Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/isl/110114

Remediation of Pollution in a River by Releasing Clean Water

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Received: 9 Jun. 2021, Revised: 1 Aug. 2021, Accepted: 4 Sep. 2021
Published online: 1 Jan. 2022

Abstract: Analytical solution has been obtained for one-dimensional advection-diffusion equation with constant coefficients by using Laplace transformation. Also numerical solution has been obtained by using explicit finite difference scheme. In this paper, the initial condition and the boundary condition at the source of pollution $(x = 0)$ were applied to describe the exponential variation in pollutant concentration. Impacts of different parameters controlling the pollutant dispersion have been studied separately with the help of graphs. This publication proved mathematically the fact that the high concentration of pollutant can be reduced by releasing clean water discharges from barrage in a river. For a real situation, our simple model can give decision support for planning restrictions to be imposed on cultivating and urban practices. According to our information, this is the first study concerning a mathematical simulation for remediation of pollution in a river by releasing clean water.

Keywords: Remediation of pollution, Advection-diffusion equation, Finite difference method, Laplace transformation, Solutions of partial differential equations

1 Introduction

It is well known in real situations, rivers are polluted by various kinds of pollutants coming from many sources. Despite the fact that the Nile is Egypt's lifeblood, it is sadly contaminated with a variety of chemical and biological pollutants, as well as agricultural waste. In summer 2020, a very big quantity of clean water came to the Naser Lake and by releasing this clean water to the River Nile, the high polluted regions can be treated. Mathematical models help governments and health organizations predict the behavior of diseases, the extent of epidemics and how pollutants will spread in the rivers [1]. Here comes the importance of this study to know how we can predict the propagation of the pollution concentration at the reasonable time.

Fick's first law is used to derive the advection-diffusion equation, which is a parabolic partial differential equation based on the theory of mass conservation. Convection and diffusion are two processes that transfer particles, energy, or physical quantities within a physical system. The advection-diffusion equation can be used to model a variety of phenomena such as heat transfer, pollutant diffusion and fluid mechanics applications [2].

The analytical and numerical solutions of the advection-diffusion equation along with an initial condition and two boundary conditions aid to understand the pollutant concentration distribution behavior through an open medium like air, rivers, lakes and porous medium. On the basis of which therapeutic operations can be carried out to reduce or eliminate damage [3]. Only very few partial differential equations have the analytical or exact solutions, anyone who wants to create and use models based on such equations and their related conditions must be able to obtain numerical solutions efficiently and accurately [4]. Zoppou and Knighe [5] provided analytical solutions for the one-dimensional transport of a pollutant in an open channel with a steady unpolluted lateral inflow uniformly distributed over its entire length. Tamora and Wadham [6] solved the advection-diffusion equation for radial flow numerically. Romao et al. [7] presented the finite difference methods

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of 3D convection diffusion equation to examine error in the numerical solution of this equation. Thongmoon and Mckibbin [8] compared some numerical methods for this equation. The numerical treatment of the mathematical model for water pollution was studied by Agusto and Bamingbola [9], they used the implicit centered difference scheme in space and a forward difference scheme in time to solve the generalized transport equation. Changjun and Shuwen [10] used a grey differential model to create a numerical simulation of river water pollution. Wadi et al. [11] studied analytical solution for one-dimensional advection-dispersion equation of the pollution concentration. Remediation of pollution in a river by unsteady aeration with arbitrary initial and boundary conditions was studied by Ibrahim et al. [12]. Manitcharoen and Pimpunchatt [13] used a mathematical model in a one-dimensional advection-dispersion equation that included terms of decay and enlargement process to study the motion of flowing pollution. Hesham et al. [14] proved experimentally that, the impacts of high organic loads in Rosetta branch of River Nile during the low demand period can be mitigated by releasing clean water of amount 30 million m^3/ day from River Nile water at the Delta barrage. They proved that this solution reduced the concentrations of ammonia and organic nitrogen below the limits of the local guidelines. Kusuma et al. [15] provided a numerical solution for mathematical model of the transport equation in a simple rectangular box domain. Suryani et al. [16] solved the diffusion-convection equation with variable coefficients and for anisotropic media by using the boundary element method. Azis et al. [17] used the boundary element method for solving a boundary value problem of homogeneous anisotropic media governed by diffusion-convection equation.

The objective of this study is to develop a mathematical model for one-dimensional advection-diffusion equation with constant coefficients by using Laplace transformation and explicit finite difference scheme. Also study the effect of different parameters controlling the pollutant dispersion along the river at any time and the effect of releasing clean water discharges from a barrage on concentration of pollutant. A river's response to an exponential varying concentration of pollutant is studied.

