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Two Analytical Approaches for Space- and Time-Fractional Coupled Burger's Equations via Elzaki Transform

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Abstract: In the present paper, we examine one- and two- dimensional fractional coupled Burger's equations (FCBEs) by two different schemes, namely iterative Elzaki transform scheme (IETM) and homotopy analysis Elzaki transform method (HAETM). These schemes provide a numerical solution of one- and two- dimensional FCBEs in the terms of power series. Several sample problems have been solved to illustrate the accuracy and efficiency of proposed schemes. In numerical studies, we show that both proposed schemes HAETM and IETM give the same results in the case of the one-dimensional FCBE, while in the two-dimensional FCBE, the solution gains by HAETM converge rapidly than the approximate result by IETM.
Keywords: Iterative Elzaki transform scheme (IETM), homotopy analysis Elzaki transform method (HAETM), Elzaki transform, fractional coupled Burger's equations (FCBEs).

1 Introduction

FC is the generalization of classical calculus, which studies non-integer order of derivatives. The beauty of FC is that fractional order derivatives and integrals are non-local. The purpose of using fractional models in differential equations in physical models due to their non-local property. This is due to the fact that the fractional-order derivatives and integrals are capable to characterize the properties of memory effects as an essential aspect in many real-world phenomena [1-4]. Recently, numerous models such as Baleanu et al. [5] in nanotechnology, Arif et al. [6] in nanofluid, Jajarmi et al. [7] and Khan et al. [8] in biology and Oldham [9] in electrochemistry have been modelled with the help of fractional order derivatives.

Motivated by this fact, several researchers have been established analytical methods by using fractional order differential operators to find the approximate solutions. Baleanu et al. [10] have been studied the wave equations and the non-linear fractional equations and Daftardar et al. [11] respectively by using fractional variational iteration method, decomposition method and iterative method. Recently, Baleanu et al. [12] have been analyzed two-dimensional partial differential equations. Analytical solutions of fractional physical models have been obtained by Khan et al. [13]. Furati et al. [14] studied on existence and uniqueness for fractional derivatives. Application of fractional derivatives in the behaviour of immune and tumour cells is given by Ghanbari et al. [15]. Time fractional coupled equation is solved by Gómez-Aguilar [16] by using homotopy analysis. Sontakke [17, 18] solved the Hirita-satsuma coupled Kdv, mKdv and Kawahara equations. Several papers studies the solution of differential nonlocal systems [19-24].

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The Burger's equation demonstrates coupling among diffusion and convection processes. This equation designates the structure of shock waves, acoustic transmission, and traffic flow. Abazari [25] gave the numerical solution of the Burger's and coupled Burger's equation. Also in last few years, numerous researches have been studied and analyzed one- and two-dimensional FCBEs with many analytical schemes such as the Sumudu decomposition method (SDM) by Ahmed et al. [26], VIM by Biazar et al. [27], Adomain decomposition method (ADM) by Alharbi et al. [28], Dehghan et al. [29] and Gorguis [30], q-HATM by Singh et al. [31] and Elzaki homotopy perturbation method by Suleman et al. [32].

The main objective of this work is to extend the application of the homotopy Elzaki transform method (HETM) and iteration Elzaki transform method (IETM) to derive explicit analytical approximate solutions of the FCBE with time- and space-Caputo fractional derivatives for the following two models:

i. One-dimensional FCBE

$$\frac{\partial^{\zeta_j}\omega}{\partial\tau^{\zeta_j}} = \frac{\partial^2\omega}{\partial\rho^2} + 2\omega\frac{\partial\omega}{\partial\rho} - \frac{\partial(\omega\vartheta)}{\partial\rho},$$

$$\frac{\partial^{\xi_j}\vartheta}{\partial\tau^{\xi_j}} = \frac{\partial^2\vartheta}{\partial\rho^2} + 2\vartheta\frac{\partial\vartheta}{\partial\rho} - \frac{\partial(\omega\vartheta)}{\partial\rho}. \quad (1)$$

ii. Two-dimensional FCBE

$$\frac{\partial^{\zeta_j}\omega(\rho,\sigma,\tau)}{\partial\tau^{\zeta_j}} + \omega(\rho,\sigma,\tau)\frac{\partial\omega(\rho,\sigma,\tau)}{\partial\rho} + \vartheta(\rho,\sigma,\tau)\frac{\partial\omega(\rho,\sigma,\tau)}{\partial\sigma} = \frac{1}{R}\left[\frac{\partial^2\omega(\rho,\sigma,\tau)}{\partial\rho^2} + \frac{\partial^2\omega(\rho,\sigma,\tau)}{\partial\sigma^2}\right],$$
$$\frac{\partial^{\xi_j}\vartheta(\rho,\sigma,\tau)}{\partial t^{\xi_j}} + \omega(\rho,\sigma,\tau)\frac{\partial\vartheta(\rho,\sigma,\tau)}{\partial\rho} + \vartheta(\rho,\sigma,\tau)\frac{\partial\vartheta(\rho,\sigma,\tau)}{\partial\sigma} = \frac{1}{R}\left[\frac{\partial^2\vartheta(\rho,\sigma,\tau)}{\partial\rho^2} + \frac{\partial^2\vartheta(\rho,\sigma,\tau)}{\partial\sigma^2}\right]. \quad (2)$$

The factors ζ_j and ξ_j, where $0 < \zeta_j, \xi_j \leq 1, j = 1,2, \ldots$, stand for the order of the fractional time and space derivative, respectively. R represents the Reynolds number.

In present study, we gain the analytical and numerical solutions of one- and two-dimensional FCBEs by using two different approaches, namely the iterative Elzaki transform method (IETM) and homotopy Elzaki transform method (HETM). The IETM is an elegant coupling of the NIM and Elzaki transform and the HAETM is an elegant combination of the HAM and Elzaki transform. Numerical comparisons with graphical representation of both suggested approaches with the exact solutions are given to illustrate the efficiency and the accuracy of the proposed approaches. The results prove that both proposed approaches are very effective and simple.
The rest of this paper has been organized as follows: In Section 2, Some preliminary definitions and properties related to fractional order Caputo derivative and the Elzaki transform are given. In Section 3, we describe the procedures of HAETM and IETM. In Section 4, two numerical problems are provided to illustrate the feasibility of proposed methods. Finally, Section 5 concludes the output of the whole paper.

