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Stability of First Order Linear General Quantum Difference Equations in a Banach Algebra

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Abstract: The general quantum difference operator D_β is defined by $D_\beta y(t) = (y(\beta(t)) - y(t))/(\beta(t) - t)$, $\beta(t) \neq t$ where the function $\beta(t)$ is strictly increasing continuous on an interval $I \subseteq \mathbb{R}$ and has a unique fixed point $s_0 \in I$. In this paper, we establish the characterizations of stability of the first order linear β-difference equations, associated with D_β, in a Banach algebra $\mathbb{E}$ with a unit $\mathfrak{e}$ and norm $\|\cdot\|$. We prove the uniform stability, asymptotic stability, exponential stability and h-stability of these equations.

Keywords: A general quantum difference operator, β-difference equations, Banach algebra, uniform stability, asymptotic stability, exponential stability, h-stability

1 Introduction and Preliminaries

Hamza et al. (2015) in [1], introduced the quantum difference calculus associated with the β-difference operator defined as:

$$D_\beta f(t) = \begin{cases} \dfrac{f(\beta(t)) - f(t)}{\beta(t) - t}, & t \neq s_0, \\ f'(s_0), & t = s_0. \end{cases}$$

The function $f : I \to \mathbb{R}$ is said to be β-differentiable on the interval $I \subseteq \mathbb{R}$, if $f'(s_0)$ exists, where $s_0 \in I$ is the unique fixed point of the function $\beta(t)$ which is strictly increasing continuous defined on I. In [1], two inequalities were presented; the first inequality is $(t - s_0)(\beta(t) - t) \leq 0$ for all $t \in I$, in this case $\lim_{k\to\infty} \beta^k(t) = s_0$; $\beta^k(t) := \underbrace{\beta \circ \beta \circ \ldots \circ \beta}_{k-times}(t)$. The Jackson q-difference operator with $\beta(t) = qt$, $q \in (0,1)$, $s_0 = 0$ and the Hahn difference operator with $\beta(t) = qt + \omega$, $q \in (0,1)$, $\omega > 0, s_0 = \frac{\omega}{1-q}$ are examples of quantum operators with $\beta(t)$ satisfy this inequality. On the other hand, the second inequality is $(t - s_0)(\beta(t) - t) \geq 0$ for all $t \in I$, in this case $\lim_{k\to\infty} \beta^k(t) = \infty$ and the backward Hahn difference operator with $\beta(t) = qt + \omega$, $q > 1, \omega \geq 0$ is an example of this inequality, see [2,3]. In [4], the different types of the function $\beta(t)$ contain finite and denumerable fixed points that one can construct the associated calculi were presented. The quantum difference operators deal with sets of non-differentiable functions. The applications of these operators can be used in several fields of mathematics and physics, see, e.g.[5,6,7,8]. In [9], some properties of the β-exponential functions $e_{A,\beta}(t)$ and $E_{A,\beta}(t)$ were defined in a Banach algebra $\mathbb{E}$ with a unit $\mathfrak{e}$. Moreover, it is proved that the first order β-initial value problems for a mapping $A : I \to \mathbb{E}$ continuous at s_0, with the form,

$$D_\beta y(t) = A(t)y(t), \;\; y(s_0) = \mathfrak{e}, \tag{1}$$

and

$$D_\beta y(t) = -A(t)y(\beta(t)), \;\; y(s_0) = \mathfrak{e}, \tag{2}$$

have respectively the unique solutions

$$e_{A,\beta}(t) = \left[\prod_{k=0}^{\infty} \left[\mathfrak{e} - A(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))\right]\right]^{-1} \tag{3}$$

and

$$E_{A,\beta}(t) = \prod_{k=0}^{\infty} \left[\mathfrak{e} + A(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))\right], \tag{4}$$

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where $A(t)$, $A(\beta^k(t))$ commute for every k and $e^{-1}_{A,\beta}(t) = E_{-A,\beta}(t)$, provided that both the infinite products in (3) and (4) converge. In addition, for a mapping $g : I \to \mathbb{E}$ continuous at s_0, the non-homogeneous β-difference equation

$$D_\beta y(t) = A(t)y(t) + g(t), \ \ y(s_0) = y_0,$$

has the unique solution

$$y(t) = e_{A,\beta}(t)\left[y_0 + \int_{s_0}^{t} E_{-A,\beta}\big(\beta(\tau)\big)g(\tau)d_\beta\tau\right].$$

Theorem 1.1. ([9]) Let $x, y : I \to \mathbb{E}$ be β-differentiable on I. Then:

(i) The product $xy : I \to \mathbb{E}$ is β-differentiable on I,

$$\begin{aligned} D_\beta(xy)(t) &= (D_\beta x(t))y(t) + x(\beta(t))D_\beta y(t) \\ &= (D_\beta x(t))y(\beta(t)) + x(t)D_\beta y(t), \end{aligned}$$

where, $(xy)(t) = x(t)y(t)$.

