shapes of the pdf and cdf of MH-CTED for selected values of λ_1 and λ_2 at $a = 2$ are illustrated in Figure (15) and Figure (16) respectively.

5.2.1 The Moment Generating Function (MGF) of MH-CTED

The moment generating function of MH-CTED is introduced in the following theorem

Theorem 5.4 Assume that X is a random variable with the MH-CTED. The MGF is defined, respectively, as

$$\begin{aligned} M_x(t) = & \left(\frac{a}{a-t}\right) + 4\pi(\lambda_1+\lambda_2)\ 3^{\left(\frac{-5}{2}+\frac{t}{a}\right)} \\ & \Gamma(1-\frac{t}{a})\mathscr{H}(t) + \lambda_1(-1)^{(\frac{t}{a})} \\ & \left(\Gamma(1-\frac{t}{a},-1) - \Gamma(1-\frac{t}{a})\right) \\ & -\lambda_2\left(\frac{2ea^2}{2a^2-3at+t^2}\right)H(t) \end{aligned} \tag{41}$$

where $\mathscr{H}(t) = {}_p\check{F}_q\left[\{1,\frac{4}{3},\frac{5}{3}\},\{\frac{4}{3}-\frac{t}{3a},\frac{5}{3}-\frac{t}{3a},2-\frac{t}{3a}\},-1\right]$ is the Regularized Hypergeometric function.

Proof. From the definition

$$\begin{aligned} M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} f(x)e^{tx}dx = \int_0^{\infty} f(x)e^{tx}dx \\ = \int_0^1 \omega^{-(\frac{t}{a})}\zeta(\omega;\lambda_1,\lambda_2)d\omega \\ = I_1 + \lambda_1(I_4 - I_2) + \lambda_2(I_4 - I_3) \end{aligned} \tag{42}$$

such that

$$I_1 = \int_0^1 \omega^{-(\frac{t}{a})} d\omega$$

$$I_2 = \int_0^1 \omega^{-(\frac{t}{a})} G_1(\omega)d\omega$$

$$I_3 = \int_0^1 \omega^{-(\frac{t}{a})} \left[2(1-\omega)G_2(\omega)\right]d\omega$$

$$I_4 = \int_0^1 \omega^{-(\frac{t}{a})} \left[3(1-\omega)^2G_3(\omega)\right]d\omega$$

Using, Wolfram Mathematica software, to get the integral I_4 and the integrals I_1, I_2 and I_3 as in Eq.(34)

$$I_4 = 4\pi(3)^{\left[\frac{-5}{2}+\frac{t}{a}\right]}\Gamma(1-\frac{t}{a})\mathscr{H}(t) \tag{43}$$

By substituting Eq.(34) and Eq.(43) in Eq.(42) we obtain the proof of the theorem .
Now, we can get the r^{th} moments of MH-CTED by the derivative of Eq.(41) and putting $r = 1,2$ to obtain the arithmetic mean and variance of MH-CTED as follows

$$\begin{aligned} E(X) &= \frac{1}{a}\Big[1 + (1.32961)\lambda_1 + (0.54331)\lambda_2\Big] \\ E(X^2) &= \frac{1}{a^2}\Big[2 + (3.65364)\lambda_1 + (1.6882)\lambda_2\Big] \\ Var[X] &= \frac{1}{a^2}\Big[1 + (1.1)\lambda_1 - (0.3)\lambda_2 + (1.3)\lambda_1\lambda_2 \\ &\quad + (1.69)\lambda_1^2 + (0.25)\lambda_2^2\Big] \end{aligned} \tag{44}$$

Table (4) shows the mean and variance of MH-CTED for various sets of model parameters. The arithmetic mean and variance decrease as the inverse scale parameter a increases when the transmuted parameters λ_1 and λ_2 are held constant. When the inverse scale parameter a and the transmuted parameter $\lambda_1(\lambda_2)$ are held constant, the mean and variance increase as the transmuted parameter $\lambda_2(\lambda_1)$ increases.

5.2.2 The Parameter Estimation of MH-CTED

The maximum likelihood estimate for MH-CTED parameters is discussed in this part.
Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from MH-CTED. Then the likelihood function is given by

$$\begin{aligned} L &= \prod_{i=0}^{n} f(x_i; a, \lambda_1, \lambda_2) \\ &= a^n \prod_{i=0}^{n} \omega_i \ \ \zeta(\omega_i; \lambda_1, \lambda_2); \end{aligned}$$

where $\omega_i = e^{-ax_i}$. Therefore, the log-likelihood function is

$$l = \ln L = n \ln a + \sum_{i=0}^{n} \Big[\ln \omega_i + \ln \zeta(\omega_i; \lambda_1, \lambda_2) \Big] \tag{45}$$

by taking the derivative of the log-likelihood function in Eq.(44) with respect to the unknown parameters a, λ_1, λ_2 and equating them to zero. We obtain the following likelihood equations.