2 Preliminaries

Definition 2.1 The fractional derivative of $\mu \in C_{-1}^{s}$ in Caputo [33] sense has been defined as:

$$D_\tau^\zeta\mu(x,\tau) = \begin{cases} \frac{\partial^s\mu(t)}{\partial\tau^s}, \zeta = s \in N, \\ \frac{1}{\Gamma s-\zeta}\int_0^t (\tau-\upsilon)^{s-\zeta-1}\mu^{(s)}(\upsilon)d\upsilon, & s-1<\zeta<s, s\in N. \end{cases} \quad (3)$$

where s is the smallest integer that exceeds ζ.
Some basic arithmetic properties of the Caputo's fractional derivative are given as:

$$D_\tau^\zeta C = 0, \quad \text{(C is a constant)},$$
$$D_\tau^\zeta\left(\delta f(\tau) + \beta g(\tau)\right) = \delta D_\tau^\zeta f(\tau) + \beta D_\tau^\zeta g(\tau),$$

where δ and β are constants, and

$$D^\zeta\tau^\varsigma = \begin{cases} 0, & \varsigma \leq \zeta - 1, \\ \frac{\Gamma(\varsigma+1)\tau^{\varsigma-\zeta}}{\Gamma(\varsigma-\zeta+1)}, & \varsigma \geq \zeta - 1. \end{cases}$$

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Definition 2.2 Fractional integral of a function $\mu(\tau) \in C_\eta(\eta \geq -1)$ with order $\zeta > 0$, preliminary introduced by Riemann-Liouville and expressed [34, 35] as:

$$\begin{cases} J^\zeta \mu(\tau) = \frac{1}{\Gamma\zeta}\int_0^t (\tau - \upsilon)^{\zeta-1} \mu(\zeta)d\zeta, \\ J^0\mu(\tau) = \mu(\tau). \end{cases} \tag{4}$$

Definition 2.3: The Elzaki transform (ET) or modified Sumudu transform definition for the function f(τ) is given as:

$$E[f(\tau)] = T(p) = p\int_0^\infty f(\tau)\, e^{\frac{-\tau}{p}}\, d\tau, \quad \tau > 0.$$

The ET is a very efficient and strong scheme to solve the integral equation that the Sumudu transform method cannot match.
The Elzaki transform of $D^\zeta \mu(x,\tau)$ is given as follows [36, 37, 38]:

$$E[D^\zeta \mu(x,\tau)] = \frac{T(p)}{p^\zeta} - \sum_{r=0}^{s-1} p^{2-\zeta+r}\, \mu^{(r)}(0,\tau), \qquad s-1 < \zeta \leq s, s \in N. \tag{5}$$

3 Basic idea of HAETM and IETM

We consider a general non-linear time fractional differential equation and Applying two analytical methods, namely the IETM and the HAETM, to obtain series solution.

$$\begin{cases} {}^cD_\tau^\zeta \omega(\rho,\tau) = h(\rho,\tau) + R[\omega(\rho,\tau)] + N[\omega(\rho,\tau)], \\ \omega(\rho,0) = \Psi(\rho). \end{cases} \tag{6}$$

where $0 < \zeta \leq 1$.

3.1 Homotopy analysis Elzaki transform method (HAETM):

Taking the Elzaki transform (ET) on both sides of (6), we get:

$$E[\omega] = \chi(\rho,p) + p^\zeta\big[E[R\omega(\rho,\tau)] + E[N\omega(\rho,\tau)]\big], \tag{7}$$

where,

$$\chi(\rho,p) = p^2\omega(\rho,0) + p^\zeta E[h(\rho,\tau)].$$

On simplifying, (7) reduces to:

$$E[\omega] - p^\zeta \sum_{k=0}^{m-1} p^{2-\zeta+k}\omega^{(k)}(\rho,0) + p^\zeta\big[E[R\omega(\rho,\tau)] + E[N\omega(\rho,\tau)] - E[h(\rho,\tau)]\big] = 0, \tag{8}$$

Next, a non-linear operator defines as:

$$N[\varphi(\rho,\tau;q)] = E[\varphi(\rho,\tau;q)] - p^\zeta \sum_{k=0}^{m-1} p^{2-\zeta+k}\, \Psi^k(\rho) + p^\zeta\big[E[R\varphi(\rho,\tau;q)] + E[N\varphi(\rho,\tau;q)] - E[h(\rho,\tau)]\big], \tag{9}$$

here $q \in [0,1]$ and $\varphi(\rho,\tau;q)$ represents a real function.

We build a homotopy as:

$$(1-q)E[\varphi(\rho,\tau;q)] - \omega_0(\rho,\tau) = \hbar q H(\rho,\tau) N[\varphi(\rho,\tau;q)], \tag{10}$$

where E represents the Elzaki transform, $q \in [0,1]$ be embedding parameter, $\hbar$ and $H(\rho,\tau)$ are nonzero auxiliary parameter and auxiliary function, respectively. $\omega_0(\rho,\tau)$ is the initial guess of $\omega(\rho,\tau)$. Apparently, when $q = 0$ and $q = 1$; the result holds:

$$\varphi(\rho,\tau;0) = \omega_0(\rho,\tau), \qquad \varphi(\rho,\tau;1) = \omega(\rho,\tau), \tag{11}$$

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When q upsurges from 0 to 1, $\varphi(\rho,\tau)$ differs from initial guess $\omega_0(\rho,\tau)$ to the solution $\omega(\rho,\tau)$. It is expanding about q by Taylor's theorem:

$$\varphi(\rho,\tau;q) = w_0 + \sum_{\ell=1}^{\infty} q^{\ell}\, w_{\ell}(\rho,\tau), \tag{12}$$

where

$$\omega_{\ell}(\rho,\tau) = \frac{1}{\ell!}\frac{\partial^{\ell}\varphi(\rho,\tau;q)}{\partial q^{\ell}}\bigg|_{q=0}, \tag{13}$$

with properly choice of initial guess $\omega_0(\rho,\tau)$, $\hbar$ and $H(\rho,\tau)$ series (12) converges at $q=1$ and we get:

$$\omega(\rho,\tau) = \omega_0 + \sum_{\ell=1}^{\infty} \omega_{\ell}(\rho,\tau), \tag{14}$$

equation (14) gives one of the solution of original equation. Next, we define vectors as:

$$\overrightarrow{\omega_{\ell}} = \{\omega_0(\rho,\tau), \omega_1(\rho,\tau), \ldots\ldots\ldots\ldots, \omega_{\ell}(\rho,\tau)\}, \tag{15}$$