(ii) Let y be invertible, then xy^{-1} is β-differentiable at t and

$$\begin{aligned} D_\beta(xy^{-1})(t) &= (D_\beta x(t))(y(\beta(t)))^{-1} \\ &\quad - x(t)(y(\beta(t)))^{-1}(D_\beta y(t))(y(t))^{-1}, \end{aligned}$$

provided that for every $t \in I$, $(y(t))^{-1}$ exists.

(iii) $D_\beta(y^{-1})(t) = -(y(\beta(t)))^{-1}(D_\beta y(t))(y(t))^{-1}$, provided that for every $t \in I$, $(y(t))^{-1}$ exists.

Lemma 1.2. ([9]) If $y : I \to \mathbb{E}$ is a continuous mapping at s_0, then the sequence $\{y(\beta^k(t))\}_{k=0}^{\infty}$ converges uniformly to $y(s_0)$ on every compact interval $J \subseteq I$ containing s_0.

Theorem 1.3. ([9]) If $y : I \to \mathbb{E}$ is a continuous mapping at s_0, then the series $\sum_{k=0}^{\infty} \| \big(\beta^k(t) - \beta^{k+1}(t)\big) y(\beta^k(t))\|$ is uniformly convergent on every compact interval $J \subseteq I$ containing s_0.

Lemma 1.4. ([9]) Let $y : I \to \mathbb{E}$ be β-differentiable and $D_\beta y(t) = 0$ for all $t \in I$, then $y(t) = y(s_0)$ for all $t \in I$.

In [10], the theory of the linear β-difference equations was build up. Also, the β-Laplace transform associated with D_β was deduced in [11]. Moreover, the β-Sturm Liouville problem was investigated in [12]. In [13], the β-variational calculus was presented. Furthermore, the β-convolution theorem and some properties were proved in [14]. To proceed the study of the β-calculus, we study the stability of the linear β-difference equations. Indeed, the stability of the differential and difference equations has important role in different fields such as engineering, mathematical biology, pharmacometrics, control systems and physical systems, see, e.g. [15,16,17]. Recently, the characterizations of stability has been studied in fractional differential equations, dynamic equations and difference equations, see [18,19,20]. Furthermore, there are many types of stability such as the uniform stability, asymptotic stability, uniform asymptotic stability, global stability, global asymptotic stability, exponential stability, uniform exponential stability and h-stability. In [21,22], different types of stability of the linear dynamic equations were investigated on time scales. Also, Hamza et al studied the characterizations of stability of the linear Hahn difference equations in a Banach space and Banach algebras in [23,24].

In this paper, we introduce in a Banach algebra $\mathbb{E}$ with a unit $\mathfrak{e}$ and norm $\|\cdot\|$, the concepts of some types of the stability of the zero solution, $y = 0$, of the β-difference equation:

$$D_\beta y(t) = F(t,y), \quad y(\tau) = y_\tau \in \mathbb{E}, \ \ t,\tau \in I, \ t \geq \tau. \quad (5)$$

We assume $F(t,0) = 0$ for all $t \in I$, consequently, $y = 0$ is a solution of equation (5). Furthermore, we study the uniform stability, the asymptotic stability, the exponential stability and the h-stability of the homogeneous β-difference equation:

$$D_\beta y(t) = A(t)y(t), \ y(\tau) = y_\tau \in \mathbb{E}, \text{ for all } t \geq \tau, \ t,\tau \in I, \quad (6)$$

and the non-homogeneous β-difference equation:

$$D_\beta y(t) = A(t)y(t) + g(t), \ y(\tau) = y_\tau \in \mathbb{E}, \text{ for all } t \geq \tau, \ t,\tau \in I, \quad (7)$$

where $A, g : I \to \mathbb{E}$ are continuous mappings at s_0.

Throughout this paper, D_β means applying the β-derivative with respect to the variable t. Also, $e_{A,\beta}(t,\tau) = e_{A,\beta}(t)e^{-1}_{A,\beta}(\tau)$ and then $e_{A,\beta}(\tau,t) = e^{-1}_{A,\beta}(t,\tau)$.

In the following Section 2, we introduce the definitions of some types of stability for the β-difference equation (5). Moreover, we study the stability, the uniform stability, the asymptotic stability, the global asymptotic stability, the exponential stability, the uniform exponential stability, the h-stability and the uniform h-stability of the β-difference equations (6) and (7).