2 Formulation of the problem

We consider the unsteady flow in a river as being one-dimensional characterized by a single spatial distance x (m) measured from the source of pollution $(x=0)$. The water pollution or the concentration of the pollutant $\mathrm{C}(\mathrm{x},\mathrm{t})\left(\mathrm{kg\,m^{-3}}\right)$ is assumed to vary with time t (days) along the length of the river. The equation governing one-dimensional advective-dispersive transport can be written as ([18] and [19])

$$\frac{\partial \mathrm{C}}{\partial \mathrm{t}} = D\frac{\partial^2 \mathrm{C}}{\partial \mathrm{x}^2} - u\frac{\partial \mathrm{C}}{\partial \mathrm{x}} - \gamma C + \mu, \tag{1}$$

where D is the dispersion coefficient in x direction $\left(\mathrm{m}^2\,\mathrm{day}^{-1}\right)$, u is the average flow velocity in x direction $\left(\mathrm{m\ day}^{-1}\right)$, γ is the pollutant decay rate $\left(\mathrm{day}^{-1}\right)$ and μ is the Zero-order source term $\left(\mathrm{kg\,m^{-3}\,day^{-1}}\right)$.

In our study, we will assume that the river is initially contains pollutants. Hence the initial and boundary conditions associated with equation (1) are:

$$C(x,0) = c_0\, e^{\frac{-x}{k}} \quad , \quad x \geq 0, \tag{2}$$

$$C(0,t) = c_1 + c_2\, e^{-wt}, \quad t > 0, \tag{3}$$

$$\frac{\partial C}{\partial x} = 0, \qquad x \to \infty \ , \quad t \geq 0, \tag{4}$$

where c_o is the initial concentration at $x=0$ and $t=0\left(\mathrm{kg\,m^{-3}}\right)$, k is the initial pollutant-decay length (m), c_1 is the steady state $(\mathrm{t} \to \infty)$ concentration $\left(\mathrm{kg\,m^{-3}}\right)$, c_2 is a constant whose dimension is $\left(\mathrm{kg\,m^{-3}}\right)$ and w is the unsteadiness parameter (day $^{-1}$).

3 The analytical solution

Using the following transformation defined by [20]

$$C(x,t) = K(x,t)\, e^{\left(\frac{ux}{2D} - \lambda t\right)} + \frac{\mu}{\gamma}, \tag{5}$$

where λ is constant which is given by $\lambda = \frac{u^2}{4D} + \gamma$. Equation (5) transforms equations (1-4) into:

$$\frac{\partial K}{\partial t} = D\,\frac{\partial^2 K}{\partial x^2}\,, \tag{6}$$

$$K(x,0) = c_o\, e^{-\left(\frac{u}{2D}+\frac{1}{k}\right)x} - \frac{\mu}{\gamma}\, e^{\frac{-u}{2D}x} \quad , \quad x \geq 0, \tag{7}$$

$$K(0,t) = \left(c_1 - \frac{\mu}{\gamma}\right) e^{\lambda t} + c_2\, e^{(\lambda - w)t}, \quad t > 0, \tag{8}$$

$$\frac{\partial K(x,t)}{\partial x} + \frac{u}{2D}\, K(x,t) = 0, \qquad x \to \infty \ , \quad t \geq 0. \tag{9}$$

Applying Laplace transformation on equations (6, 8 and 9) and using equation (7) gives:

$$\frac{d^2\overline{K}(x,P)}{dx^2} - \frac{P}{D}\overline{K}(x,P) = \frac{-1}{D}\left(\begin{array}{c} c_O\, e^{-\left(\frac{u}{2D}+\frac{1}{k}\right)x} \\ -\frac{\mu}{\gamma}\, e^{\frac{-u}{2D}x} \end{array}\right), \tag{10}$$

$$\overline{K}(0,P) = \left(c_1 - \frac{\mu}{\gamma}\right)\frac{1}{P-\lambda} + c_2\frac{1}{P-\lambda+w}, \ P > 0, \tag{11}$$

$$\frac{d\overline{K}(x,P)}{dx} + \frac{u}{2D}\overline{K}(x,P) = 0, \ x \to \infty, \ P \geq 0, \tag{12}$$

where P is the Laplace transform variable and $\overline{K}$ is Laplace transform of K. Thus, the general solution of the ordinary differential equation (10) subject to conditions (11 and 12), may be written as:

$$\overline{K}(x,P) = -\frac{4D\,\mu\,e^{-\frac{ux}{2D}}}{\gamma\left(4DP-u^2\right)} - \frac{4D\,k^2\,c_o\,e^{-\frac{x}{k}-\frac{ux}{2D}}}{4D^2-4Dk^2P+4Dku+k^2u^2} + e^{-\frac{\sqrt{P}\,x}{\sqrt{D}}}\left(\begin{array}{c}\frac{4D\mu}{(4DP-u^2)\gamma}+\left(c_1-\frac{\mu}{\gamma}\right)\frac{1}{P-\lambda}\\ +\frac{c_2}{P+w-\lambda}+\frac{4Dk^2c_o}{4D^2-4Dk^2P+4Dku+k^2u^2}\end{array}\right). \tag{13}$$

