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Numerical Approximation of a Nonlinear Fractional Elliptic System

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Abstract: This paper proposes a fractional numerical approximation of elliptic system with fractional order ($s \in]0,1[$) via the finite difference method (FDM), the analysis is performed on two general dimensional domains with homogeneous boundary conditions. A convergent series solution and the corresponding error estimates are obtained. The performance of FDM is tested with known exact solution, which confirm the theoretical predictions.

Keywords: Finite difference method, rate of convergence, exact solutions, fractional-order derivatives

1 Introduction

In the mathematical modeling of real life phenomena, the study of fractional differential equations has gained notable importance among interested researchers [1,2,3,4]. It is realized that the use of fractional calculus methods is quite prominent in modeling various processes (see [5,6,7,8]). In many cases these equations can be solved analytically and some cases where the equations have no analytical solutions, one way is to use numerical methods, therefore, many numerical methods are used for the fractional differential equations [9,10,11].

The finite difference methods for the nonlinear fractional problem were extended in some sense [12] and many authors contributed to develop the finite difference approximations.

The aim of this study is to prove a general convergence result of the finite difference approximation of a nonlinear fractional elliptic system involving fractional Laplacian in two dimension.

The fractional Laplacian is widely-spread in the modern study of fractional partial differential systems. It has a variety of definitions, though they can be distilled down to the follwing two:

$$(-\Delta)^s\varphi = (2\pi|\xi|^{2s}\widehat{\varphi})^{\vee},$$

and

$$(-\Delta)^s\varphi(z) = K(n,s)P.V.\int_{\mathbb{R}^n}\frac{\varphi(z)-\varphi(w)}{|z-w|^{n+2s}}dw.$$

We recall that the Riesz potentials of order α for $0<\alpha<n$ and $n\in\mathbb{N}^*$ is defined by

$$I_\alpha\varphi = I_\alpha * \varphi,$$

where

$$I_\alpha(z) = \frac{\gamma(n,\alpha)}{|z|^{n-\alpha}},$$

and the constant

$$\gamma(n,\alpha) = \frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}}2^\alpha\Gamma(\frac{\alpha}{2})}.$$

Through analytic continuation, the Riesz potential can be extended to negative exponents. Thus the auteurs in [13] arrives at the next formula for the fractional laplacian

$$(-\Delta)^s\varphi = I_{-2s}\varphi,$$

and others propositions which are used in our work.

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After the Mathematical preliminaries and the analytic study that the reader can extend into by checking our article [14], we perform numerical study and we close with numerical experiment illustrating the convergence results.

2 Mathematical preliminaries

The main motivation of our study is the efficient numerical solution of the boundary value system

$$\begin{cases} (-\Delta)^s \varphi(z) = p(z,\varphi(z),\phi(z)) & \text{in} \quad \Omega, \\ (-\Delta)^s \phi(z) = k(z,\varphi(z),\phi(z)) & \text{in} \quad \Omega, \\ \varphi = \phi = 0 & \text{on} \quad \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $s \in]0,1[$ and $p,k : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions satisfying the Carathéodory conditions (i.e : $p(.,x)$, $k(.,y)$ are measurables for each $x,y \in \mathbb{R}^2$ and $p(z,.)$, $k(w,.)$ are continuous for almost every $z,w \in \Omega$), and Lipschitz continuous functions with respect to the second variable, i.e., there are constants c_1, $c_2 \in \mathbb{R}^+$ for almost every $x \in \Omega$ and for any $x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$\begin{cases} \|p(z,x_1) - p(z,x_2)\|_{L^2(\Omega)} \le c_1 \|x_1 - x_2\|_{L^2(\Omega)\times L^2(\Omega)}, \\ \|k(z,y_1) - k(z,y_2)\|_{L^2(\Omega)} \le c_2 \|y_1 - y_2\|_{L^2(\Omega)\times L^2(\Omega)}. \end{cases} \tag{2}$$