2 Main results

Lemma 2.1. The homogeneous β-difference equation (6) has a unique solution $e_{A,\beta}(t,\tau)y_\tau$ and the non-homogeneous β-difference equation (7) has a unique solution

$$y(t) = e_{A,\beta}(t,\tau)\left[y_\tau + \int_{\tau}^{t} e_{A,\beta}\big(\tau,\beta(\xi)\big)g(\xi)d_\beta\xi\right]. \quad (8)$$

Proof. By Equation (1), $D_\beta e_{A,\beta}(t) = A(t)e_{A,\beta}(t)$. Then,

$$\begin{aligned} D_\beta e_{A,\beta}(t,\tau) &= D_\beta\left[e_{A,\beta}(t)e^{-1}_{A,\beta}(\tau)\right] \\ &= D_\beta\left[e_{A,\beta}(t)\right]e^{-1}_{A,\beta}(\tau) \\ &= A(t)\left[e_{A,\beta}(t)e^{-1}_{A,\beta}(\tau)\right] \\ &= A(t)e_{A,\beta}(t,\tau). \end{aligned}$$

Now,

$$\begin{aligned} D_\beta(e_{A,\beta}(t,\tau)y_\tau) &= D_\beta(e_{A,\beta}(t,\tau))y_\tau + e_{A,\beta}(\beta(t),\tau)D_\beta(y_\tau) \\ &= A(t)e_{A,\beta}(t,\tau)y_\tau, \end{aligned}$$

where $D_\beta(y_\tau) = 0$ since y_τ is constant.
Also, $e_{A,\beta}(t,\tau)y_\tau|_{t=\tau} = e_{A,\beta}(\tau,\tau)y_\tau = y_\tau$. Thus, $e_{A,\beta}(t,\tau)y_\tau$ is a solution of the homogeneous equation (6). To prove the uniqueness, let equation (6) has another solution $z(t) \neq e_{A,\beta}(t,\tau)$. Then

$$\begin{aligned} D_\beta\left[e_{A,\beta}(\tau,t)z(t)\right] &= [D_\beta e_{A,\beta}(\tau,t)]z(t) \\ &\quad + e_{A,\beta}(\tau,\beta(t))D_\beta z(t) \\ &= -e_{A,\beta}(\tau,\beta(t))A(t)z(t) \\ &\quad + e_{A,\beta}(\tau,\beta(t))A(t)z(t) \\ &= 0, \end{aligned}$$

and therefore, $e_{A,\beta}(\tau,t)z(t)$ is a constant for all $t \in I$.
Hence, using the initial condition $e_{A,\beta}(\tau,t)z(t)|_{t=\tau} = e_{A,\beta}(\tau,\tau)z(\tau) = y_\tau$.
Consequently, $z(t) = e_{A,\beta}(t,\tau)y_\tau$ is the unique solution of the β-IVP (6).
On the other hand, from equation (8)

$$\begin{aligned} D_\beta y(t) &= D_\beta\left(e_{A,\beta}(t,\tau)\right)y_\tau + D_\beta\left(e_{A,\beta}(t,\tau)\right) \\ &\qquad \int_\tau^t e_{A,\beta}\left(\tau,\beta(\xi)\right)g(\xi)d_\beta\xi \\ &\quad + e_{A,\beta}\left(\beta(t),\tau\right)e_{A,\beta}\left(\tau,\beta(t)\right)g(t) \\ &= A(t)e_{A,\beta}(t,\tau)y_\tau + A(t)e_{A,\beta}(t,\tau) \\ &\qquad \int_\tau^t e_{A,\beta}\left(\tau,\beta(\xi)\right)g(\xi)d_\beta\xi + g(t) \\ &= A(t)y(t) + g(t). \end{aligned}$$

Then, $y(t)$ is a solution of (7). To prove the uniqueness of the solution, let $x(t) \neq y(t)$ be another solution of equation (7). Suppose that $z(t) = e_{A,\beta}(\tau,t)x(t)$, and hence $x(t) = e_{A,\beta}(t,\tau)z(t)$. Then,

$$\begin{aligned} A(t)e_{A,\beta}(t,\tau)z(t) + g(t) &= D_\beta\left[e_{A,\beta}(t,\tau)z(t)\right] \\ &= D_\beta(e_{A,\beta}(t,\tau))z(t) \\ &\quad + e_{A,\beta}(\beta(t),\tau)D_\beta(z(t)) \\ &= A(t)e_{A,\beta}(t,\tau)z(t) \\ &\quad + e_{A,\beta}(\beta(t),\tau)D_\beta(z(t)). \end{aligned}$$

Consequently,

$$D_\beta(z(t)) = e_{A,\beta}(\tau,\beta(t))g(t).$$

This yields that

$$z(t) = y_\tau + \int_\tau^t e_{A,\beta}\left(\tau,\beta(\xi)\right)g(\xi)d_\beta\xi,$$

$z(\tau) = e_{A,\beta}(\tau,\tau)y(\tau) = y_\tau$. Hence, $x(t) = y(t)$. □

Definition 2.2. A solution $y(t,\tau,y_\tau)$ of the β-difference equation (6) is said to be bounded if there is a constant $\kappa(\tau) > 0$ that depends on τ and y_τ such that

$$\|y(t,\tau,y_\tau)\| \leq \kappa(\tau)\|y_\tau\|, \quad t \in I.$$

Definition 2.3. We say that the family $\{e_{A,\beta}(t,\tau) : t,\tau \in I,\ t \geq \tau\}$ is stable if it is bounded i.e. if there is $\kappa(\tau) > 0$ such that

$$\|e_{A,\beta}(t,\tau)\| \leq \kappa(\tau) \ \text{ for all } t,\tau \in I,\ t \geq \tau.$$

2.1 Types of stability

In the following, we introduce the definitions of some types of stability for the β-difference equation (5).