Now, applying inverse of Laplace transformation on equation (13) and using equation (5), hence the analytical solution of advection-dispersion equation (1) associated with the initial and boundary conditions (2-4) may be written in terms of (x,t) as:

$$\begin{aligned}C(x,t) = {} & \frac{\mu}{\gamma}+\frac{\mu}{\gamma}e^{-\gamma t}\left(\begin{array}{c}-1+\frac{1}{2}\operatorname{erfc}\left[\frac{x-ut}{2\sqrt{Dt}}\right]\\ +\frac{1}{2}e^{\frac{ux}{D}}\operatorname{erfc}\left[\frac{x+ut}{2\sqrt{Dt}}\right]\end{array}\right)\\ & +\frac{c_2\,e^{-w\,t}}{2}\left(\begin{array}{c}e^{\frac{\left(u-u\sqrt{1+\frac{4D(\gamma-w)}{u^2}}\right)x}{2D}}\operatorname{erfc}\left[\frac{x-tu\sqrt{1+\frac{4D(\gamma-w)}{u^2}}}{2\sqrt{Dt}}\right]\\ +e^{\frac{\left(u+u\sqrt{1+\frac{4D(\gamma-w)}{u^2}}\right)x}{2D}}\operatorname{erfc}\left[\frac{x+tu\sqrt{1+\frac{4D(\gamma-w)}{u^2}}}{2\sqrt{Dt}}\right]\end{array}\right)\\ & +\frac{\left(c_1-\frac{\mu}{\gamma}\right)}{2}\left(\begin{array}{c}e^{\left(\frac{u-u\sqrt{1+\frac{4\gamma D}{u^2}}}{2D}\right)x}\operatorname{erfc}\left[\frac{x-tu\sqrt{1+\frac{4\gamma D}{u^2}}}{2\sqrt{Dt}}\right]\\ +e^{\left(\frac{u+u\sqrt{1+\frac{4\gamma D}{u^2}}}{2D}\right)x}\operatorname{erfc}\left[\frac{x+tu\sqrt{1+\frac{4\gamma D}{u^2}}}{2\sqrt{Dt}}\right]\end{array}\right)\\ & +\mathrm{e}^{-\gamma t}C_0\left(\begin{array}{c}\mathrm{e}^{\left(\frac{Dt}{k^2}+\frac{tu-x}{k}\right)}-\frac{1}{2}\mathrm{e}^{\left(\frac{Dt}{k^2}+\frac{tu-x}{k}\right)}\operatorname{erfc}\left[\frac{x-ut}{2\sqrt{Dt}}-\frac{\sqrt{Dt}}{k}\right]\\ -\frac{1}{2}\mathrm{e}^{\left(\frac{Dt}{k^2}+\frac{tu+x}{k}+\frac{ux}{D}\right)}\operatorname{erfc}\left[\frac{x+ut}{2\sqrt{Dt}}+\frac{\sqrt{Dt}}{k}\right]\end{array}\right),\end{aligned} \tag{14}$$

where erfc is the complementary error function. We confirmed that equation (14) satisfies equation (1) also it satisfies the initial and boundary conditions (2-4).

4 Special cases

The analytical solution (equation (14)) has practical applications in many field problems as follows:
(I) The special case for which $c_1 = c_2 = 0$ is derived from equation (14) as:

$$\begin{aligned}C(x,t) = {} & \frac{\mu}{\gamma}+\frac{\mu e^{-\gamma t}}{\gamma}\left(\begin{array}{c}-1+\frac{1}{2}\operatorname{erfc}\left[\frac{x-ut}{2\sqrt{Dt}}\right]\\ +\frac{1}{2}e^{\frac{ux}{D}}\operatorname{erfc}\left[\frac{x+ut}{2\sqrt{Dt}}\right]\end{array}\right)\\ & -\frac{\mu}{2\gamma}\left(\begin{array}{c}e^{\left(\frac{u-u\sqrt{1+\frac{4\gamma D}{u^2}}}{2D}\right)x}\operatorname{erfc}\left[\frac{x-tu\sqrt{1+\frac{4\gamma D}{u^2}}}{2\sqrt{Dt}}\right]\\ +e^{\left(\frac{u+u\sqrt{1+\frac{4\gamma D}{u^2}}}{2D}\right)x}\operatorname{erfc}\left[\frac{x+tu\sqrt{1+\frac{4\gamma D}{u^2}}}{2\sqrt{Dt}}\right]\end{array}\right)\\ & +\mathrm{e}^{-\gamma t}C_\mathrm{o}\left(\begin{array}{c}\mathrm{e}^{\left(\frac{Dt}{k^2}+\frac{tu-x}{k}\right)}\\ -\frac{1}{2}\mathrm{e}^{\left(\frac{Dt}{k^2}+\frac{tu-x}{k}\right)}\operatorname{erfc}\left[\frac{x-ut}{2\sqrt{Dt}}-\frac{\sqrt{Dt}}{k}\right]\\ -\frac{1}{2}\mathrm{e}^{\left(\frac{Dt}{k^2}+\frac{tu+x}{k}+\frac{ux}{D}\right)}\operatorname{erfc}\left[\frac{x+ut}{2\sqrt{Dt}}+\frac{\sqrt{Dt}}{k}\right]\end{array}\right).\end{aligned} \tag{15}$$