We consider the space

$$U = E^{s,2}(\Omega) \times E^{s,2}(\Omega), \tag{3}$$

with the norm

$$\|(\varphi,\phi)\|_U^2 = \|\varphi\|_{E^{s,2}(\Omega)}^2 + \|\phi\|_{E^{s,2}(\Omega)}^2,$$

where $E^{s,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|.\|_{H^s}}$, ($E^{s,2}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ compared to the $H^s(\Omega)$ norm), if Ω is a bounded Lipschitz open set, then

$$E^{s,2}(\Omega) = \{\varphi \in H^s(\mathbb{R}^n), \text{ such that } \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega\},$$

such that

$$H^s(\mathbb{R}^n) = \{\varphi \in L^2(\mathbb{R}^n) : \frac{|\varphi(z) - \varphi(w)|}{|z-w|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n)\};$$

thus $E^{s,2}(\Omega)$ is a Hilbert space with respect to the scalar product

$$\langle \varphi, \phi \rangle = K(n,s) \iint_{\mathbb{R}^{2n}} \frac{(\varphi(z) - \varphi(w))(\phi(z) - \phi(w))}{|z-w|^{n+2s}} dwdz.$$

The norm in $E^{s,2}(\Omega)$ is

$$\|\varphi\|_{E^{s,2}(\Omega)} = \left[\iint_{\mathbb{R}^{2n}} \frac{|\varphi(z) - \varphi(w)|^2}{|z-w|^{n+2s}} dwdz \right]^{\frac{1}{2}}.$$

All along the paper and without further mention, we always assume that $n = 2$ and $\Omega =]0,1[\times]0,1[$.

3 Analytical study

The contraction principle is applied to have the following result.

Theorem 1(Existence and uniqueness). *Let the Carathéodory functions p, k be Lipschitzian continuous with respect to the second variable with constants $c_i > 0$ $(i = 1,2)$ such that $|c| < c_{emb}^{-2}$ ($c = (c_1,c_2)$ and c_{emb} is the embedding constant). Then, there is a unique fixed point $(\varphi,\phi) \in U$, i.e., (φ,ϕ) is a unique weak solution of (1).*

For more details and the proof, you can see [14].

4 Numerical study

We devote this Section to the description of the numerical scheme that we are going to employ. In order to solve numerically (1), we will develop a finite difference scheme on a uniform mesh. To this purpose, let us first introduce a partition of $\Omega =]0,1[\times]0,1[$ as follows:

$$\Omega = \{(x_i,y_j) : 0 = x_0 < ... < x_i < ... < x_{N+1} = 1$$
$$\text{and } 0 = y_0 < ... < y_i < ... < y_{M+1} = 1\},$$

with $x_{i+1} = x_i + h$, $x_i = x_0 + ih$ where $i = 0,...N$ and $y_{j+1} = y_j + k$, $y_j = j_0 + jk$ where $j = 0,...M$ (N and M are non-null positive constants), in the rest of this paper, we take $N = M$ and $h = k$.

For $n = 2$, we call (i,j) an interior grid point if all of its neighbors $(i-1,j)$, $(i+1,j)$, $(i,j-1)$ and $(i,j+1)$ are in Ω. The matrix $A \in M^{N\times N}$ denotes the approximation of the operator $(-\Delta)^s$ with the standard five-star difference scheme such that for each (φ,ϕ) indexed according to the grid points we have

$$(x_i,y_j) \to \varphi(x_i,y_j) = \varphi_{ij},$$
$$(x_i,y_j) \to \phi(x_i,y_j) = \phi_{ij},$$

and from [13], we get that the system (1) is equivalent to

$$-\frac{\partial^s}{\partial x^s}\frac{\partial^s \varphi}{\partial x^s} - \frac{\partial^s}{\partial y^s}\frac{\partial^s \varphi}{\partial y^s} = p(x,y,\varphi,\phi),$$
$$-\frac{\partial^s}{\partial x^s}\frac{\partial^s \phi}{\partial x^s} - \frac{\partial^s}{\partial y^s}\frac{\partial^s \phi}{\partial y^s} = k(x,y,\varphi,\phi),$$