$$\begin{aligned} n &= \sum_{i=0}^{n} \left[\frac{\eta_i \omega_i}{\zeta(\omega_i; \lambda_1, \lambda_2)} - 1 \right] \ln \omega_i \\ 0 &= \sum_{i=0}^{n} \frac{3(1-\omega_i)^2 G_3(\omega_i) - e^{\omega_i}}{\zeta(\omega_i; \lambda_1, \lambda_2)} \\ 0 &= \sum_{i=0}^{n} \frac{3(1-\omega_i)^2 G_3(\omega_i) - 2(1-\omega_i) G_2(\omega_i)}{\zeta(\omega_i; \lambda_1, \lambda_2)} \end{aligned} \tag{46}$$

where

$$\begin{aligned} \eta_i =& 3(\lambda_1 + \lambda_2)\left[2(1-\omega_i) - 3(1-\omega_i)^4\right] G_3(\omega_i) \\ &+ 2\lambda_2 \left[2(1-\omega_i)^2 - 1\right] G_2(\omega_i) + \lambda_1 G_1(\omega_i) \end{aligned}$$

6 Application of MH-QTED and MH-CTED

In this section, the MH-QTED and MH-CTED are applied to a real-world data set. Table 2 displays the lifespans of 20 electronic components; this data is right-skewed. It has been used by ([10],[14], [15]). The summary statistics of the data are reported in Table 3, while Table 5 demonstrates the maximum likelihood estimates, the log-likelihood value (-Log(L)), the Kolmogorov-Smirnov (ks) test statistic, and its corresponding p-value, for the fitted distributions, which include Exponential ED, Quadratic Transmuted Exponential QTED, and Cubic Transmuted Exponential that was proposed by Rahman et al. [10] CTED-R, in addition to the two proposed distributions MH-QTED and MH-CTED.

Table 2: Lifetimes of 20 Electronic Component

0.03	0.12	0.22	0.35	0.73	0.79	1.25	1.41
1.52	1.79	1.8	1.94	2.38	2.4	2.87	2.99
3.14	3.17	4.72	5.09				

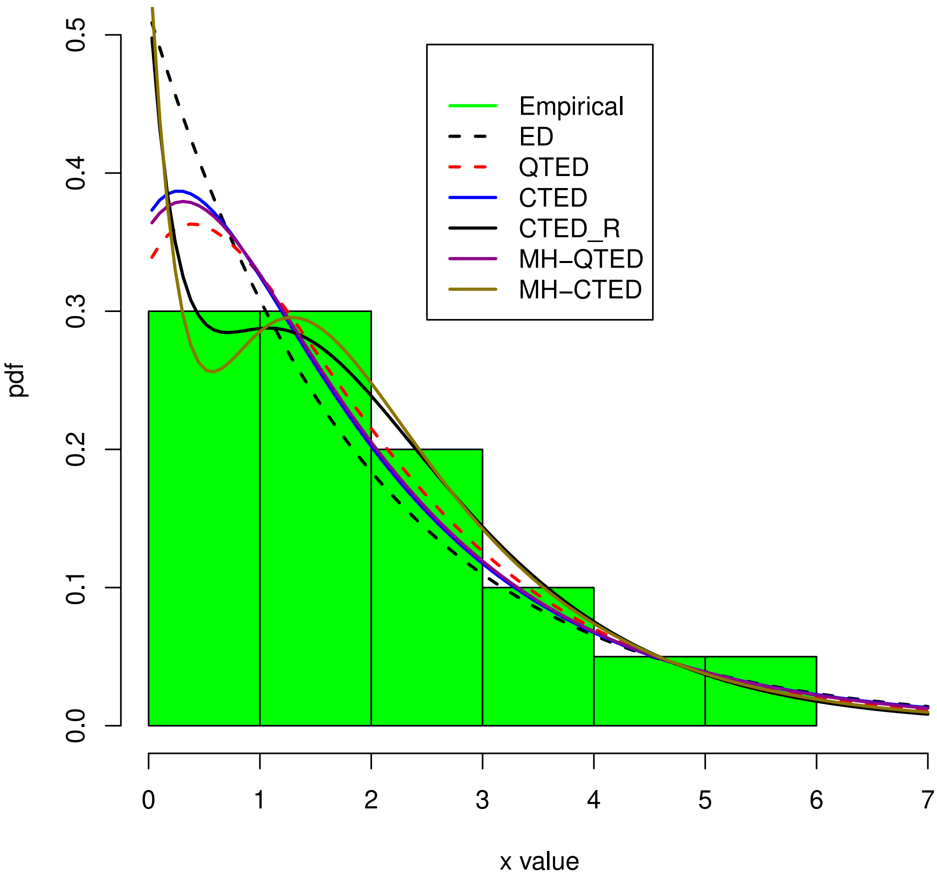

Fig. 17: The pdf of the selected distributions for the Electronic data

Table 3: Summary Statistics for the Electronic data

Data	n	Mean	Median	Skewness	kurtosis
Electronic data	20	1.936	1.795	0.653	0.008

Looking at the values of the ks and p-value in Table 5 we see that all distributions fit the electronic data adequacy and the MH-CTED is the best one (see Figure 17). Moreover, MH-CTED has the minimum value of -log(L) (31.693). We can infer that the proposed MH-CTED is the most suitable model for electronic data.