Equation (15) gives $C(0,t) = 0$, this satisfies the boundary condition which is given by equation (3) when $c_1 = c_2 = 0$.
(II) The special case for which $c_o = 0$ and $c_2 = 0$ is derived from equation (14) as:

$$\begin{aligned}C(x,t) = {} & \frac{\mu}{\gamma}-\frac{\mu}{\gamma}e^{-\gamma t}\left(1-\frac{1}{2}\operatorname{erfc}\left[\frac{x-ut}{2\sqrt{Dt}}\right]-\frac{1}{2}e^{\frac{ux}{D}}\operatorname{erfc}\left[\frac{x+ut}{2\sqrt{Dt}}\right]\right)\\ & +\frac{1}{2}\left(c_1-\frac{\mu}{\gamma}\right)\left(\begin{array}{c}e^{\left(\frac{u-u\sqrt{1+\frac{4\gamma D}{u^2}}}{2D}\right)x}\operatorname{erfc}\left[\frac{x-tu\sqrt{1+\frac{4\gamma D}{u^2}}}{2\sqrt{Dt}}\right]\\ +e^{\left(\frac{u+u\sqrt{1+\frac{4\gamma D}{u^2}}}{2D}\right)x}\operatorname{erfc}\left[\frac{x+tu\sqrt{1+\frac{4\gamma D}{u^2}}}{2\sqrt{Dt}}\right]\end{array}\right).\end{aligned} \tag{16}$$

Equation (16) is the same as that obtained by Kumar [20] (when $m = 0$).
(III) The special case for which $k \to \infty$ is derived from equation (14) as:

$$\begin{aligned}C(x,t) = {} & \frac{\mu}{\gamma}+\left(c_0-\frac{\mu}{\gamma}\right)e^{-\gamma t}\left(\begin{array}{c}1-\frac{1}{2}\operatorname{erfc}\left[\frac{x-ut}{2\sqrt{Dt}}\right]\\ -\frac{1}{2}e^{\frac{ux}{D}}\operatorname{erfc}\left[\frac{x+ut}{2\sqrt{Dt}}\right]\end{array}\right)\\ & +\frac{\left(c_1-\frac{\mu}{\gamma}\right)}{2}\left(\begin{array}{c}e^{\left(\frac{u-u\sqrt{1+\frac{4\gamma D}{u^2}}}{2D}\right)x}\operatorname{erfc}\left[\frac{x-tu\sqrt{1+\frac{4\gamma D}{u^2}}}{2\sqrt{Dt}}\right]\\ +e^{\left(\frac{u+u\sqrt{1+\frac{4\gamma D}{u^2}}}{2D}\right)x}\operatorname{erfc}\left[\frac{x+tu\sqrt{1+\frac{4\gamma D}{u^2}}}{2\sqrt{Dt}}\right]\end{array}\right)\\ & +\frac{c_2\,e^{-w\,t}}{2}\left(\begin{array}{c}e^{\left(\frac{u-u\sqrt{1+\frac{4D(\gamma-w)}{u^2}}}{2D}\right)x}\operatorname{erfc}\left[\frac{x-tu\sqrt{1+\frac{4D(\gamma-w)}{u^2}}}{2\sqrt{Dt}}\right]\\ +e^{\left(\frac{u+u\sqrt{1+\frac{4D(\gamma-w)}{u^2}}}{2D}\right)x}\operatorname{erfc}\left[\frac{x+tu\sqrt{1+\frac{4D(\gamma-w)}{u^2}}}{2\sqrt{Dt}}\right]\end{array}\right).\end{aligned} \tag{17}$$

Equation (17) is the same as that given by Genuchten and Alves [18] (problem C13 when $R = 1$).
(IV) The special case for which $\mu = 0$, $\gamma = 0$ and $k \to \infty$ is

derived from equation (14) as:

$$C(x,t) = c_0 + \frac{(c_1 - c_0)}{2} \left(\begin{array}{c} \operatorname{erfc}\left[\frac{x-ut}{2\sqrt{Dt}}\right] \\ +e^{\frac{ux}{D}} \operatorname{erfc}\left[\frac{x+ut}{2\sqrt{Dt}}\right] \end{array} \right) + \frac{c_2\, e^{-wt}}{2} \left(\begin{array}{c} e^{\frac{\left(u-u\sqrt{1-\frac{4wD}{u^2}}\right)x}{2D}} \operatorname{erfc}\left[\frac{x-tu\sqrt{1-\frac{4wD}{u^2}}}{2\sqrt{Dt}}\right] \\ +e^{\frac{\left(u+u\sqrt{1-\frac{4WD}{u^2}}\right)x}{2D}} \operatorname{erfc}\left[\frac{x+tu\sqrt{1-\frac{4wD}{u^2}}}{2\sqrt{Dt}}\right] \end{array} \right). \tag{18}$$

Equation (18) is the same as that given by Genuchten and Alves [18] (problem A9 when $R = 1$).