which lead to

$$-(I_{2-2s} * \frac{\partial^2 \varphi}{\partial x^2}) - (I_{2-2s} * \frac{\partial^2 \varphi}{\partial y^2}) = p(x,y,\varphi,\phi),$$
$$-(I_{2-2s} * \frac{\partial^2 \phi}{\partial x^2}) - (I_{2-2s} * \frac{\partial^2 \phi}{\partial y^2}) = k(x,y,\varphi,\phi),$$

but we have

$$\frac{\partial^2 \varphi}{\partial x^2} \simeq \frac{\varphi_{i+1,j} - 2\varphi_{ij} + \varphi_{i-1,j}}{h^2},$$
$$\frac{\partial^2 \varphi}{\partial y^2} \simeq \frac{\varphi_{i,j+1} - 2\varphi_{ij} + \varphi_{i,j-1}}{h^2},$$

then, we can write our system as follows

$$I_{2-2s} * \frac{-\varphi_{i-1,j} - \varphi_{i+1,j} + 4\varphi_{ij} - \varphi_{i,j-1} - \varphi_{i,j+1}}{h^2} = p(x_i, y_j, \varphi_{ij}, \phi_{ij}),$$

$$I_{2-2s} * \frac{-\phi_{i-1,j} - \phi_{i+1,j} + 4\phi_{ij} - \phi_{i,j-1} - \phi_{i,j+1}}{h^2} = k(x_i, y_j, \varphi_{ij}, \phi_{ij}),$$

according to the definition of convolution product for sequences (by replacing the Lebesgue measure by the counting measure), we can write

$$\sum_{n=1}^{N}\sum_{m=1}^{M} \frac{1}{|(x_i,y_j)-(x_n,y_m)|^{n-2+2s}} \frac{-\varphi_{i-1,j} - \varphi_{i+1,j} + 4\varphi_{ij} - \varphi_{i,j-1} - \varphi_{i,j+1}}{h^2} = p(x_i, y_j, \varphi_{ij}, \phi_{ij}),$$

$$\sum_{n=1}^{N}\sum_{m=1}^{M} \frac{1}{|(x_i,y_j)-(x_n,y_m)|^{n-2+2s}} \frac{-\phi_{i-1,j} - \phi_{i+1,j} + 4\phi_{ij} - \phi_{i,j-1} - \phi_{i,j+1}}{h^2} = k(x_i, y_j, \varphi_{ij}, \phi_{ij}),$$

and after a simple calculus and the fact that $n = 2$ we arrive to the numerical scheme of the system (1)

$$\begin{cases} \sum_{n=1}^{N}\sum_{m=1}^{M} \frac{1}{((i-n)^2+(j-m)^2)^s} \frac{-\varphi_{i-1,j}-\varphi_{i+1,j}+4\varphi_{ij}-\varphi_{i,j-1}-\varphi_{i,j+1}}{h^2} = h^{2-2s} p(x_i, y_j, \varphi_{ij}, \phi_{ij}), \\ \sum_{n=1}^{N}\sum_{m=1}^{M} \frac{1}{((i-n)^2+(j-m)^2)^s} \frac{-\phi_{i-1,j}-\phi_{i+1,j}+4\phi_{ij}-\phi_{i,j-1}-\phi_{i,j+1}}{h^2} = h^{2-2s} k(x_i, y_j, \varphi_{ij}, \phi_{ij}), \\ \varphi_{i,0} = \varphi_{i,N+1} = \varphi_{0,j} = \varphi_{N+1,j} = \phi_{i,0} = \phi_{i,N+1} = \phi_{0,j} = \phi_{N+1,j}. \end{cases} \tag{4}$$