Definition 2.4. The β-difference equation (5) is called stable if for all $\varepsilon > 0, \tau \in I$, there is $\delta = \delta(\varepsilon,\tau) > 0$ such that for a solution $y(t,\tau,y_\tau)$, if $\|y_\tau\| < \delta$ implies that $\|y(t,\tau,y_\tau)\| < \varepsilon$, for all $t \geq \tau, t,\tau \in I$. The stability of the β-difference equation (5) is equivalent to the stability of the zero solution, $y = 0$. Furthermore, the β-difference equation (5) is said to be stable if all of its solutions are stable.

Definition 2.5. The β-difference equation (5) is called uniformly stable if for all $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that if $\|y_\tau\| < \delta$ implies that $\|y(t,\tau,y_\tau)\| < \varepsilon$, for all $t \geq \tau, t,\tau \in I$.

Definition 2.6. The β-difference equation (5) is called asymptotically stable if it is stable and there is $\delta = \delta(\tau) > 0$ such that if $\|y_\tau\| < \delta(\tau)$ implies that $\lim_{t\to\infty}\|y(t,\tau,y_\tau)\| = 0$.

Definition 2.7. The β-difference equation (5) is called uniformly asymptotically stable if it is uniformly stable and there is $\delta > 0$, such that if $\|y_\tau\| < \delta$ implies that $\lim_{t\to\infty}\|y(t,\tau,y_\tau)\| = 0$.

Definition 2.8. The β-difference equation (5) is called globally asymptotically stable if it is stable and for a solution $y(t) = y(t,\tau,y_\tau)$ of equation (5), we have

$$\lim_{t\to\infty}\|y(t,\tau,y_\tau)\| = 0.$$

Definition 2.9. The β-difference equation (5) is called exponentially stable if there exist finite constants $\lambda > 0$ and $\kappa = \kappa(\tau) > 0$ such that

$$\|y(t,\tau,y_\tau)\| \leq \kappa \|y_\tau\| e_{-\lambda,\beta}(t,\tau), \quad \text{for all } t \geq \tau, \ t,\tau \in I.$$

Definition 2.10. The β-difference equation (5) is called uniformly exponentially stable if κ independent of $\tau \in I$.

Definition 2.11. Let $h : I \to \mathbb{R}$ be a positive bounded function. The β-difference equation (5) is called h-stable if for a solution $y(t) = y(t,\tau,y_\tau)$ of equation (5), we have

$$\|y(t,\tau,y_\tau)\| \leq \kappa(\tau)\|y_\tau\| h(t) h^{-1}(\tau), \quad \text{for all } t \geq \tau, \ t,\tau \in I,$$

where $\kappa = \kappa(\tau) \geq 1$ and $h^{-1}(\tau) = \frac{1}{h(\tau)}$.

Definition 2.12. The β-difference equation (5) is called h-uniformly stable if $\kappa \geq 1$ independent of $\tau \in I$.

2.2 Stability of the β-difference equations

In the following theorems we study the stability and the uniform stability of the homogeneous β-difference equation (6) and the non-homogeneous β-difference equation (7). We show that the β-difference equation (6) is said to be stable if and only if its solution $y(t) = e_{A,\beta}(t,\tau)y_\tau$ is bounded for all $t \geq \tau \in I$.

Theorem 2.13. The following statements are equivalent.

(a) The homogenous β-difference equation (6) is stable.
(b) There is $\kappa(\tau) > 0$ such that

$$\|e_{A,\beta}(t,\tau)\| \leq \kappa(\tau) \quad \text{for all } t,\tau \in I, \ \ t \geq \tau.$$

(c) For all $\tau \in I$, there is $\kappa(\tau) > 0$, such that for a solution $y(t) = y(t,\tau,y_\tau)$ of the homogenous β-difference equation (6), we have

$$\|y(t)\| \leq \kappa(\tau)\|y_\tau\|, \ \ t \geq \tau, \ \ \tau \in I.$$

Proof. $(a) \Rightarrow (b)$**.** Suppose that equation (6) is stable. Let $\varepsilon = 1$, there is $\delta > 0$ such that for a solution $y(t) = y(t,\tau,y_\tau)$, we have

$$\|y_\tau\| < \delta \Rightarrow \|e_{A,\beta}(t,\tau)y_\tau\| < 1, \quad \text{for all } \ t \geq \tau, \ \ t \in I.$$