(V) The special case for which $c_2 = 0$ and $k \to \infty$ is derived from equation (14) as:

$$C(x,t) = \frac{\mu}{\gamma} + \left(c_0 - \frac{\mu}{\gamma}\right) e^{-\gamma t} \left(\begin{array}{c} 1 - \frac{1}{2} \operatorname{erfc}\left[\frac{x-ut}{2\sqrt{Dt}}\right] \\ -\frac{1}{2} e^{\frac{ux}{D}} \operatorname{erfc}\left[\frac{x+ut}{2\sqrt{Dt}}\right] \end{array} \right) + \frac{\left(c_1 - \frac{\mu}{\gamma}\right)}{2} \left(\begin{array}{c} e^{\left(\frac{u-u\sqrt{1+\frac{4\gamma D}{u^2}}}{2D}\right)x} \operatorname{erfc}\left[\frac{x-tu\sqrt{1+\frac{4\gamma D}{u^2}}}{2\sqrt{Dt}}\right] \\ +e^{\left(\frac{u+u\sqrt{1+\frac{4\gamma D}{u^2}}}{2D}\right)x} \operatorname{erfc}\left[\frac{x+tu\sqrt{1+\frac{4\gamma D}{u^2}}}{2\sqrt{Dt}}\right] \end{array} \right). \tag{19}$$

Equation (19) is the same as that given by Genuchten and Alves [18] (problem C5 when $R = 1$ and $0 < t < t_0$).

(VI) The special case for which $\mu = 0$, $\gamma = 0$, $c_2 = 0$ and $k \to \infty$ is derived from equation (14) as:

$$C(x,t) = c_0 + \frac{(c_1 - c_0)}{2} \left(\begin{array}{c} \operatorname{erfc}\left[\frac{x-ut}{2\sqrt{Dt}}\right] \\ +e^{\frac{ux}{D}} \operatorname{erfc}\left[\frac{x+ut}{2\sqrt{Dt}}\right] \end{array} \right). \tag{20}$$

Equation (20) is the same as that given by Genuchten and Alves [18] (problem A1 when $R = 1$ and $0 < t < t_0$).

5 Numerical solution

The explicit finite difference method (EFDM) is applied to solve equation (1) associated with the initial and boundary conditions (2-4). The central difference scheme was used for $\frac{\partial^2 C}{\partial x^2}$ and $\frac{\partial C}{\partial x}$. The forward difference scheme was used for $\frac{\partial C}{\partial t}$. With these substitutions, equation (1) can be written as :

$$C_{i,j+1} = r_1\, C_{i-1,j} + r_2\, C_{i,j} + r_3\, C_{i+1,j} + \mu\, \Delta t, \tag{21}$$

where i and j refer to the discrete step lengths Δx and Δt for the coordinate x and time t, respectively, and

$$r_1 = \frac{D\, \Delta t}{(\Delta x)^2} + \frac{u\, \Delta t}{2(\Delta x)}, \tag{22}$$

$$r_2 = 1 - \frac{2D\, \Delta t}{(\Delta x)^2} - \gamma\, \Delta t, \tag{23}$$

$$r_3 = \frac{D\, \Delta t}{(\Delta x)^2} - \frac{u\, \Delta t}{2(\Delta x)}. \tag{24}$$

Equation (21) represents a formula for $C(i, j+1)$ at the $(i, j+1)^{th}$ mesh point in terms of known values along the j^{th} time row. The truncation error for equation (21) is $O(\Delta t, (\Delta x)^2)$. Using a small-enough values of Δx and Δt, the truncation error can be reduced until the accuracy achieved is within the error tolerance [21]. The initial condition (2) can be expressed in the finite difference form as

$$C_{i,0} = c_o\, e^{\frac{-x_i}{k}} \quad , \quad x \geq 0,\, t = 0. \tag{25}$$

The boundary conditions (3) and (4) can be written in the finite difference form as

$$C_{0,j} = c_1 + c_2\, e^{-w\, t_j} \quad , \quad x = 0,\, t > 0, \tag{26}$$

$$C_{N,j} = C_{N-1,j} \quad , \quad x \to \infty,\, t \geq 0, \tag{27}$$

where $t_j = j\Delta t$ and $x_i = i\, \Delta x$. $N = x_\infty / \Delta x$ is the grid dimension in the x direction and x_∞ is the distance in the direction x at which $\frac{\partial C}{\partial x} \to 0$.

6 Results and discussions

The solution obtained in equation (14) is illustrated in figures (1-3) for the common input data $0 \leq x \leq 1(\mathrm{m})$, $\mathrm{k} = 1(\mathrm{m})$, $c_0 = 0.01\left(\mathrm{kg\,m^{-3}}\right)$, $\mathrm{c_1} = 1\left(\mathrm{kg\,m^{-3}}\right)$, $\mathrm{c_2} = 0.01\left(\mathrm{kg\,m^{-3}}\right)$ and $\mathrm{w} = 0.001\left(\mathrm{day}^{-1}\right)$. Figure (1) shows the variation of $C(x,t)$ with time for the values $t = 0$, 0.02, 0.04 and 0.06 (day), where $D = 1\left(\mathrm{m^2\,day^{-1}}\right)$, $\mathrm{u} = 1\left(\mathrm{m\,day^{-1}}\right)$, $\gamma = 0.4\left(\mathrm{day^{-1}}\right)$ and $\mu = 0.1\left(\mathrm{kg\,m^{-3}\,day^{-1}}\right)$. From figure (1), it is clear that:

1- At any cross section $x =$ constant, as t increases, C increases. This is due to the fact that at any cross section $x =$ constant, as t increases, the accumulation of the pollutant increases. This result agrees with that obtained by Kumar et a1. [3], Wadi et al. [11], Kumar [20], Andallah and Khatun [22] and finally with Yadav and Kumar [23].