Hence (4) is equivalent to the matrix system

$$BU = b,$$

where $U = (\varphi, \phi)$, $b = (p, k)$ and

$$B = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},$$

as an example if we take $N = 2$, then

$$A = \begin{pmatrix} -2 & (4-\frac{1}{2^s}) & (4-\frac{1}{2^s}) & (\frac{4}{2^s}-2) \\ (4-\frac{1}{2^s}) & -2 & (\frac{4}{2^s}-2) & (4-\frac{1}{2^s}) \\ (4-\frac{1}{2^s}) & (\frac{4}{2^s}-2) & -2 & (4-\frac{1}{2^s}) \\ (\frac{4}{2^s}-2) & (4-\frac{1}{2^s}) & (4-\frac{1}{2^s}) & -2 \end{pmatrix}$$

and

$$U = \begin{pmatrix} \varphi_{11} \\ \varphi_{12} \\ \varphi_{21} \\ \varphi_{22} \\ \phi_{11} \\ \phi_{12} \\ \phi_{21} \\ \phi_{22} \end{pmatrix}, b = \begin{pmatrix} p_{11} \\ p_{12} \\ p_{21} \\ p_{22} \\ k_{11} \\ k_{12} \\ k_{21} \\ k_{22} \end{pmatrix}.$$

Consequently to get the approaches solution of the system (4) it's equivalent to solve the system

$$U = B^{-1}b.$$

4.1 Error estimate

In this part, we study the consistency, the stability and the convergence of the scheme (4) but first, it is necessary to prove the existence and the unicity of the solution and for that we must prove the next theorem.

Theorem 2. *Under the assumption (2) the system (4) admits a unique solution.*

We use the discrete maximum principle to prove that the application BU is injective in finite dimension, than the matrix B is invertible which lead us to the existence and uniqueness of solution.

4.1.1 Consistency

We say that a method is consistent with the differential equation and boundary conditions if

$$\|R\|_\infty \le Ch^t,$$

where R is the rest and C is positive constant.

Remark. If $\|R\|_\infty \le Ch^t$, we say that the method is consistent of order t where t is a positive real constant.

Proposition 1. *The scheme (4) is consistent of order $(2-2s)$, moreover*

$$\|R\|_\infty \le Ch^{2-2s}.$$

Proof. We have

$$R_{ij} = \begin{cases} \frac{h^{2-2S}}{4!}(N \times N - 1)(S+T), \\ \frac{h^{2-2S}}{4!}(N \times N - 1)(L+M), \end{cases}$$

where

$$S = \frac{\partial^4 \varphi}{\partial x^4}(\alpha_1, y) + \frac{\partial^4 \varphi}{\partial x^4}(\alpha_2, y),$$

$$T = \frac{\partial^4 \varphi}{\partial y^4}(x, \beta_1) + \frac{\partial^4 \varphi}{\partial y^4}(x, \beta_2),$$

$$L = \frac{\partial^4 \phi}{\partial x^4}(\alpha_1, y) + \frac{\partial^4 \phi}{\partial x^4}(\alpha_2, y),$$

$$M = \frac{\partial^4 \phi}{\partial y^4}(x, \beta_1) + \frac{\partial^4 \phi}{\partial y^4}(x, \beta_2),$$

then for every N

$$\|R\|_\infty \le \frac{h^{2-2s}}{6} C_N,$$

such that $C_N = (N \times N - 1)W$, where $W = max(max|\partial^4 \varphi|, max|\partial^4 \phi|)$.
Consequently

$$\|R\|_\infty \le \frac{h^{2-2s}}{6} min(C_N). \tag{5}$$

4.1.2 Stability

A numerical scheme is said to be stable if

$$\|U\|_\infty \leq K\|b\|_\infty,$$

where K is positive constant.

The stability result is given in the following statement, which is proved in several steps.

Theorem 3. *The (4) scheme is stable for the $\|.\|_\infty$ norm. In particular:*

$$\|B^{-1}\|_\infty \leq \frac{1}{8}. \tag{6}$$

Proof. • Step 1: In the first step, we prove that $B^{-1} \geq 0$.

- Step 2: We give an exact solution.
- Step 3: We calculate the critical points.
- Step 4: Finally, we conclude that $\|B^{-1}\|_\infty \leq \frac{1}{8}$.