We construe ℓ^{th} order deformation equation by differentiating (10) ℓ -times w. r. t. q and setting $q=0$, we get:

$$E[\omega_{\ell}(\rho,\tau) - \kappa_{\ell}\omega_{\ell-1}(\rho,\tau)] = \hbar H(\rho,\tau)\Re_{\ell}(\overrightarrow{\omega_{\ell-1}}), \tag{16}$$

taking inverse Elzaki transform of (16), we have:

$$\omega_{\ell}(\rho,\tau) = \kappa_{\ell}\omega_{\ell-1}(\rho,\tau) + \hbar E^{-1}[H(\rho,\tau)\Re_{\ell}(\overrightarrow{\omega_{\ell-1}})], \tag{17}$$

where

$$\Re_{\ell}(\overrightarrow{\omega_{\ell-1}}) = \frac{1}{(\ell-1)!}\frac{\partial^{\ell-1}N[\varphi(\rho,\tau;q)]}{\partial q^{\ell-1}}\bigg|_{q=0}, \tag{18}$$

and

$$\kappa_{\ell} = \begin{cases} 0, \ell \le 1 \\ 1, \ell > 1\,. \end{cases} \tag{19}$$

3.2 IETM for TFPDEs

Taking the Elzaki transform (ET) on both sides of (6) and on simplifying, we get:

$$E[\omega] = p^{\zeta}\sum_{k=0}^{m-1} p^{2-\zeta+k}\omega^{(k)}(\rho,0) + p^{\zeta}\big[E[R\omega(\rho,\tau)] + E[N\omega(\rho,\tau)] - E[h(\rho,\tau)]\big], \tag{20}$$

On simplifying (20), we get:

$$E[\omega] = \chi(\rho,p) + p^{\zeta}\big[E[R\omega(\rho,\tau)] + E[N\omega(\rho,\tau)]\big], \tag{21}$$

where

$$\chi(\rho,p) = p^2\omega(\rho,0) + p^{\zeta}E[h(\rho,\tau)], \tag{22}$$

applying the inverse of Elzaki transform of (21), we get:

$$\omega(\rho,\tau) = \chi(\rho,\tau) + E^{-1}\left[p^{\zeta}E[R\omega(\rho,\tau) + N\omega(\rho,\tau)]\right]. \tag{23}$$

By iterative method, we consider the solution in form of infinite series as:

$$\omega(\rho,\tau) = \sum_{\ell=0}^{\infty} \omega_{\ell}(\rho,\tau). \tag{24}$$

Define the linear operator R as:

$$R(\sum_{\ell=0}^{\infty}\omega_\ell(\rho,\tau)) = \sum_{\ell=0}^{\infty} R[\omega_\ell(\rho,\tau)], \tag{25}$$

and nonlinear operator N is immobilized as:

$$N(\sum_{\ell=0}^{\infty}\omega_\ell(\rho,\tau)) = N[\omega_0(\rho,\tau)] + \sum_{\ell=0}^{\infty}\{N\sum_{i=0}^{\ell}[\omega_i(\rho,\tau)] - N\sum_{i=0}^{\ell-1}[\omega_i(\rho,\tau)]\}. \tag{26}$$

Put the value in (24) from (25) and (26), we have:

$$\sum_{\ell=0}^{\infty}\omega_\ell(\rho,\tau) = \omega_0(\rho,\tau)$$

$$+E^{-1}\left[p^\zeta E\left[R\sum_{\ell=0}^{\infty}[\omega_\ell(\rho,\tau)] + N[\omega_0(\rho,\tau)] + \sum_{\ell=0}^{\infty}\{N\sum_{i=0}^{\ell}[\omega_i(\rho,\tau)] - N\sum_{i=0}^{\ell-1}[\omega_i(\rho,\tau)]\}\right]\right], \tag{27}$$

Comparing the coefficients of $\omega_\ell(\rho,\tau)$, for $\ell = 0,1,2,3,\ldots\ldots.$ to both sides of (27), we get:

$$\omega_0(\rho,\tau) = \Psi(\rho),$$

$$\omega_1(\rho,\tau) = E^{-1}\left[p^\zeta E\left[R[\omega_o(\rho,\tau)] + N[\omega_o(\rho,\tau)]\right]\right],$$

$$\omega_{\ell+1}(\rho,\tau) = E^{-1}\left[p^\zeta E\left[R[\omega_\ell(\rho,\tau)] + \{N\sum_{i=0}^{\ell}[\omega_i(\rho,\tau)] - N\sum_{i=0}^{\ell-1}[\omega_i(\rho,\tau)]\}\right]\right], \ell \geq 1\,. \tag{28}$$

Thus the series solution of (6) is:

$$\omega(\rho,\tau) = \sum_{\ell=0}^{\infty}\omega_\ell(\rho,\tau). \tag{29}$$

4 Numerical problems

In this section, two numerical problems are given to illustrate the efficiency and accuracy of HAETM and IETM, respectively.

Example 4.1. Consider one-dimensional FCB equation as:

$$\begin{aligned}\frac{\partial^\zeta\omega}{\partial\tau^\zeta} &= \frac{\partial^2\omega}{\partial\rho^2} + 2\omega\frac{\partial\omega}{\partial\rho} - \frac{\partial(\omega\vartheta)}{\partial\rho}, 0<\zeta\leq 1,\\ \frac{\partial^\xi\vartheta}{\partial\tau^\xi} &= \frac{\partial^2\vartheta}{\partial\rho^2} + 2\vartheta\frac{\partial\vartheta}{\partial\rho} - \frac{\partial(\omega\vartheta)}{\partial\rho}, 0<\xi\leq 1.\end{aligned} \tag{30}$$

with the initial conditions

$$\omega(\rho,0) = \omega_0 = \sin\rho\,, \qquad \vartheta(\rho,0) = \vartheta_0 = \sin\rho. \tag{31}$$

The exact solution of (30), when $\zeta = \xi = 1$ is

$$\omega(\rho,\tau) = e^{-\tau}\sin\rho\,, \qquad \vartheta(\rho,\tau) = e^{-\tau}\sin\rho.$$

