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Inferences for two Weibull Fréchet Populations under Joint Progressive Type-II Censoring with Applications in Engineering Chemistry

Ethar M. Shokr[1], *Rashad M. EL-Sagheer*[2,*], *Moaiad Khder*[3], *and Beih S. El-Desouky*[1]

[1]Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[2]Mathematics Department, Faculty of Science, Al-Azhar University,Nasr city (11884), Cairo, Egypt
[3]Department of Computer Science, College of Arts & Science, Road 23, East Al-Ekir, Applied Science University, Kingdom of Bahrain

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Abstract: Comparative lifetime experiments are vital when the interest is in learning the overall benefits of two competing products with respect to their reliability. In this article, point and interval estimations for the unknown parameters of two Weibull- Fréchet populations based on joint progressive Type-II censoring samples are discussed. The point estimations for the two distribution parameters are obtained by the maximum likelihood, Bayes and parametric bootstrap methods. Moreover, approximate confidence intervals and credible intervals of the estimators have been obtained and compared with four bootstrap confidence intervals. Furthermore, Bayes estimators have been developed under squared error loss and linear exponential loss functions using independent gamma prior distributions when Gibbs sampler within the Metropolis-Hasting algorithm is applied to generate Markov chain Monte Carlo samples from the posterior density functions. A real data set is studied to illustrate the application of the proposed criteria. Finally, extensive simulation experiments have been performed to study the performances of the different methods.

Keywords: Weibull Fréchet model, Joint progressive Type-II censoring, MLE, Parametric Bootstrap, MCMC approach

1 Introduction

Censoring scheme is a popular problem in life testing experiments. Therefore, in the past few years many censoring data have appeared. The oldest censoring schemes are called Type-I, and the other is Type-II. In practice, there are usually two random variables, the time, and the number of failures of items. This strategy of censoring schemes shows how the examiner imagines the experiment based on a predetermined time. A random number of items is accounted for the first Type-I of a censoring scheme, which means it may be assumed the exact time of stopping experiment. While the predetermined number of failure items and a random time in the Type-II censoring scheme. In these two types of censoring schemes, items cannot be removed from an experiment until the final stage or the number of items fail. These schemes allow the detection of some items that are defective after running the experiment.

The mixture of these types of censoring schemes is called hybrid censoring scheme. To remove items from the test at any stage of the experiment is called as a progressive censoring scheme. There are different types of progressive censoring schemes. The design of the progressive Type-I censoring scheme is to test time τ and determine the number m of failure items, and suppose$n$ independent items are tested under censoring scheme $\mathbf{R} = (R_1, R_2, ...R_m)$. The failure item is removed at $min(\tau, T_m)$, where T_m is the stopping time of the number of failure items m. After each failure time (T_i, R_i), survival items are removed from the experiment, where $i = 1, 2, ..., J$ and $J \leq m$. In a progressive Type-II censoring scheme, the number m of failure items and $\mathbf{R} = (R_1, R_2, ...R_m)$ are determined, and suppose that n independent items are examined and the experiment is stopped at T_m. After each failure time (T_i, R_i) survival items are removed from the test, where $i = 1, 2, ..., m$.

The joint censoring scheme completely helpful and viable in executing comparative life tests of products

* Corresponding author e-mail: rashadmath27@gmail.com

coming from various units within the same facility. It can be briefly described as follows: assume that products are being manufactured by two various lines and we chose two independent samples of sizes m and n from these lines then placed altogether on a life-testing experiment. Furthermore, to provide cost and time, a joint progressive type-II censoring scheme (JP-II-CS) is implemented and the life-testing terminates when a specified number of failures occur. The JP-II-CS has been discussed before in the literature. For example, see Rasouli and Balakrishnan [1], Doostparast et al. [2], Balakrishnan et al. [3], Mondal and Kundu [4], Mondal and Kundu [5], Krishna and Goel [6], Goel and Krishna[7], Aljohani [8], Abo-Kasem and Nassar [9] and Abo-Kasem and Nassar [10].