4.1.3 Convergence

A scheme is said to be convergent if $\|e\| \to 0$ as $h \to 0$ where e is the error between the exact solution and the approximate solution. Combining the ideas introduced above, we arrive at the conclusion that if we have the consistency and the stability we get the convergence of the scheme.

Theorem 4. *The (4) scheme is convergent, moreover*

$$\|e\|_\infty \leq \frac{h^{2-2s}}{48} min(C_N).$$

Proof. This is easily proved by using (5) and (6).

These facts will be confirmed by the numerical simulations that we are going to present in Section 5 below, by observing the behavior of the approximate solution, the exact solution, and the norm of the error e in the infinity. In this way, as predicted by Theorem 4, we obtain a numerical evidence of the properties of null and the convergence of system (1), in accordance with the theoretical results in Section 4.

5 Numerical results

In this Section, we present the numerical simulations corresponding to the scheme previously described, and we provide a complete discussion of the results obtained.

First of all, in order to numerically test the accuracy of our method, we use the following system

$$\begin{cases} (-\Delta)^{\frac{1}{2}}\varphi(x,y) = p(x,y,\varphi(x,y),\phi(x,y)) & \text{in} \quad \Omega, \\ (-\Delta)^{\frac{1}{2}}\phi(x,y) = k(x,y,\varphi(x,y),\phi(x,y)) & \text{in} \quad \Omega, \\ \varphi = \phi = 0 & \text{on} \quad \mathbb{R}^2 \setminus \Omega, \end{cases} \tag{7}$$

where $\Omega =]0,1[\times]0,1[$ and

$$p(x,y,\varphi(x,y),\phi(x,y)) = -4x^3(y-1)^4 - 4x^4(y-1)^3,$$
$$k(x,y,\varphi(x,y),\phi(x,y)) = -4y^3(x-1)^4 - 4y^4(x-1)^3,$$

we can write p and k in terms of φ and ϕ as follows

$$\begin{aligned} p(x,y,\varphi(x,y),\phi(x,y)) &= -4x(y-1)^2\varphi^{\frac{1}{2}} - 4x^2(y-1)\phi^{\frac{1}{2}} \\ &\quad -4x^2y^2(y-1) - 8x^2(y-1)y^2(x-1) \\ &\quad -4x^4(y-1) + 8x^4y(y-1), \\ k(x,y,\varphi(x,y),\phi(x,y)) &= -4y^2(x-1)\phi^{\frac{1}{2}} - 4y(x-1)^2\varphi^{\frac{1}{2}} \\ &\quad -4yx^2(x-1)^2 - 8yx^2(x-1)^2(y-1) \\ &\quad -4y^3(x-1)^2 + 8y^3x(x-1)^2. \end{aligned}$$

In this particular case, the solution can be computed exactly and it reads as follows,

$$\varphi(x,y) = x^4(y-1)^4,$$
$$\phi(x,y) = y^4(x-1)^4.$$

According to the matrix transformation method we proceeded as follows.

- The domain was discretized using a uniform square-grid with the grid size h.
- The standard five-point approximation of the operator $(-\Delta)^s$ was applied to obtain the matrix B.
- Gauss Seidel method was applied.
- To get the approximate solution, we use the solutions of the quadratic equation.

The next results were shown after a lot of mathematical calculations.

In Fig. 1 and Fig. 2, we show a comparison between the exact solution and the computed numerical approximation. Here we consider $N = 5$ then $h = 0.1667$ and $s = \frac{1}{2}$.

One can notice that the computed solution is to a certain extent not different from the exact solution, Fig. 3 and Fig. 4 where $N = 60$ and $h = 0.0164$ shown that very well (there is a different but we can't detect it by the aye we need to zoom the figure to see it).

However, one should be careful with such result and a more precise analysis of the error should be carried.