Since $\|y_\tau\| < \delta$, let $0 \neq z_0 \in \mathbb{E}$, and take $y_\tau = \delta z_0/(2\|z_0\|)$. Therefore,

$$\|e_{A,\beta}(t,\tau)z_0\| < 2\|z_0\|/\delta, \ \ t \geq \tau, \ \ t \in I.$$

Then, by the uniform bounded-ness theorem, [25], there is $\kappa(\tau) > 0$ such that

$$\|e_{A,\beta}(t,\tau)\| \leq \sup_{\|z_0\|=1} 2\|z_0\|/\delta = \kappa(\tau) \text{ for all } t,\tau \in I, \ t \geq \tau.$$

$(b) \Rightarrow (c)$**.** There is $\kappa(\tau) > 0$ such that $\|e_{A,\beta}(t,\tau)\| \leq \kappa(\tau)$, for all $t \in I$, $t \geq \tau$. Therefore, for a solution $y(t) = y(t,\tau,y_\tau)$ of equation (6), we have

$$\begin{aligned}\|y(t)\| &= \|e_{A,\beta}(t,\tau)y_\tau\| \\ &\leq \kappa(\tau)\|y_\tau\|, \ \ \text{for all } \ t \geq \tau, \ t,\tau \in I.\end{aligned}$$

$(c) \Rightarrow (a)$**.** Assume that there is $\kappa(\tau) > 0$, $\tau \in I$ such that

$$\|y(t)\| \leq \kappa(\tau)\|y_\tau\|, \ \ t \in I.$$

Let $\varepsilon > 0$, and take $\delta = \frac{\varepsilon}{\kappa(\tau)}$, $\tau \in I$. For any $y_\tau \in \mathbb{E}$ such that $\|y_\tau\| < \delta$, we get

$$\begin{aligned}\|y(t)\| &\leq \kappa(\tau)\|y_\tau\| \\ &= \frac{\varepsilon}{\delta}\|y_\tau\| < \varepsilon, \ \ t \geq \tau, \ \ t,\tau \in I.\end{aligned}$$

□

Corollary 2.14. If there exists $\gamma = \gamma(\tau) \geq 0$ such that

$$\int_\tau^t \|g(\xi)\|\kappa(\beta(\xi))d_\beta\xi \leq \gamma, \ \ t,\tau \in I.$$

Then, the homogeneous β-difference equation (6) is stable if and only if the non-homogeneous β-difference equation (7) is stable.

Proof. Suppose that equation (6) is stable. Then, by Theorem 2.13, there is $\kappa(\tau) > 0$ such that

$$\|e_{A,\beta}(t,\tau)\| \leq \kappa(\tau) \ \ \text{for all } t,\tau \in I, \ \ t \geq \tau.$$

Let $y_l(t)$ be a solution of equation (7) with initial value y_τ. Then by using equation (8), we get

$$\begin{aligned}\|y_l(t)\| &\leq \kappa(\tau)\|y_\tau\| + \int_\tau^t \|g(\xi)\|\kappa(\beta(\xi))d_\beta\xi \\ &\leq \kappa\|y_\tau\| + \gamma.\end{aligned}$$

Let $\varepsilon > 0$, and take $\delta = \frac{\varepsilon}{\kappa(\tau)}$, $\tau \in I$ and $\gamma = 0$. For any $y_\tau \in \mathbb{E}$ such that $\|y_\tau\| < \delta$, we get

$$\begin{aligned}\|y_l(t)\| &\leq \kappa(\tau)\|y_\tau\| \\ &< (\frac{\varepsilon}{\delta})\delta = \varepsilon, \ \ t \geq \tau, \ \ t,\tau \in I.\end{aligned}$$

Therefore, the non-homogeneous β-difference equation (7) is stable. Conversely, assume that equation (7) is stable. Then, for all $\varepsilon > 0, \tau \in I$, there is $\delta = \delta(\varepsilon,\tau) > 0$ such that for a solution $y_l(t)$ of equation (7) if $\|y_\tau\| < \delta$ implies that $\|y_l(t)\| < \varepsilon$, for all $t \geq \tau, t,\tau \in I$, and then,

$$\|y_l(t)\| \leq \|e_{A,\beta}(t,\tau)y_\tau\| + \gamma.$$

Consequently, $\|y(t)\| = \|e_{A,\beta}(t,\tau)y_\tau\| < \varepsilon$. Hence, the homogeneous β-difference equation (6) is stable. □

The proofs of the following Theorem 2.15 and Corollary 2.16 will be omitted since they are similar to the proofs of Theorem 2.13 and Corollary 2.14.

Theorem 2.15. The following statements are equivalent

(i_1) The homogeneous β-difference equation (6) is uniformly stable.
(i_2) There is $\kappa > 0$ independent of τ such that

$$\|e_{A,\beta}(t,\tau)\| \le \kappa \text{ for all } t,\tau \in I, \quad t \ge \tau.$$

(i_3) There is $\kappa > 0$ such that for a solution $y(t) = y(t,\tau,y_\tau)$ of the homogeneous β-difference equation (6), we have

$$\|y(t)\| \le \kappa \|y_\tau\|, \quad t \ge \tau, \ t \in I.$$

□

Corollary 2.16. If there exists $\gamma \ge 0$ such that

$$\int_\tau^t \kappa \|g(\xi)\| d_\beta \xi \le \gamma, \quad t,\tau \in I.$$

Then, the homogeneous β-difference equation (6) is uniformly stable if and only if the non-homogeneous β-difference equation (7) is uniformly stable. □

In the following, we present the asymptotic stability and global asymptotic stability of the β-difference equations (6) and (7).