Suppose that $\{X_1,\cdots,X_m\}$ and $\{Y_1,\cdots,Y_n\}$ be the lifetimes of m,n pattern of product A and B, respectively, are i.i.d. random variables (r.v.) from distribution function $F(x),G(x)$ and density function $f(x),g(x)$. Moreover, assume $W_{(1)},\cdots,W_{(N)}$ be the order statistics of the $N = m+n$ random variable of the joint sample$\{X_1,\cdots,X_m;Y_1,\cdots,Y_n\}$. Thereafter, under the JP-II-CS, the observable data consists of (Z,W,S) where $W=(W_1,\cdots,W_r)$ with $1<r<N$ be the total number of failures and is a pre-fixed integer before the experiment, $Z=(z_1,\cdots,z_r)$ with $z_i=1$ or 0 accordingly as W_i comes from X or Y failure, respectively, and $S=(s_1,\cdots,s_r)$, where the JP-II-CS $R=(R_1,\cdots,R_r)$ has the decomposition $S+T=(s_1,\cdots,s_r)+(t_1,\cdots,t_r)$ are the number of elements withdrawn at the time of the ith failure are related to X and Y samples, respectively, and these are unknown and random variables. Also $r=m_r+n_r$ where $m_r=\sum_{i=1}^{r} z_i$ and $n_r=\sum_{i=1}^{r}(1-z_i)$.

More specifically, assume, the lifetime of m element $X_1,\cdots,X_m$ of product A, are i.i.d. r.v. from Weibull Fréchet distribution $WFr(\lambda_1,\theta_1,\alpha_1,\beta_1)$ population, which discussed in Afify et al. [11], with density and distribution functions, respectively, as

$$f(x) = \lambda_1\theta_1\beta_1\alpha_1^{\beta_1}x^{-\beta_1-1}\exp\left[-\theta_1\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right] \times\left\{1-\exp\left[-\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right]\right\}^{-\theta_1-1} \times\exp\left(-\lambda_1\left\{\exp\left[\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right]-1\right\}^{-\theta_1}\right), \quad (1)$$

and

$$F(x) = 1-\exp\left(-\lambda_1\left\{\exp\left[\left(\frac{\alpha_1}{x}\right)^{\beta_1}\right]-1\right\}^{-\theta_1}\right). \quad (2)$$

Similarly, suppose the lifetime of n element $Y_1,\cdots,Y_n$ of product B, be i.i.d. r. v. from Weibull Fréchet distribution $WFr(\lambda_2,\theta_2,\alpha_2,\beta_2)$ population with density and distribution functions, respectively, as

$$g(y) = \lambda_2\theta_2\beta_2\alpha_2^{\beta_2}y^{-\beta_2-1}\exp\left[-\theta_2\left(\frac{\alpha_2}{y}\right)^{\beta_2}\right] \times\left\{1-\exp\left[-\left(\frac{\alpha_2}{y}\right)^{\beta_2}\right]\right\}^{-\theta_2-1} \times\exp\left(-\lambda_2\left\{\exp\left[\left(\frac{\alpha_2}{y}\right)^{\beta_2}\right]-1\right\}^{-\theta_2}\right), \quad (3)$$

and

$$G(y) = 1-\exp\left(-\lambda_2\left\{\exp\left[\left(\frac{\alpha_2}{y}\right)^{\beta_2}\right]-1\right\}^{-\theta_2}\right), \quad (4)$$

where $\lambda_i,\theta_i,\alpha_i,\beta_i>0,\ x,y>0,\ i=1,2$ α_i are the scale parameter and $\lambda_i,\theta_i,\beta_i$ are the shape parameters. The WFr distribution has been shown to be useful for modeling and analyzing the life time data in medical and biological sciences, engineering, etc. This distribution is a very flexible model that approaches to different distributions when its parameters are changed. It contains the following new special models:

1. $WFr(\lambda,\theta,\alpha,1)$ follows the Weibull inverse exponential model.
2. $WFr(\lambda,1,\alpha,\beta)$ is the exponential Fréchet model.
3. $WFr(\lambda,\theta,\alpha,2)$ refers to the Weibull inverse Rayleigh model.
4. $WFr(\lambda,1,\alpha,2)$ reduces to the exponential inverse Rayleigh model.
5. $WFr(\lambda,1,\alpha,1)$ follows the exponential inverse exponential model.

The rest of the paper is organized as follows: Section 2 exhibits the details of the JP-II-CS model and derives the maximum likelihood estimators (MLE) of the unknown parameters. Approximate confidence intervals (ACIs) based on the MLEs is presented in Section 3. In Section 4, different bootstrap confidence intervals (CIs) are constructed. Section 5, provides the Bayesian analysis using square error, LINEX loss functions. We analyze a real data sets to illustrate the estimation methods discussed in this paper in Section 6. Furthermore simulation study is debated in Section 7. Eventually, conclusion of the work is inserted in Section 8.