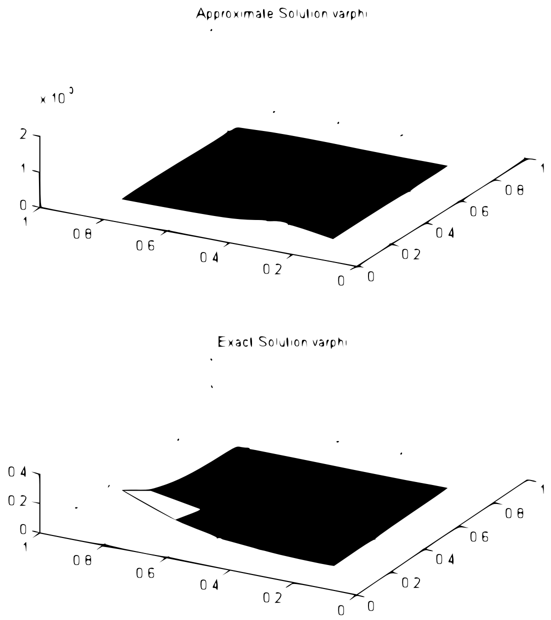

Fig. 1: The surface graph of the exact solution varphi and the fifth-order approximate solution.

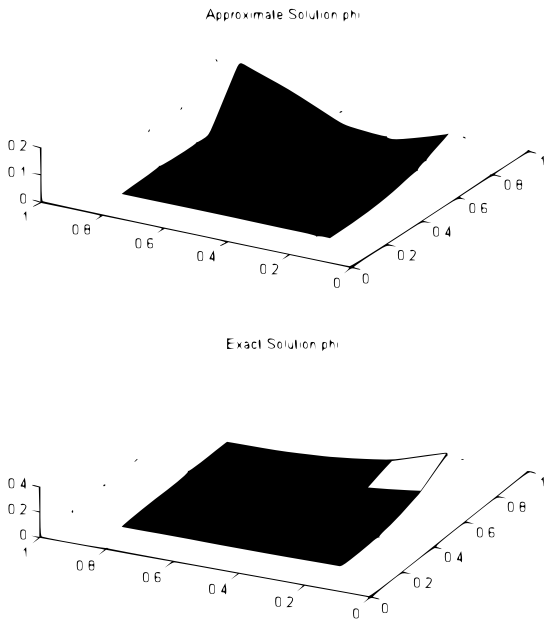

Fig. 2: The surface graph of the exact solution phi and the fifth-order approximate solution.

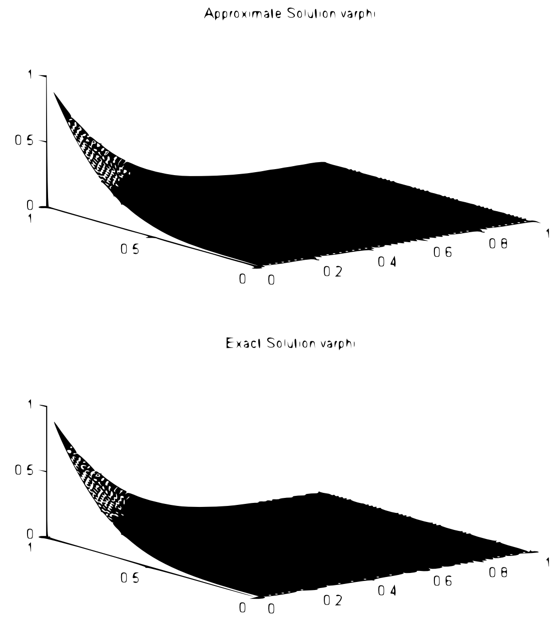

Fig. 3: The surface graph of the exact solution varphi and the sixty-order approximate solution.