Theorem 2.17. The following statements are equivalent

(i) The homogeneous β-difference equation (6) is asymptotically stable.
(ii) $\lim_{t\to\infty} \|e_{A,\beta}(t,\tau)y\| = 0$ for every $y \in \mathbb{E}$, $\tau \in I$.
(iii) The homogeneous β-difference equation (6) is globally asymptotically stable.

Proof. $(i) \Rightarrow (ii)$**.** Suppose that equation (6) is asymptotically stable. Then, there is $\delta(\tau) > 0$ such that for a solution $y(t) = y(t,\tau,y_\tau)$ of equation (6), with initial value y_τ, we have

$$\|y_\tau\| < \delta(\tau) \Rightarrow \lim_{t\to\infty} \|y(t)\| = 0.$$

Let $0 \ne y \in \mathbb{E}$. Take $y_\tau = \delta(\tau)y/(2\|y\|)$. Therefore,

$$\lim_{t\to\infty} \|e_{A,\beta}(t,\tau)\delta(\tau)y/(2\|y\|)\| = 0.$$

Then, $\lim_{t\to\infty} \|e_{A,\beta}(t,\tau)y\| = 0$.

$(ii) \Rightarrow (iii)$**.** Let $\lim_{t\to\infty} \|e_{A,\beta}(t,\tau)y\| = 0$ for every $y \in \mathbb{E}$, $\tau \in I$. Then, by the uniform bounded-ness theorem, there is $\kappa(\tau) > 0$ such that

$$\|e_{A,\beta}(t,\tau)\| \le \kappa(\tau) \text{ for all } t,\tau \in I, \quad t \ge \tau.$$

Thus, by Theorem 2.13, equation (6) is stable. Therefore, the homogeneous β-difference equation (6) is globally asymptotically stable.

$(iii) \Rightarrow (i)$**.** Assume that equation (6) is globally asymptotically stable. Then equation (6) is stable and for a solution $y(t) = y(t,\tau,y_\tau)$ of equation (6), we have

$$0 = \lim_{t\to\infty} \|y(t,\tau,y_\tau)\| = \lim_{t\to\infty} \|e_{A,\beta}(t,\tau)y_\tau\|,$$

By Theorem 2.13, we have

$$\|e_{A,\beta}(t,\tau)y_\tau\| \le \kappa(\tau)\|y_\tau\|.$$

Let $\varepsilon > 0$, and take $\delta(\tau) = \frac{\varepsilon}{\kappa(\tau)}$, $\tau \in I$. For any $y_\tau \in \mathbb{E}$ such that $\|y_\tau\| < \delta(\tau)$, we get

$$\|e_{A,\beta}(t,\tau)y_\tau\| < \varepsilon.$$

implies that $\lim_{t\to\infty} \|y(t)\| = 0$. Therefore, the homogeneous β-difference equation (6) is asymptotically stable. □

Corollary 2.18. If there exists $\gamma = \gamma(\tau) \ge 0$ such that

$$\int_\tau^t \|g(\xi)\| \|e_{A,\beta}(t,\beta(\xi))\| d_\beta \xi \le \gamma, \quad t,\tau \in I.$$

Then, the homogeneous β-difference equation (6) is globally asymptotically stable if and only if the non-homogeneous β-difference equation (7) is globally asymptotically stable.

Proof. Suppose that equation (6) be globally asymptotically stable. Then, $\lim_{t\to\infty} \|e_{A,\beta}(t,\tau)\| = 0$. Let $y_l(t)$ be a solution of equation (7). Therefore,

$$\begin{aligned}\|y_l(t)\| &\le \|e_{A,\beta}(t,\tau)y_\tau\| + \int_\tau^t \|g(\xi)\| \|e_{A,\beta}(t,\beta(\xi))\| d_\beta \xi \\ &\le \|e_{A,\beta}(t,\tau)\| \|y_\tau\| + \gamma,\end{aligned}$$

and then $\lim_{t\to\infty} \|y_l(t)\| = 0$. Hence, the non-homogeneous β-difference equation (7) is globally asymptotically stable. Conversely, suppose that equation (7) is globally asymptotically stable. Then, for a solution $y_l(t)$ of equation (7), we have $\lim_{t\to\infty} \|y_l(t)\| = 0$ and so,

$$0 = \lim_{t\to\infty} \|y_l(t)\| \le \lim_{t\to\infty} \|e_{A,\beta}(t,\tau)y_\tau\| + \gamma.$$

Then, $\lim_{t\to\infty} \|e_{A,\beta}(t,\tau)y_\tau\| = 0$. Therefore, by Theorem 2.17, the homogeneous β-difference equation (6) is globally asymptotically stable. □

Now, we introduce the exponential stability and the uniform exponential stability of the β-difference equations (6) and (7).