4.1.1 Solution by using HAETM

By using the proposed scheme and (31), (30) reduced into:

$$\omega_\ell(\rho,\tau) = (\kappa_\ell+\hbar)\omega_{\ell-1}(\rho,\tau) - (1-\kappa_\ell)\hbar\{e^{-\tau}\sin\rho\} - \hbar E^{-1}\left\{p^\zeta E\left[\frac{\partial^2\omega}{\partial\rho^2} + 2\sum_{i=0}^{\ell-1}\omega_i\frac{\partial\omega_{\ell-i}}{\partial\rho}\right] - \frac{\partial}{\partial\rho}\left(\sum_{i=0}^{\ell-1}\omega_i\vartheta_{\ell-1-i}\right)\right\}, \tag{32}$$

and

$$\vartheta_\ell(\rho,\tau) = (\kappa_\ell+\hbar)\vartheta_{\ell-1}(\rho,\tau) - (1-\kappa_\ell)\{e^{-\tau}\sin\rho\} - \hbar E^{-1}\left\{p^\xi E\left[\frac{\partial^2\vartheta}{\partial\rho^2} + 2\sum_{i=0}^{\ell-1}\vartheta_i\frac{\partial\vartheta_{\ell-i}}{\partial\rho}\right] - \frac{\partial}{\partial\rho}\left(\sum_{i=0}^{\ell-1}\omega_i\vartheta_{\ell-1-i}\right)\right\}. \tag{33}$$

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By solving (32) and (33), we get the iterative terms of $\omega_\ell(\rho,\tau)$ and $\vartheta_\ell(\rho,\tau)$ follows as:

$$\omega_0(\rho,\tau) = sin\,\rho, \qquad \vartheta_0(\rho,\tau) = sin\,\rho,$$

$$\omega_1(\rho,\tau) = \hbar\, sin\,\rho \frac{\tau^\zeta}{\Gamma(\zeta+1)}, \quad \vartheta_1(\rho,\tau) = \hbar\, sin\,\rho \frac{\tau^\xi}{\Gamma(\xi+1)},$$

$$\omega_2(\rho,\tau) = \frac{\hbar(\hbar+1)\, sin\,\rho\,\tau^\zeta}{\Gamma(\zeta+1)} + \hbar^2(1-2\,cos\,\rho)\,sin\,\rho \frac{\tau^{2\zeta}}{\Gamma(2\zeta+1)} + \hbar^2(2\,sin\,\rho\,cos\,\rho)\frac{\tau^{\zeta+\xi}}{\Gamma(\zeta+\xi+1)},$$

$$\vartheta_2(\rho,\tau) = \frac{\hbar(1+\hbar)\, sin\,\rho\,\tau^\xi}{\Gamma(\xi+1)} + \hbar^2(1-2\,cos\,\rho)\,sin\,\rho \frac{\tau^{2\xi}}{\Gamma(2\xi+1)} + \hbar^2(2\,sin\,\rho\,cos\,\rho)\frac{\tau^{\zeta+\xi}}{\Gamma(\zeta+\xi+1)}, \tag{34}$$

And so on.

Hence, the solution of Eq. (30) is given as:

$$\omega(\rho,\tau) = \sum_{\ell=0}^{\infty}\omega_\ell(\rho,\tau), \qquad \vartheta(\rho,\tau) = \sum_{\ell=0}^{\infty}\vartheta_\ell(\rho,\tau). \tag{35}$$

4.1.2 Solution by using IETM:

Taking the Elzaki transform on (30), we get:

$$E[\omega(\rho,\tau)] = p^2\omega(\rho,0) + p^\zeta E\left[\frac{\partial^2\omega}{\partial\rho^2} + 2\omega\frac{\partial\omega}{\partial\rho} - \frac{\partial(\omega\vartheta)}{\partial\rho}\right],$$

$$E[\vartheta(\rho,\tau)] = p^2\vartheta(\rho,o) + p^\xi E\left[\frac{\partial^2\vartheta}{\partial\rho^2} + 2\vartheta\frac{\partial\vartheta}{\partial\rho} - \frac{\partial(\omega\vartheta)}{\partial\rho}\right]. \tag{36}$$

Taking the inverse Elzaki transform of (36), we have:

$$\omega(\rho,\tau) = \omega(\rho,o) + E^{-1}\left[p^\zeta E\left[\frac{\partial^2\omega}{\partial\rho^2} + 2\omega\frac{\partial\omega}{\partial\rho} - \frac{\partial(\omega\vartheta)}{\partial\rho}\right]\right],$$

$$\vartheta(\rho,\tau) = \vartheta(\rho,o) + E^{-1}\left[p^\xi E\left[\frac{\partial^2\vartheta}{\partial\rho^2} + 2\vartheta\frac{\partial\vartheta}{\partial\rho} - \frac{\partial(\omega\vartheta)}{\partial\rho}\right]\right], \tag{37}$$

In view of (27), we get:

$$\omega_0(\rho,\tau) = sin\,\rho, \qquad \vartheta_0(\rho,\tau) = sin\,\rho,$$

$$\omega_1(\rho,\tau) = -\,sin\,\rho \frac{\tau^\zeta}{\Gamma(\zeta+1)}, \qquad \vartheta_1(\rho,\tau) = -\,sin\,\rho \frac{\tau^\xi}{\Gamma(\xi+1)},$$

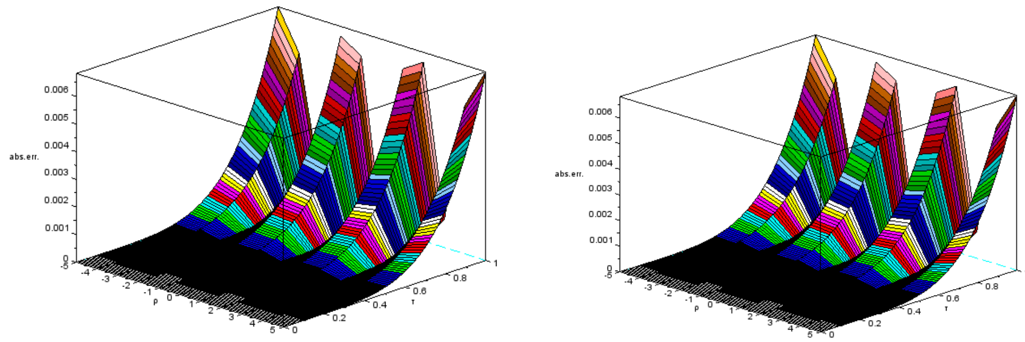
$$\omega_2(\rho,\tau) = \sin\rho(1-2\cos\rho)\frac{\tau^{2\zeta}}{\Gamma(2\zeta+1)} + 2\sin\rho\cos\rho\frac{\tau^{\zeta+\xi}}{\Gamma(\zeta+\xi+1)} - 2\sin\rho\cos\rho\frac{\Gamma(\zeta+\xi+1)}{\Gamma(\zeta+1)\Gamma(\xi+1)}\frac{\tau^{2\zeta+\xi}}{\Gamma(2\zeta+\xi+1)}$$
$$+ 2\sin\rho\cos\rho\frac{\Gamma(2\zeta+1)}{(\Gamma(\zeta+1))^2}\frac{\tau^{3\zeta}}{\Gamma(3\zeta+1)},$$