2 Maximum Likelihood Inferences

The likelihood of (Z,W,S) according to Rasouli and Balakrishnan [1] is given by

$$L(\lambda_1,\lambda_2,\theta_1,\theta_2,\alpha_1,\alpha_2,\beta_1,\beta_2;z,w,s) = C\prod_{i=1}^{r}\left[(f(w_i))^{z_i}(g(w_i)^{1-z_i})\right]\left[(\bar{F}(w_i))^{s_i}(\bar{G}(w_i))^{t_i}\right], \quad (5)$$

where $w_1\le w_2\le\cdots\le w_r$, $\bar{F}=1-F$, $\bar{G}=1-G$, $\sum_{i=1}^{r}s_i = m-m_r$, $\sum_{i=1}^{r}t_i=n-n_r$, $\sum_{i=1}^{r}R_i=\sum_{i=1}^{r}s_i+\sum_{i=1}^{r}t_i$ and $C=$

$D_1 D_2$ with

$$D_1 = \prod_{j=1}^{r} \left\{ \left(m - \sum_{i=1}^{j-1} z_i - \sum_{i=1}^{j-1} s_i\right) z_j + \left(n - \sum_{i=1}^{j-1} (1-z_i) - \sum_{i=1}^{j-1} t_i\right)(1-z_i) \right\},$$

$$D_2 = \prod_{j=1}^{r} \left\{ \frac{\binom{m-\sum_{i=1}^{j-1} z_i - \sum_{i=1}^{j-1} s_i}{s_i} \binom{n-\sum_{i=1}^{j-1}(1-z_i) - \sum_{i=1}^{j-1} t_i}{t_i}}{\binom{m+n-j-\sum_{i=1}^{j-1} R_i}{R_j}} \right\}.$$

Now, substituting Eqs. 1, 2, 3 and 4 in Eq. 5, we get

$$\begin{aligned} L = {} & C(\lambda_1\ \theta_1\ \beta_1\ \alpha_1^{\beta_1})^{m_r} (\lambda_2\ \theta_2\ \beta_2\ \alpha_2^{\beta_2})^{n_r} \\ & \times \prod_{i=1}^{r} (w_i^{z_i(-\beta_1-1)}) \prod_{i=1}^{r} (w_i^{(1-z_i)(-\beta_2-1)}) \\ & \times \exp\left[-\theta_1 \sum_{i=1}^{r} \left(\frac{\alpha_1}{w_i}\right)^{\beta_1} z_i\right] \\ & \times \exp\left[-\theta_2 \sum_{i=1}^{r} \left(\frac{\alpha_2}{w_i}\right)^{\beta_2} (1-z_i)\right] \\ & \times \prod_{i=1}^{r} \left\{1 - \exp\left[-\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]\right\}^{-z_i(\theta_1+1)} \\ & \times \prod_{i=1}^{r} \left\{1 - \exp\left[-\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]\right\}^{-(1-z_i)(\theta_2+1)} \\ & \times \exp\left(-\lambda_1 \sum_{i=1}^{r} (z_i+s_i) \left(\left\{\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] - 1\right\}^{-\theta_1}\right) \\ & \times \exp\left(-\lambda_2 \sum_{i=1}^{r} (1-z_i+t_i) \left(\left\{\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right] - 1\right\}^{-\theta_2}\right). \end{aligned} \tag{6}$$

Hence, the log-likelihood function $\ell = \ln(L(\lambda_1, \lambda_2, \theta_1, \theta_2, \alpha_1, \alpha_2, \beta_1, \beta_2, z, w, s))$ is given by:

$$\begin{aligned} \ell = {} & \ln c + m_r \ln\lambda_1 + m_r \ln\theta_1 + m_r \ln\beta_1 + m_r\beta_1 \ln\alpha_1 \\ & - (\beta_1+1) \sum_{i=1}^{r} z_i \ln w_i + n_r \ln\lambda_2 + n_r \ln\theta_2 + n_r \ln\beta_2 \\ & + n_r\beta_2 \ln\alpha_2 - (\beta_2+1) \sum_{i=1}^{r} (1-z_i) \ln w_i \\ & - \theta_1 \sum_{i=1}^{r} z_i \left(\frac{\alpha_1}{w_i}\right)^{\beta_1} - \theta_2 \sum_{i=1}^{r} (1-z_i) \left(\frac{\alpha_2}{w_i}\right)^{\beta_2} \\ & - \lambda_1 \sum_{i=1}^{r} (z_i+s_i) \left(\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] - 1\right)^{-\theta_1} \\ & - \lambda_2 \sum_{i=1}^{r} (1-z_i+t_i) \left(\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right] - 1\right)^{-\theta_2} \\ & - (\theta_1+1) \sum_{i=1}^{r} z_i \ln\left[1 - \exp\left[-\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]\right] \\ & - (\theta_2+1) \sum_{i=1}^{r} (1-z_i) \ln\left[1 - \exp\left[-\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]\right] \end{aligned} \tag{7}$$