Theorem 2.19. The following statements are equivalent

(i) The homogeneous β-difference equation (6) is exponentially stable.
(ii) There is $\lambda > 0$ and $\kappa(\tau) > 0$ such that

$$\|e_{A,\beta}(t,\tau)\| \le \kappa(\tau)e_{-\lambda,\beta}(t,\tau) \text{ for all } t \ge \tau.$$

Proof. $(i) \Rightarrow (ii)$ Assume that equation (6) is exponentially stable. Then, there is $\kappa(\tau) > 0$ such that

$$\begin{aligned}\|y(t)\| &= \|e_{A,\beta}(t,\tau)y_\tau\| \\ &\le \kappa(\tau)e_{-\lambda,\beta}(t,\tau)\|y_\tau\|, \quad \text{for all } t \ge \tau.\end{aligned}$$

Hence, $\|e_{A,\beta}(t,\tau)\| \leq \kappa(\tau)e_{-\lambda,\beta}(t,\tau)$.
$(ii) \Rightarrow (i)$ Let $y(t) = y(t,\tau,y_\tau)$ be a solution of equation (6) with $y_\tau \in \mathbb{E}$. Then, we get

$$\begin{aligned}\|y(t)\| &= \|e_{A,\beta}(t,\tau)y_\tau\| \\ &\leq \kappa(\tau)e_{-\lambda,\beta}(t,\tau)\|y_\tau\|, \quad \text{for all } t \geq \tau.\end{aligned}$$

Therefore, the homogeneous β-difference equation (6) is exponentially stable. □

Corollary 2.20. If there exists $\gamma = \gamma(\tau) \geq 0$ such that

$$\int_\tau^t \|g(\xi)\|\kappa(\beta(\xi))e_{-\lambda,\beta}(\tau,\beta(\xi))d_\beta\xi \leq \gamma, \quad t \in I.$$

Then, the homogeneous β-difference equation (6) is exponentially stable if and only if the non-homogeneous β-difference equation (7) is exponentially stable.

Proof. Let equation (6) be exponentially stable. By Theorem 2.19, there is $\lambda > 0$ and $\kappa(\tau) > 0$ such that

$$\|e_{A,\beta}(t,\tau)\| \leq \kappa(\tau)e_{-\lambda,\beta}(t,\tau) \text{ for all } t \geq \tau.$$

Assume that $y_l(t)$ is a solution of equation (7) with initial value y_τ. Using equation (8), we have

$$\begin{aligned}\|y_l(t)\| &\leq \kappa(\tau)e_{-\lambda,\beta}(t,\tau)\|y_\tau\| \\ &\quad + \int_\tau^t \|g(\xi)\|\kappa(\beta(\xi))e_{-\lambda,\beta}(t,\beta(\xi))d_\beta\xi \\ &\leq \{\kappa\|y_\tau\| + \gamma\}e_{-\lambda,\beta}(t,\tau).\end{aligned}$$

Therefore, the non-homogeneous β-difference equation (7) is exponentially stable. Conversely, assume that equation (7) is exponentially stable. Then, there exist $\lambda > 0$ and $\kappa = \kappa(\tau) > 0$ such that

$$\|y_l(t)\| \leq \kappa\|y_\tau\|e_{-\lambda,\beta}(t,\tau), \quad \text{for all } t \geq \tau, \ t,\tau \in I.$$

Consequently, with $\gamma = 0$,

$$\|y(t)\| = \|e_{A,\beta}(t,\tau)y_\tau\| \leq \kappa\|y_\tau\|e_{-\lambda,\beta}(t,\tau).$$

Then the homogeneous β-difference equation (6) is exponentially stable. □

The proofs of the following Theorem 2.21 and Corollary 2.22 are the same technique as the proofs of Theorem 2.19 and Corollary 2.20, therefore they will be omitted.

Theorem 2.21. The following statements are equivalent

(i) The homogeneous β-difference equation (6) is uniformly exponentially stable.
(ii) There is $\lambda > 0$ and $\kappa > 0$ independent of τ such that

$$\|e_{A,\beta}(t,\tau)\| \leq \kappa e_{-\lambda,\beta}(t,\tau), \quad \text{for all } t \geq \tau.$$

□

Corollary 2.22. If there exists $\gamma \geq 0$ such that

$$\int_\tau^t \|g(\xi)\|\kappa e_{-\lambda,\beta}(\tau,\beta(\xi))d_\beta\xi \leq \gamma, \quad t \in I.$$

Then, the homogeneous β-difference equation (6) is uniformly exponentially stable if and only if the non-homogeneous β-difference equation (7) is uniformly exponentially stable. □

Next, we present the h-stability and the uniform h-stability of the homogeneous and non-homogeneous β-difference equations (6) and (7).