$$\vartheta_2(\rho,\tau) = sin\,\rho\,(1-2\,cos\,\rho)\frac{\tau^{2\xi}}{\Gamma(2\xi+1)} + 2\,sin\,\rho\,cos\,\rho\frac{\tau^{\zeta+\xi}}{\Gamma(\zeta+\xi+1)} - 2\,sin\,\rho\,cos\,\rho\frac{\Gamma(\zeta+\xi+1)}{\Gamma(\zeta+1)\Gamma(\xi+1)}\frac{\tau^{\zeta+2\xi}}{\Gamma(\zeta+2\xi+1)}$$
$$+2\,sin\,\rho\,cos\,\rho\frac{\Gamma(2\xi+1)}{(\Gamma(\xi+1))^2}\frac{\tau^{3\xi}}{\Gamma(3\xi+1)}, \tag{38}$$

And so on.

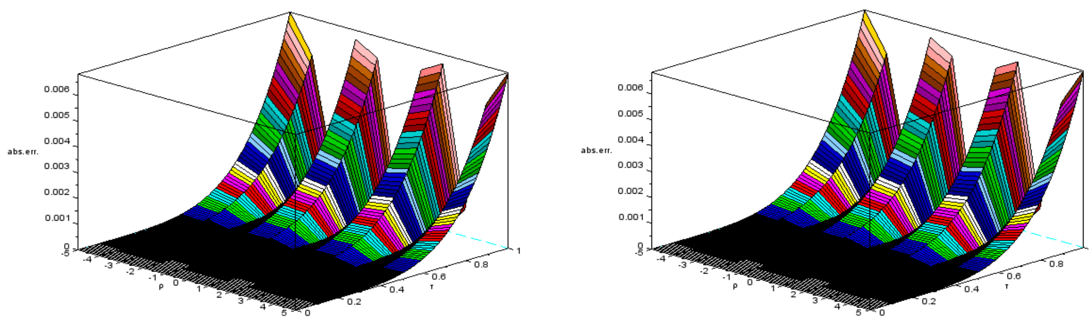
Finally, we have the series solution as:

$$\omega(\rho,\tau) = \sum_{\ell=0}^{\infty} \omega_\ell(\rho,\tau), \qquad \vartheta(\rho,\tau) = \sum_{\ell=0}^{\infty} \vartheta_\ell(\rho,\tau). \tag{39}$$



(a) (b)

Fig. 1. Surface of absolute error for $\omega(\rho,\tau)$ by (a) HAETM, and (b) IETM for $-5 \le \rho \le 5, 0 \le \tau \le 1$ and $\zeta = \xi = 1$.



(a) (b)

Fig. 2: Surface of absolute error for $\vartheta(\rho,\tau)$ by (a) HAETM, and (b) IETM for $-5 \le \rho \le 5, 0 \le \tau \le 1$ and $\zeta = \xi = 1$.

Table 1: The fifth order absolute errors for $\omega(\rho,\tau)$, when $-5 \le \rho \le 5$.

τ	HAETM	IETM
0.01	$1.9098963E-12$	$1.9098963E-12$
0.05	$5.9294056E-9$	$5.9294056E-9$
0.10	$1.8818028E-7$	$1.8818028E-7$
0.50	$5.5141181E-4$	$5.5141181E-4$
1.00	$1.6348008E-2$	$1.6348008E-2$

Table 2: The fifth order absolute errors for $\vartheta(\rho,\tau)$, when $-5 \le \rho \le 5$.

τ	HAETM	IETM
0.01	$1.9098963E-12$	$1.9098963E-12$
0.05	$5.9294056E-9$	$5.9294056E-9$
0.10	$1.8818028E-7$	$1.8818028E-7$
0.50	$5.5141181E-4$	$5.5141181E-4$
1.00	$1.6348008E-2$	$1.6348008E-2$

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Numerical outcomes of example 4.1 are given in table 1 and 2 which represent the absolute errors for our proposed schemes when $-5 \le \rho \le 5$. Figure 1 and 2 represents the Surface of absolute errors for $\omega(\rho,\tau)$and $\vartheta(\rho,\tau)$ by HAETM, and IETM for $-5 \le \rho \le 5, 0 \le \tau \le 1$ and $\zeta = \xi = 1$. The numerical outcomes show that both schemes are very effective and the absolute errors are very small at different values of τ for fifth order approximations.

Example 4.2. Consider two-dimensional FCB equations as:

$$\begin{aligned}\frac{\partial^\zeta \omega}{\partial \tau^\zeta} + \omega\frac{\partial \omega}{\partial \rho} + \vartheta\,\frac{\partial \omega}{\partial \sigma} &= \frac{1}{R}\left[\frac{\partial^2 \omega}{\partial \rho^2} + \frac{\partial^2 \omega}{\partial \sigma^2}\right],\\ \frac{\partial^\xi \vartheta}{\partial \tau^\xi} + \omega\frac{\partial \vartheta}{\partial \rho} + \vartheta\,\frac{\partial \vartheta}{\partial \sigma} &= \frac{1}{R}\left[\frac{\partial^2 \vartheta}{\partial \rho^2} + \frac{\partial^2 \vartheta}{\partial \sigma^2}\right].\end{aligned} \tag{40}$$

with the initial conditions:

$$\omega(\rho,\sigma,0) = \frac{3}{4} - \frac{1}{4\left(1+e^{(R(-\rho+\sigma))}\right)}, \qquad \vartheta(\rho,\sigma,0) = \frac{3}{4} + \frac{1}{4\left(1+e^{(R(-\rho+\sigma))}\right)}, \tag{41}$$