Then the normal equations to obtain estimates of unknown parameters are given by:

$$\frac{m_r}{\lambda_1} - \sum_{i=1}^{r} (z_i+s_i) \left(\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] - 1\right)^{-\theta_1} = 0, \tag{8}$$

$$\frac{n_r}{\lambda_2} - \sum_{i=1}^{r} (1-z_i+t_i) \left(\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right] - 1\right)^{-\theta_2} = 0, \tag{9}$$

$$\begin{aligned} & \frac{m_r}{\theta_1} - \textstyle\sum_{i=1}^{r} z_i \left(\frac{\alpha_1}{w_i}\right)^{\beta_1} + \lambda_1 \sum_{i=1}^{r} (z_i+s_i) \\ & \times \left(\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] - 1\right)^{-\theta_1} \ln\left(\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] - 1\right) \\ & - \textstyle\sum_{i=1}^{r} z_i \ln\left(1 - \exp\left[-\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]\right) = 0, \end{aligned} \tag{10}$$

$$\begin{aligned} & \frac{n_r}{\theta_2} - \textstyle\sum_{i=1}^{r} (1-z_i) \left(\frac{\alpha_2}{w_i}\right)^{\beta_2} + \lambda_2 \sum_{i=1}^{r} (1-z_i+t_i) \\ & \times \left(\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right] - 1\right)^{-\theta_2} \ln\left(\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right] - 1\right) \\ & - \textstyle\sum_{i=1}^{r} (1-z_i) \ln\left(1 - \exp\left[-\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]\right) = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} & \frac{m_r\beta_1}{\alpha_1} - \theta_1 \textstyle\sum_{i=1}^{r} \left(\frac{\beta_1 z_i}{w_i}\right) \left(\frac{\alpha_1}{w_i}\right)^{\beta_1-1} - (1+\theta_1) \\ & \times \textstyle\sum_{i=1}^{r} \frac{z_i\beta_1 \left(\frac{\alpha_1}{w_i}\right)^{\beta_1-1} \exp\left[-\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]}{\left(1-\exp\left[-\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]\right) \times w_i} + \lambda_1\theta_1\beta_1 \\ & \times \textstyle\sum_{i=1}^{r} \frac{(z_i+s_i)}{w_i} \left(\frac{\alpha_1}{w_i}\right)^{\beta_1-1} \exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] \\ & \times \left(\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] - 1\right)^{-\theta_1-1} = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} & \frac{n_r\beta_2}{\alpha_2} - \theta_2 \textstyle\sum_{i=1}^{r} \left(\frac{\beta_2(1-z_i)}{w_i}\right) \left(\frac{\alpha_2}{w_i}\right)^{\beta_2-1} \\ & -(1+\theta_2) \textstyle\sum_{i=1}^{r} \frac{(1-z_i)\beta_2 \left(\frac{\alpha_2}{w_i}\right)^{\beta_2-1} \exp\left[-\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]}{\left(1-\exp\left[-\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]\right) \times w_i} \\ & +\lambda_2\theta_2\beta_2 \textstyle\sum_{i=1}^{r} \frac{(1-z_i+t_i)}{w_i} \left(\frac{\alpha_2}{w_i}\right)^{\beta_2-1} \exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right] \\ & \times \left(\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right] - 1\right)^{-\theta_2-1} = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} & \frac{m_r}{\beta_1} + m_r \ln(\alpha_1) - \sum_{i=1}^{r} z_i \ln(w_i) \\ & -\theta_1 \sum_{i=1}^{r} z_i \left(\frac{\alpha_1}{w_i}\right)^{\beta_1} \ln\left(\frac{\alpha_1}{w_i}\right) \\ & - (1+\theta_1) \sum_{i=1}^{r} \frac{z_i \left(\frac{\alpha_1}{w_i}\right)^{\beta_1} \exp\left[-\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] \ln\left(\frac{\alpha_1}{w_i}\right)}{1 - \exp\left[-\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]} \\ & + \lambda_1\theta_1 \sum_{i=1}^{r} (z_i+s_i) \ln\left(\frac{\alpha_1}{w_i}\right) \exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] \\ & \times \left(\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right] - 1\right)^{-\theta_1-1} \left(\frac{\alpha_1}{w_i}\right)^{\beta_1} = 0, \end{aligned} \tag{14}$$