Theorem 2.23. The following statements are equivalent

(a) The homogeneous β-difference equation (6) is h-stable.
(b) There exists $\kappa = \kappa(\tau) \geq 1$ such that

$$\|e_{A,\beta}(t,\tau)\| \leq \kappa(\tau)h(t)h^{-1}(\tau), \quad \text{for all } t \geq \tau.$$

Proof. $(a) \Rightarrow (b)$. Suppose that equation (6) is h-stable. There exists $\kappa = \kappa(\tau) \geq 1$ such that for a solution $y(t) = y(t,\tau,y_\tau)$, $y_\tau \in \mathbb{E}$ of equation (6) satisfies

$$\|y(t)\| = \|e_{A,\beta}(t,\tau)y_\tau\| \leq \kappa(\tau)\|y_\tau\|h(t)h^{-1}(\tau), \quad \text{for all } t \geq \tau.$$

Therefore, we have

$$\|e_{A,\beta}(t,\tau)\| \leq \kappa(\tau)h(t)h^{-1}(\tau).$$

$(b) \Rightarrow (a)$. Suppose that $\|e_{A,\beta}(t,\tau)\| \leq \kappa(\tau)h(t)h^{-1}(\tau)$. For $\kappa = \kappa(\tau) \geq 1$, then

$$\begin{aligned}\|y(t)\| &= \|e_{A,\beta}(t,\tau)y_\tau\| \\ &\leq \|e_{A,\beta}(t,\tau)\|\|y_\tau\| \\ &\leq \kappa(\tau)\|y_\tau\|h(t)h^{-1}(\tau), \quad \text{for all } t \geq \tau.\end{aligned}$$

Hence, the homogeneous β-difference equation (6) is h-stable. □

Corollary 2.24. If there exists $\gamma = \gamma(\tau) \geq 0$ such that

$$\int_\tau^t \|g(\xi)\|\kappa(\beta(\xi))h(\tau)h^{-1}(\beta(\xi))d_\beta\xi \leq \gamma, \quad t \in I.$$

Then, the homogeneous β-difference equation (6) is h-stable if and only if the non-homogeneous β-difference equation (7) is h-stable.

Proof. Assume that equation (6) is h-stable.
By Theorem 2.23, there exists $\kappa = \kappa(\tau) \geq 1$ such that

$$\|e_{A,\beta}(t,\tau)\| \leq \kappa(\tau)h(t)h^{-1}(\tau), \quad \text{for all } t \geq \tau.$$

Let $y_I(t)$ be a solution of equation (7) with initial value y_τ. By using equation (8), we get

$$\begin{aligned}\|y_I(t)\| &\le \kappa(\tau)h(t)h^{-1}(\tau)\|y_\tau\| \\ &+ \int_\tau^t \|g(\xi)\|\kappa(\beta(\xi))h(t)h^{-1}(\beta(\xi))d_\beta\xi \\ &\le \{\kappa\|y_\tau\|+\gamma\}h(t)h^{-1}(\tau).\end{aligned}$$

Therefore, the non-homogeneous β-difference equation (7) is h-stable. Conversely, suppose that equation (7) is h-stable. Then, for $\kappa=\kappa(\tau)\ge 1$ we have

$$\|y_I(t)\| \le \kappa(\tau)\|y_\tau\|h(t)h^{-1}(\tau), \quad \text{for all } t\ge\tau,\ t,\tau\in I.$$

Consequently, with $\gamma=0$,

$$\|y(t)\| = \|e_{A,\beta}(t,\tau)y_\tau\| \le \kappa(\tau)\|y_\tau\|h(t)h^{-1}(\tau).$$

Hence, the homogeneous β-difference equation (6) is h-stable. □

The proofs of the following Theorem 2.25 and Corollary 2.26 are the same technique as the proofs of Theorem 2.23 and Corollary 2.24, accordingly they will be omitted.

Theorem 2.25. The following statements are equivalent

(a) The homogeneous β-difference equation (6) is uniformly h-stable.
(b) There exists $\kappa\ge 1$ independent of τ such that

$$\|e_{A,\beta}(t,\tau)\| \le \kappa h(t)h^{-1}(\tau), \quad \text{for all } t\ge\tau.$$

□

Corollary 2.26. If there exists $\gamma\ge 0$ such that

$$\int_\tau^t \|g(\xi)\|\kappa h(\tau)h^{-1}(\beta(\xi))d_\beta\xi \le \gamma, \quad t\in I.$$

Then, the homogeneous β-difference equation (6) is uniformly h-stable if and only if the non-homogeneous β-difference equation (7) is uniformly h-stable. □

3 Conclusion

In this paper, we established the characterizations of stability such as uniform stability, asymptotic stability, global asymptotic stability, uniform exponential stability and h-stability of linear quantum difference equations associated with D_β in a Banach algebra, where $D_\beta f(t) = \frac{f(\beta(t))-f(t)}{\beta(t)-t}$, $\beta(t)$ is strictly increasing and continuous function has a unique fixed point $s_0\in I$.

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