The exact solution of (40), when $\zeta = \xi = 1$ is

$$\omega(\rho,\sigma,\tau) = \frac{3}{4} - \frac{1}{4\left(1+e^{\left(\frac{R(-4\rho+4\sigma-\tau)}{32}\right)}\right)}, \qquad \vartheta(\rho,\sigma,\tau) = \frac{3}{4} + \frac{1}{4\left(1+e^{(R(-4\rho+4\sigma-\tau)/32)}\right)}. \tag{42}$$

4.2.1 Solution by using HAETM

Using the previous mentioned discussion, we have:

$$\begin{aligned}\omega_0(\rho,\sigma,\tau) &= \frac{3}{4} - \frac{1}{4\left(1+e^{(R(-\rho+\sigma))}\right)},\\ \vartheta_0(\rho,\sigma,\tau) &= \frac{3}{4} + \frac{1}{4\left(1+e^{(R(-\rho+\sigma))}\right)},\end{aligned} \tag{43}$$

$$\begin{aligned}\omega_1(\rho,\sigma,\tau) &= \frac{\hbar e^{(R/8(-\rho+\sigma))}R\tau^\zeta}{64\left(1+e^{(R/8(-\rho+\sigma))}\right)^3\Gamma(\zeta+1)} + \frac{\hbar e^{(R/4(-\rho+\sigma))}R\tau^\zeta}{64\left(1+e^{(R/8(-\rho+\sigma))}\right)^3\Gamma(\zeta+1)} - \frac{\hbar e^{\left(\frac{R}{8(-\rho+\sigma)}\right)}R\tau^\zeta}{128\left(1+e^{\left(\frac{R}{8(-\rho+\sigma)}\right)}\right)^2\Gamma(\zeta+1)},\\ \vartheta_1(\rho,\sigma,\tau) &= -\frac{\hbar e^{(R/8(-\rho+\sigma))}R\tau^\xi}{64\left(1+e^{(R/8(-\rho+\sigma))}\right)^3\Gamma(\xi+1)} - \frac{\hbar e^{(R/4(-\rho+\sigma))}R\tau^\xi}{64\left(1+e^{(R/8(-\rho+\sigma))}\right)^3\Gamma(\xi+1)} + \frac{\hbar e^{(R/8(-\rho+\sigma))}R\tau^\xi}{128\left(1+e^{(R/8(-\rho+\sigma))}\right)^2\Gamma(\xi+1)},\end{aligned} \tag{44}$$

And so on.

Hence, the solution of (40) is given as:

$$\begin{aligned}\omega(\rho,\sigma,\tau) &= \sum_{\ell=0}^{\infty}\omega_\ell(\rho,\sigma,\tau),\\ \vartheta(\rho,\sigma,\tau) &= \sum_{\ell=0}^{\infty}\vartheta_\ell(\rho,\sigma,\tau).\end{aligned} \tag{45}$$

4.2.2 Solution by using IETM:

Using the previous discussion, we have:

$$\begin{aligned}\omega_0(\rho,\sigma,\tau) &= \frac{3}{4} - \frac{1}{4\left(1+e^{(R(-\rho+\sigma))}\right)},\\ \vartheta_0(\rho,\sigma,\tau) &= \frac{3}{4} + \frac{1}{4\left(1+e^{(R(-\rho+\sigma))}\right)},\end{aligned} \tag{46}$$

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$$\omega_1(\rho,\sigma,\tau) = -\frac{e^{\left(R/8(-\rho+\sigma)\right)}R\tau^{\zeta}}{64\left(1+e^{\left(R/8(-\rho+\sigma)\right)}\right)^3\Gamma(\zeta+1)} - \frac{e^{\left(R/4(-\rho+\sigma)\right)}R\tau^{\zeta}}{64\left(1+e^{\left(R/8(-\rho+\sigma)\right)}\right)^3\Gamma(\zeta+1)} + \frac{e^{\left(R/8(-\rho+\sigma)\right)}R\tau^{\zeta}}{128\left(1+e^{\left(R/8(-\rho+\sigma)\right)}\right)^2\Gamma(\zeta+1)},$$

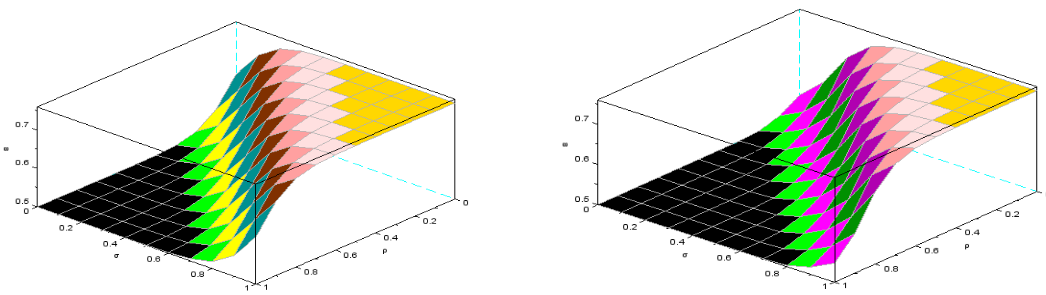
$$\vartheta_1(\rho,\sigma,\tau) = \frac{e^{\left(R/8(-\rho+\sigma)\right)}R\tau^{\xi}}{64\left(1+e^{\left(R/8(-\rho+\sigma)\right)}\right)^3\Gamma(\xi+1)} + \frac{e^{\left(R/4(-\rho+\sigma)\right)}R\tau^{\xi}}{64\left(1+e^{\left(R/8(-\rho+\sigma)\right)}\right)^3\Gamma(\xi+1)} - \frac{e^{\left(R/8(-\rho+\sigma)\right)}R\tau^{\xi}}{128\left(1+e^{\left(R/8(-\rho+\sigma)\right)}\right)^2\Gamma(\xi+1)}, \quad (47)$$

And so on.