2- At $t =$ constant, as x increases, C decreases. This result agrees with that obtained by Andallah and Khatun [22] and Yadav and Kumar [23].

3- The maximum value of pollutant concentration is at $\mathrm{x} = 0$, while the minimum value of pollutant concentration is at $\mathrm{x} = 1$. This result agrees with that obtained by Yadav and Kumar [23].

4- Figure (1) and numerical studies show that the variation of $C(x,t)$ with the time at $x = 0$ is very small for example for $\mathrm{t} = 0.02, 0.04$ and 0.06 (day), the corresponding values of $\mathrm{C}(0,\mathrm{t})$ are $1.0099998, 1.0099996$ and 1.0099994 respectively.

Figure (2) shows the variation of $C(x,t)$ with μ for the values $\mu = 0.1, 2$ and $4\left(\mathrm{kg\,m^{-3}\,day^{-1}}\right)$, where $\mathrm{t} = 0.02$ (day), $\mathrm{D} = 1\left(\mathrm{m^2\,day^{-1}}\right)$, $\mathrm{u} = 1\left(\mathrm{m\ day^{-1}}\right)$ and $\gamma = 0.4\,($

day $^{-1}$). From figure (2), it is clear that at any cross section $x =$ constant, $C(x,t)$ increases as μ increases. Figure (3) shows the variation of $C(x,t)$ with D for the values $\mathrm{D} = 1, 1.5$ and $2\left(\mathrm{m}^2\ \mathrm{day}^{-1}\right)$, where $\mathrm{t} = 0.02$ (day), $\mu = 0.1\left(\mathrm{kg\ m}^{-3}\ \mathrm{day}^{-1}\right), \mathrm{u} = 1\left(\mathrm{m\ day}^{-1}\right)$ and $\gamma = 0.4($ day $^{-1})$. From figure (3), it is clear that at any cross section $x =$ constant, C increases as D increases. This result agrees with that obtained by Yadav and Kumar [23].

Equation (15) is illustrated in figures (4) and (5) for the common input data $\mathrm{t} = 0.5$ (day), $0 \leq \mathrm{x} \leq 50000(\mathrm{m}), \mathrm{D} = 10^6\left(\mathrm{m}^2\ \mathrm{day}^{-1}\right), \gamma = 0.4\left(\mathrm{day}^{-1}\right), \mu = 0.0001\left(\mathrm{kg\ m}^{-3}\ \mathrm{day}^{-1}\right), \mathrm{k} = 2000$ (m) and $\mathrm{c}_0 = 0.2\left(\mathrm{kg\ m}^{-3}\right)$ Let the cross section area of the river at $x = 0$ be A, then the flux of the water (the volume of water crossing A every day) will be $\mathrm{Q} = \mathrm{A\ u}$. Consequently increasing the value of u means increasing the value of Q. Let the zone of clean water measured from barrage $(x = 0)$ in the direction of the flow be denoted by x_0. Let the maximum value of C be denoted by C_m and the corresponding value of x associated with C_m be x_m. Figure (4) shows the variation of $C(x,t)$ with flow velocity for the values $u = 0, 3000, 6000, 9000, 20000, 25000, 30000\left(\mathrm{m\ day}^{-1}\right)$. Figure (5) is surface graph shows the variation of $C(x,t)$ with flow velocity for the values $0 \leq \mathrm{u} \leq 40000\left(\mathrm{m\ day}^{-1}\right)$. From figure (4), it is clear that:

1- For values of $0 \leq \mathrm{u} \leq 9000, \mathrm{C_m}$ increases as u increases while for values of $20000 \leq \mathrm{u} \leq 30000, \mathrm{C_m}$ is nearly constant.

2- As x increases, there are two opposite factors controlling the values of C. These two factors occur simultaneously. For example for $u = 3000$ in the range $0 \leq x \leq 4000$, the dominant effect on C is the accumulation, hence in this zone as x increases, C increases. On the other hand in the range $4000 \leq \mathrm{x} \leq 16000$ the dominant effect on C will be advection (advection = diffusion + convection), hence in this zone as x increases, C decreases.

Figures (4 and 5) emphasize the fact that the zone of clean water measured form $x = 0$ in the direction of the flow (i.e. x_0) increases as the quantity of the clean water entering the cross section A increases.