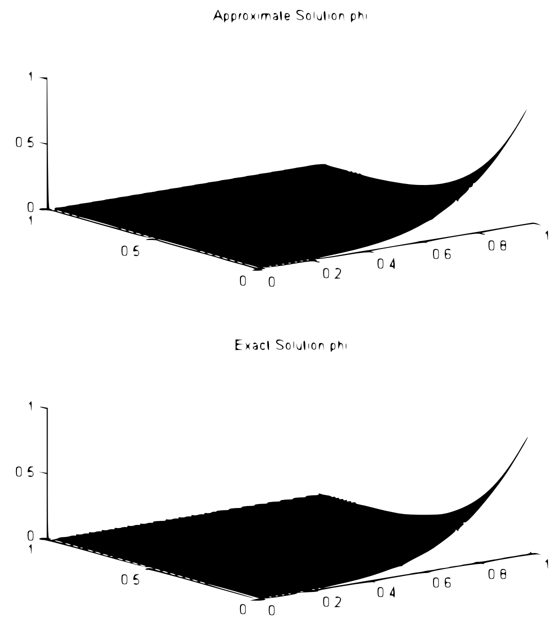

Fig. 4: The surface graph of the exact solution phi and the sixty-order approximate solution.

In the same spirit as in [12], the computation of the error can be readily done by using the Theorem 4, namely

$$\|e\|_\infty \leq \frac{h^{2-2s}}{48} min(C_N).$$

The computational results are shown for our model in Table 1. While in the two-dimensional case, the predicted convergence rate is reached shortly, in the threedimensional computations a remarkable oscillation can be detected see [12]. Since the computations are lengthy, we have tested our system only with a single parameter $s = \frac{1}{2}$.

Table 1: Computational error and estimated convergence rate r with respect to the infty-norm for the matrix transformation method applied to the finite difference approximation of (7). N: number of steps.

		φ	ϕ	
N	h	max-error	max-error	r
35	0.0278	1.3650×10^{-11}	1.1657×10^{-14}	0.0417
40	0.0244	3.0754×10^{-12}	3.5565×10^{-15}	0.0366
45	0.0217	1.0939×10^{-12}	1.2084×10^{-15}	0.0326
60	0.0164	5.3331×10^{-13}	6.9390×10^{-17}	0.0246
65	0.0152	3.6230×10^{-13}	2.7303×10^{-17}	0.0227

In Fig. 5, we present the computational errors evaluated for different values of N and h.

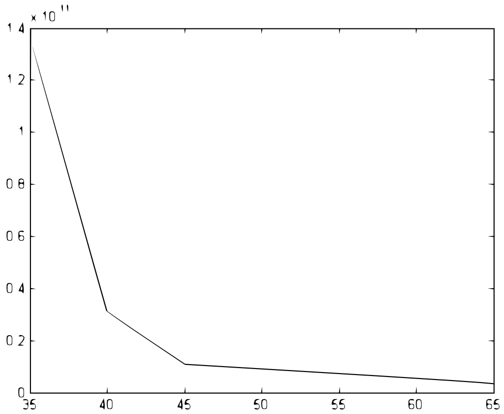

Fig. 5: Plot of the absolute error.

The rates of convergence shown are of order (in h) of $(2-2s)$.

6 Summary

We have verified the convergence of the matrix transformation method applied to the fractional elliptic system. The corresponding computation algorithm is difficult: we can't avoid the computation of a full matrix containing involved finite differences. Combined with the Gauss Seidel method, the corresponding method exhibits optimal convergence rate for $s\in(0,1)$ in the infty-norm.
The finite difference method is applied successfully for solving the nonlinear fractional elliptic systems. The fundamental objective of this article is to introduce an algorithmic form and implement a new analytical repeated algorithm derived from the finite difference method to find numerical solutions for the fractional elliptic system. Graphical and numerical consequences are introduced to illustrate the solutions. Thus, it is concluded that we can translate numericaly and find a numerical solutions for a wide class of linear and nonlinear fractional differential systems applied in physics, biologics...ect. From the results, it is clear that the numerical resolution of fractional system yields very accurate and convergent approximate solutions.

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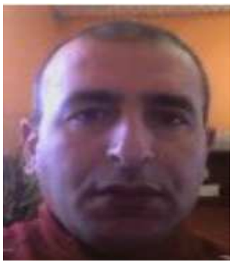

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