Hence, the solution of (40) is given as:

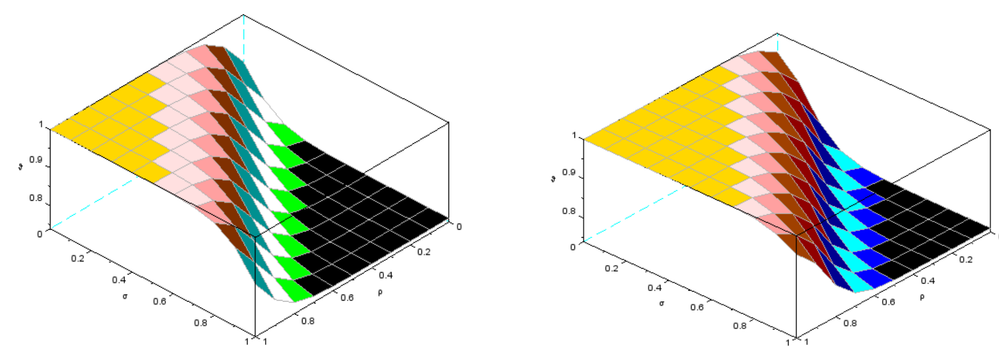
$$\omega(\rho,\sigma,\tau) = \sum_{\ell=0}^{\infty} \omega_\ell(\rho,\sigma,\tau), ,$$

$$\vartheta(\rho,\sigma,\tau) = \sum_{\ell=0}^{\infty} \vartheta_\ell(\rho,\sigma,\tau). \quad (48)$$



(a) (b)

Fig. 3: Third order distributions solutions of example 4.2 by HAETM for $\omega(\rho,\sigma,\tau)$ at (a) $\tau = 0.01$and (b) $\tau = 0.5$ with R=100.



(a) (b)

Fig. 4: Third order distributions solutions of example 4.2 by HAETM for $\vartheta(\rho,\sigma,\tau)$ at (a) $\tau = 0.01$ and (b) $\tau = 0.5$ with R=100.

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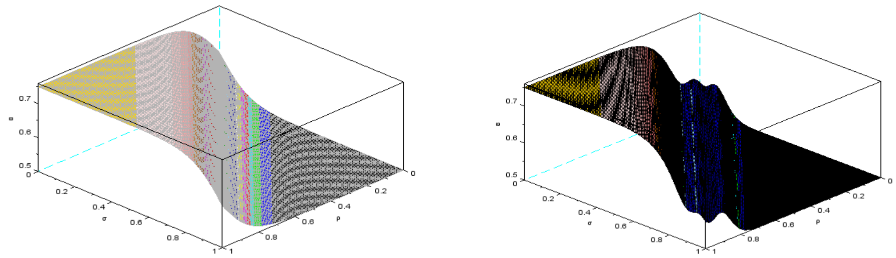
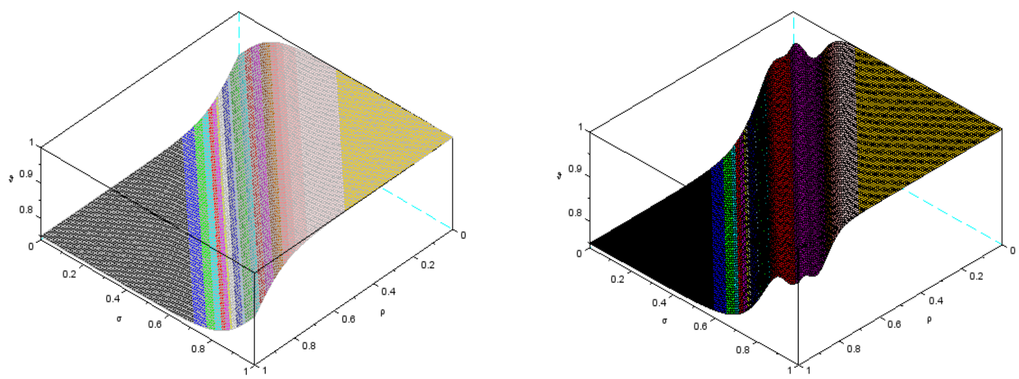


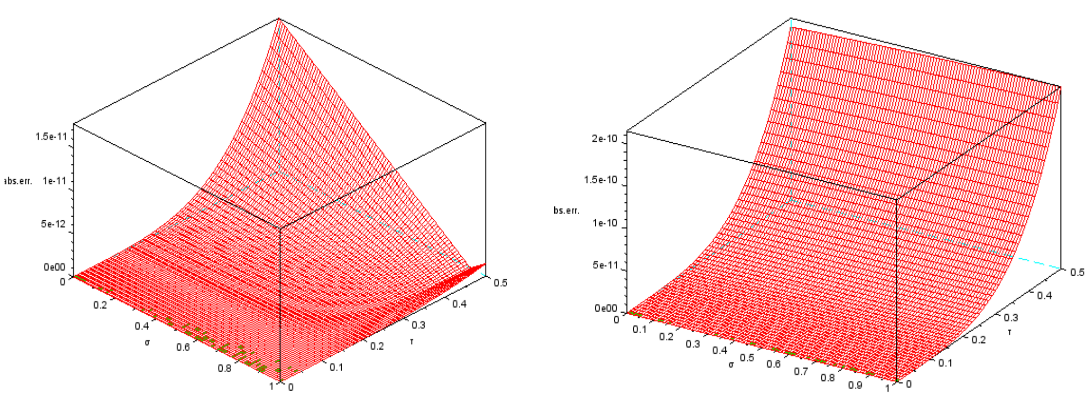
(a) (b)

Fig. 5: Third order distributions solutions of example 4.2 by IETM for $\omega(\rho,\sigma,\tau)$ at (a) $\tau = 0.01$and (b) $\tau = 0.5$ with R=100.



(a) (b)

Fig. 6: Third order distributions solutions of example 4.2 by IETM for $\vartheta(\rho,\sigma,\tau)$ at (a) $\tau = 0.01$ and (b) $\tau = 0.5$ with R=100.



(a) (b)

Fig. 7: Surfaces of absolute errors of example 4.2 for $\omega(\rho,\sigma,\tau)$ by (a) HAETM, and (b) IETM at $0 \le \sigma \le 1, 0 \le \tau \le 0.5, \rho = 0.9$ and $\zeta = \xi = 1.$