Numerical solution of equation (21) with the initial and boundary conditions (25-27) using explicit finite difference method is given in figure (6), for $\mathrm{t} = 0.02, 0.04$ and 0.06 (day). The input data are $0 \leq \mathrm{x} \leq 1(\mathrm{m}), \mathrm{k} = 1\mathrm{m}, \mathrm{c}_0 = .01\left(\mathrm{kg\ m}^{-3}\right), \mathrm{c}_1 = 1\left(\mathrm{kg\ m}^{-3}\right), \mathrm{c}_2 = 0.01\left(\mathrm{kg\ m}^{-3}\right)$ and $\mathrm{w} = 0.001\left(\mathrm{day}^{-1}\right), \mathrm{D} = 1\left(\mathrm{m}^2\ \mathrm{day}^{-1}\right), \mathrm{u} = 1\left(\mathrm{m\ day}^{-1}\right), \gamma = 0.4\left(\mathrm{day}^{-1}\right)$ and $\mu = 0.1\left(\mathrm{kg\ m}^{-3}\ \mathrm{day}^{-1}\right)$. In the numerical calculations, the step lengths $\Delta\mathrm{x} = 0.1(\mathrm{m})$ and $\Delta t = 0.002$ (day), have been used to achieve the stability of the finite difference scheme. The pollutant concentration values are shown in the longitudinal region $0 \leq \mathrm{x} \leq 1(\mathrm{m})$ in figure (6). From figure (6), it is clear that at any cross section $x =$ constant, the pollutant concentration $C(x,t)$ increases as the time (t) increases. To test the accuracy of the numerical solution, a comparison between the analytical solution given by equation (14) and numerical solution given by equation (21) is made and illustrated in figure (6). From figure (6) it is clear that there is a very good agreement between the analytical solution and numerical solution. So the explicit finite difference method is effective and accurate for solving advection-dispersion equation for point source concentration.

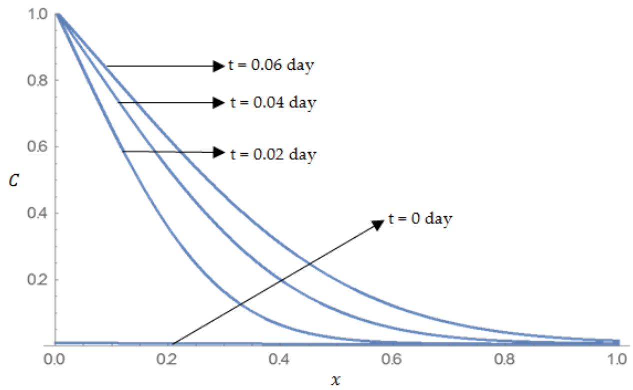

Fig. 1: The variation of $C(x,t)$ with time in equation (14) for $D = 1\left(\mathrm{m}^2\ \mathrm{day}^{-1}\right)$, $\mathrm{u} = 1\left(\mathrm{m\ day}^{-1}\right), \gamma = 0.4\left(\mathrm{day}^{-1}\right)$ and $\mu = 0.1\left(\mathrm{kg\ m}^{-3}\ \mathrm{day}^{-1}\right)$.

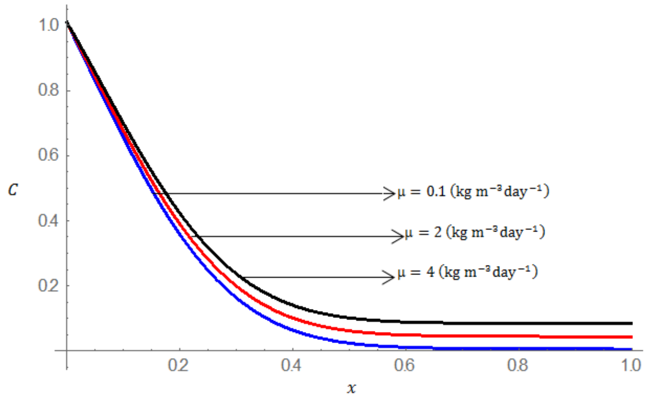

Fig. 2: The variation of $C(x,t)$ with μ in equation (14) for $\mathrm{t} = 0.02$ (day), $\mathrm{D} = 1\left(\mathrm{m}^2\ \mathrm{day}^{-1}\right), \mathrm{u} = 1\left(\mathrm{m\ day}^{-1}\right)$ and $\gamma = 0.4($ day $^{-1})$.

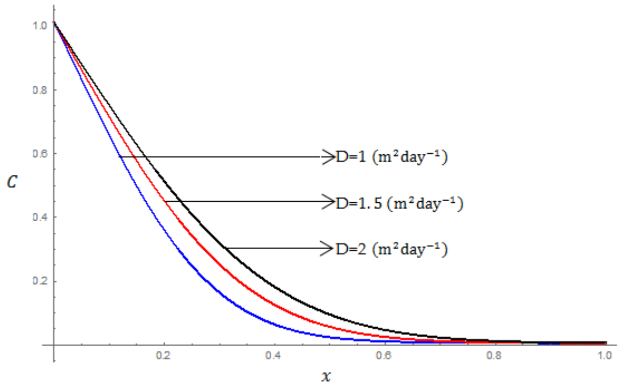

Fig. 3: The variation of $C(x,t)$ with D in equation (14) for t = 0.02 (day), $\mu = 0.1\left(\mathrm{kg\ m^{-3}\ day^{-1}}\right), \mathrm{u} = 1\left(\mathrm{m\ day^{-1}}\right)$ and $\gamma = 0.4\left(\mathrm{day}^{-1}\right)$.