7 Conclusions

In this article, a new family of transmuted distributions called MH-Transmuted Family of Distributions is introduced. The proposed family is confirmed in the quadratic and cubic distribution families. It is concluded that the cubic version of the proposed transmuted family is more flexible and capable of capturing right-skewed unimodal data than the quadratic one. Based on the quadratic and cubic distribution maps, some new distributions are proposed. Further, depending on the quadratic and cubic distribution maps, two new generalizations of the exponential distributions called MH-quadratic transmuted exponential and MH-cubic transmuted exponential are developed to evaluate the effectiveness of the proposed family of distributions. The MH-quadratic transmuted exponential and MH-cubic transmuted exponential are fitted to real-life data sets, and it has been found that the cubic transmuted exponential distribution adequately fits the selected data set as compared with the other related distributions used in the comparison.

Table 4: Mean and Variance of CTED-MH

Mean						
		$\lambda_1=-0.3$	$\lambda_1=-0.1$	$\lambda_1=0.0$	$\lambda_1=0.1$	$\lambda_1=0.3$
$a=2$	$\lambda_2=\ 0.3$	0.2191	0.352	0.4185	0.485	0.6179
	$\lambda_2=\ 0.1$	0.2734	0.4064	0.4728	0.5393	0.6723
	$\lambda_2=\ 0.0$	0.3006	0.4335	0.5	0.5665	0.6994
	$\lambda_2=-0.1$	0.3277	0.4607	0.5272	0.5936	0.7266
	$\lambda_2=-0.3$	0.3821	0.515	0.5815	0.648	0.7809
$a=4$	$\lambda_2=\ 0.3$	0.1095	0.176	0.2093	0.2425	0.309
	$\lambda_2=\ 0.1$	0.1367	0.2032	0.2364	0.2697	0.3361
	$\lambda_2=\ 0.0$	0.1503	0.2168	0.25	0.2832	0.3497
	$\lambda_2=-0.1$	0.1639	0.2303	0.2636	0.2968	0.3633
	$\lambda_2=-0.3$	0.191	0.2575	0.2907	0.324	0.3905
$a=6$	$\lambda_2=\ 0.3$	0.073	0.1173	0.1395	0.1617	0.206
	$\lambda_2=\ 0.1$	0.0911	0.1355	0.1576	0.1798	0.2241
	$\lambda_2=\ 0.0$	0.1002	0.1445	0.1667	0.1888	0.2331
	$\lambda_2=-0.1$	0.1092	0.1536	0.1757	0.1979	0.2422
	$\lambda_2=-0.3$	0.1274	0.1717	0.1938	0.216	0.2603
Variance						
$a=2$	$\lambda_2=\ 0.3$	0.0514	0.1581	0.1982	0.2295	0.2656
	$\lambda_2=\ 0.1$	0.109	0.2013	0.2342	0.2583	0.2799
	$\lambda_2=\ 0.0$	0.1356	0.2207	0.25	0.2704	0.2848
	$\lambda_2=-0.1$	0.1608	0.2386	0.2643	0.2811	f 0.2883
	$\lambda_2=-0.3$	0.2066	0.27	0.2885	0.2981	0.2908
$a=4$	$\lambda_2=\ 0.3$	0.0128	0.0395	0.0496	0.0574	0.0664
	$\lambda_2=\ 0.1$	0.0273	0.0503	0.0586	0.0646	0.07
	$\lambda_2=\ 0.0$	0.0339	0.0552	0.0625	0.0676	0.0712
	$\lambda_2=-0.1$	0.0402	0.0597	0.0661	0.0703	0.0721
	$\lambda_2=-0.3$	0.0517	0.0675	0.0721	0.0727	0.0754
$a=6$	$\lambda_2=\ 0.3$	0.0057	0.0176	0.022	0.0255	0.0295
	$\lambda_2=\ 0.1$	0.0121	0.0224	0.026	0.0287	0.0311
	$\lambda_2=\ 0.0$	0.0151	0.0245	0.0278	0.0300	0.0316
	$\lambda_2=-0.1$	0.0179	0.0265	0.0294	0.0312	0.0320
	$\lambda_2=-0.3$	0.023	0.03	0.0321	0.0323	0.0331

Table 5: Estimates of parameters, -log (L), k-s test value, and p-value for the selected distributions

Distribution	Parameters estimates			-log(L)	k-s	P-value
ED	$a=0.5167$			33.207	0.1758	0.511
QTED	a= 0.6325	λ= -0.0.4715		32.714	0.1296	0.848
CTED	a= 0.5306	λ = 0.3045		33.037	0.1457	0.739
CTED-R	a= 0.7616	λ_1= -0.3017	λ_2= -0.99	31.726	0.0986	0.979
MH-QTED (proposed)	a=0.5592	λ=0.1309		32.958	0.1419	0.765
MH-CTED (proposed)	$a=0.6735$	λ_1= -0.0579	λ_2=0.49	**31.693**	**0.0913**	**0.991**

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