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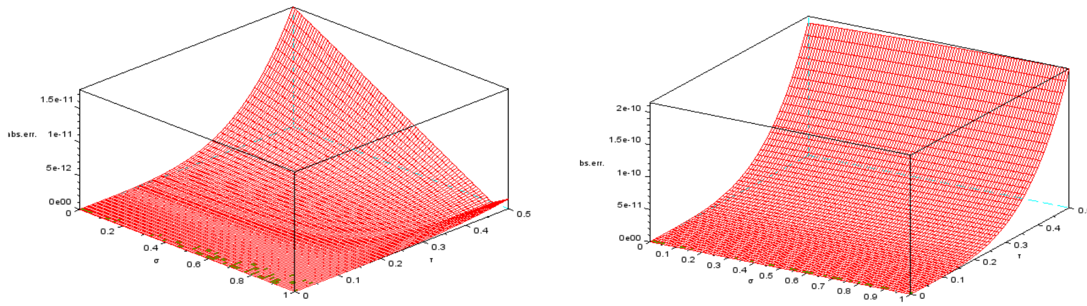
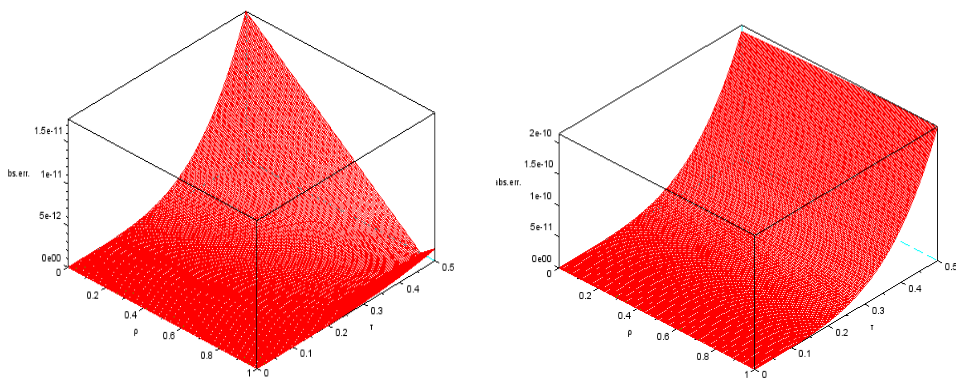


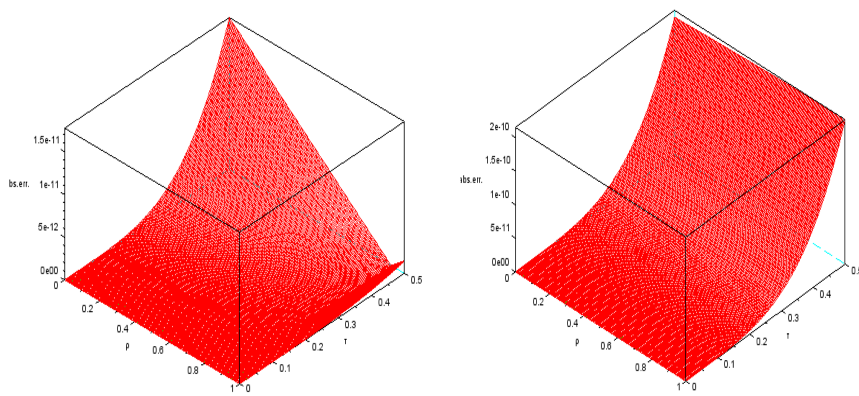
(a) (b)

Fig. 8. Surfaces of absolute errors of example 4.2 for $\vartheta(\rho,\sigma,\tau)$ by (a) HAETM, and (b) IETM at $0 \le \sigma \le 1, 0 \le \tau \le 0.5, \rho = 0.9\ and\ \zeta = \xi = 1$.



(a) (b)

Fig. 9. Surface of absolute error for $\omega(\rho,\sigma,\tau)$ by (a) HAETM, and (b) IETM for example 4.2 at $0 \le \rho \le 1, 0 \le \tau \le 0.5$, $\sigma = 0.9\ and\ \zeta = \xi = 1$.



(a) (b)

Fig. 10. Surface of absolute error for $\vartheta(\rho,\sigma,\tau)$ by (a) HAETM, and (b) IETM for example 4.2 at $0 \le \rho \le 1, 0 \le \tau \le 0.5, \sigma = 0.9\ and\ \zeta = \xi = 1$.

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Table 3: The third order absolute errors of 4.2 for $\omega(\rho,\sigma,\tau)$, when $0 \le \rho \le 1, 0 \le \sigma \le 1$.

τ	HAETM	IETM
0.05	0.0	$2.1316E-14$
0.10	$1.1102230E-16$	$3.4173E-13$
0.20	$4.9960036E-15$	$5.4738E-12$
0.30	$3.7747583E-14$	$2.7736E-11$
0.40	$1.5898394E-13$	$8.7738E-11$
0.50	$4.8505644E-13$	$2.144E-10$

Table 4: The third order absolute errors of 4.2 for $\vartheta(\rho,\sigma,\tau)$, when $0 \le \rho \le 1, 0 \le \sigma \le 1$.

τ	HAETM	IETM
0.05	0.0	$2.1316E-14$
0.10	$1.1102230E-16$	$3.4173E-13$
0.20	$4.9960036E-15$	$5.4738E-12$
0.30	$3.7747583E-14$	$2.7736E-11$
0.40	$1.5898394E-13$	$8.7738E-11$
0.50	$4.8505644E-13$	$2.144E-10$

Numerical outcomes of example 4.2 for third order approximations are given in tables 3 and 4, which represents the absolute errors for our proposed schemes for $\omega(\rho,\sigma,\tau)$and $\vartheta(\rho,\sigma,\tau)$, when $0 \le \rho \le 1, 0 \le \sigma \le 1$, and R=1 respectively. One can say that the estimated solutions gained by proposed schemes converge faster than the approximate solutions obtained using schemes in [8, 11] to exact solution. Figures 3-6 represents the distributions of approximation solutions of our proposed schemes for both $\omega(\rho,\sigma,\tau)$ and $\vartheta(\rho,\sigma,\tau)$ at $\tau = 0.01$, $\tau = 0.5$ respectively, with R=100. Surface of absolute error for $\omega(\rho,\sigma,\tau)$ and $\vartheta(\rho,\sigma,\tau)$ at $\zeta = \xi = 1$, $0 \le \tau \le 0.5$, gained by IETM and HAETM when $0 \le \sigma \le 1, \rho = 0.9$, is exhibited in Figs. 7 and 8, respectively and for $0 \le \rho \le 1, \sigma = 0.9$, is exhibited in Fig. 9 and 10, respectively. It is well-known that in these two cases the same numerical results are obtained.

5 Conclusion

The main objective of this article was to construct efficient approximate analytical and numerical solutions to the one- and two-dimensional FCBEs. We successfully achieved this target by using two approaches: namely, HAETM and IETM. The gained solutions approved the reliability and accuracy of the proposed approaches. The numerical and graphical studies showed that both approaches HAETM and IETM offered the same results in the case of the one-dimensional FCBE, but in the case of two-dimensional FCBE, the solution gained by HAETM converge faster than the approximate solution by IETM.

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