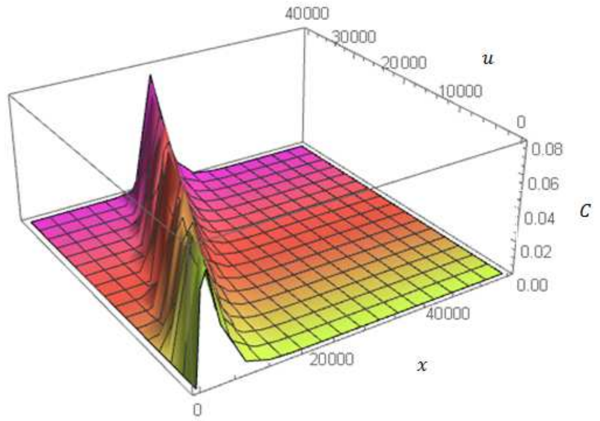

Fig. 5: The surface graph of the variation of $C(x,t)$ with u in equation (15) for t = 0.5 (day), $0 \leq \mathrm{x} \leq 50000(\mathrm{m}), \mathrm{D} = 10^6\left(\mathrm{m^2\ day^{-1}}\right), \gamma = 0.4\left(\mathrm{day^{-1}}\right), \mu = 0.0001\left(\mathrm{kg\ m^{-3}\ day^{-1}}\right), \mathrm{k} = 2000$ m and $\mathrm{c_0} = 0.2\left(\mathrm{kg\ m^{-3}}\right)$.

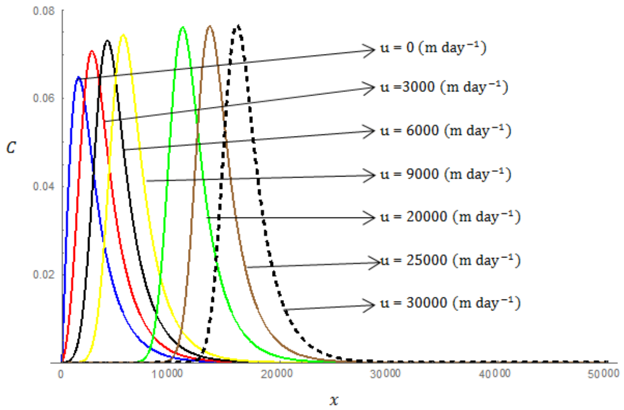

Fig. 4: The variation of $C(x,t)$ with u in equation (15) for t = 0.5 (day), $0 \leq \mathrm{x} \leq 50000(\mathrm{m}), \mathrm{D} = 10^6\left(\mathrm{m^2\ day^{-1}}\right), \gamma = 0.4\left(\mathrm{day^{-1}}\right), \mu = 0.0001\left(\mathrm{kg\ m^{-3}\ day^{-1}}\right), \mathrm{k} = 2000$ m and $\mathrm{c_0} = 0.2\left(\mathrm{kg\ m^{-3}}\right)$.

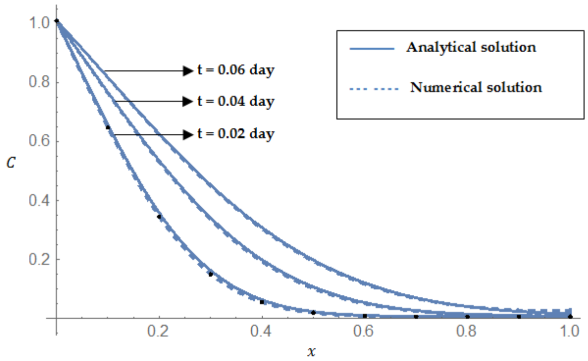

Fig. 6: The comparison between the analytical solution (equation (14)) and the numerical solution (equation (21)) for $D = 1\left(\mathrm{m^2\ day^{-1}}\right)$, $\mathrm{u} = 1\left(\mathrm{m\ day^{-1}}\right), \gamma = 0.4\left(\mathrm{day^{-1}}\right)$ and $\mu = 0.1\left(\mathrm{kg\ m^{-3}\ day^{-1}}\right)$.

7 Conclusions

The analytical solution obtained generalize the earlier solutions obtained by Kumar [20] (when $m = 0$) and that given by Genuchten and Alves [18] (in problems A1, A9, C5 and C13). Numerical solution for the same problem also obtained by using explicit finite difference scheme. When comparing the analytical solution with the numerical solution, we found a very good agreement between them. Impacts of different parameters controlling the pollutant dispersion have been studied separately with the help of graphs. We found that at any cross section $x =$ constant, $C(x,t)$ increases with the increase of either t, D or μ. At constant time t, $C(x,t)$ decreases as x increases. Figures (4 and 5) emphasize the fact that we can reduce the high concentration of pollutant by releasing clean water discharges from barrage in a river. For a real situation, our simple model can give decision support for planning restrictions to be imposed on cultivating and urban practices.

Acknowledgements

This project was supported financially by the Academy of Scientific Research and Technology (ASRT), Egypt, Grant No (6366), (ASRT) is the 2nd affiliation of this research. We would like to thank the many colleagues who contributed in different ways to this work.

Conflict of Interest

The authors declare that they have no conflict of interest.

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