and

$$\frac{n_r}{\beta_2}+n_r\ln(\alpha_2)-\sum_{i=1}^{r}(1-z_i)\ln(w_i)-\theta_2\sum_{i=1}^{r}(1-z_i)$$
$$\times\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\ln\left(\frac{\alpha_2}{w_i}\right)-(1+\theta_2)$$
$$\times\sum_{i=1}^{r}\frac{(1-z_i)\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\exp\left[-\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]\ln\left(\frac{\alpha_2}{w_i}\right)}{1-\exp\left[-\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]}$$
$$+\lambda_2\theta_2\sum_{i=1}^{r}(1-z_i+t_i)\ln\left(\frac{\alpha_2}{w_i}\right)\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]$$
$$\times\left(\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]-1\right)^{-\theta_2-1}\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}=0. \tag{15}$$

Thus, from Eqs. 8 and 9 the MLEs of a_i, $i=1,2$ can be given, respectively, by

$$\hat{\lambda}_1=m_r\left[\sum_{i=1}^{r}(z_i+s_i)\left(\exp\left[\left(\frac{\alpha_1}{w_i}\right)^{\beta_1}\right]-1\right)^{-\theta_1}\right]^{-1}, \tag{16}$$

and

$$\hat{\lambda}_2=n_r\left[\sum_{i=1}^{r}(1-z_i+t_i)\left(\exp\left[\left(\frac{\alpha_2}{w_i}\right)^{\beta_2}\right]-1\right)^{-\theta_2}\right]^{-1}. \tag{17}$$

By using a suitable numerical method as Newton-Raphson iteration method to solve these equations, we get the unknown parameters $\hat{\lambda}_i,\hat{\theta}_i,\hat{\alpha}_i$ and $\hat{\beta}_i$, $i=1,2$.

3 Approximate Confidence Intervals

Based on the asymptotic normality of the MLEs, we construct the approximate confidence intervals (ACIs) for the unknown parameters ξ_i, $i=1,2,...,8$. The asymptotic distribution of the MLEs of the parameters is $(\hat{\xi}_i-\xi_i)\sim N(0,\hat{v})$, where $\hat{v}$ is the variance-covariance matrix of the MLEs which can be computed as the inverse of the observed Fisher information matrix.

Let $I(\xi_i)=I_{i,j}(\xi_i)$ where $i,j=1,2,\cdots,8$ and $(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6,\xi_7,\xi_8)=(\lambda_1,\lambda_2,\theta_1,\theta_2,\alpha_1,\alpha_2,\beta_1,\beta_2)$ be the Fisher information matrix of the parameter ξ_i, where

$$\hat{I}(\xi_i)=(-E\left(\frac{\partial^2\ell}{\partial\xi_i\partial\xi_j}\right)).$$

Therefore, the asymptotic variance-covariance matrix is given by

$$\hat{V}=V_{i,j}=(cov(\hat{\xi}_i,\hat{\xi}_j))=\hat{I}^{-1}(\hat{\xi}_i). \tag{18}$$

Thus, the $(1-\delta)100\%$ ACIs for ξ_i, $i=1,\cdots,8$ are given by

$$[\hat{\xi}_i\pm Z_{\delta/2}\sqrt{var(\hat{\xi}_i)}], \tag{19}$$

where $var(\hat{\xi}_i)=cov(\hat{\xi}_i,\hat{\xi}_i)$.

4 Bootstrap Confidence Intervals

It is known that the normal approximation is inappropriate when the sample is small, and then bootstrap method can improve the accuracy of the confidence intervals. In this section, we propose a re-sampling technique: Bootstrap procedure, to obtain a more widely used confidence intervals as follows bootstrap-p CI (Boot-p), see Efron and Tibshirani [12], bootstrap-t (Boot-t), see Hall [13], Bias-Corrected bootstrap CI (Boot-BC) and bias-corrected accelerated bootstrap CI (Boot-BCa), see DiCiccio and Efron [14]. Furthermore the algorithms for these Bootstrap methods are given bellow:

4.1 Boot-p CI

First, solving Eqs. (8)– (15) by the joint original sample W, and using the values of $\xi=(\lambda_1,\lambda_2,\theta_1,\theta_2,\alpha_1,\alpha_2,\beta_1,\beta_2)$ and denoted them by $\hat{\xi}=(\hat{\lambda}_1,\hat{\lambda}_2,\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta}_1,\hat{\beta}_2)$ to generate a bootstrap sample $w^*=(w_1^*,\cdots,w_r^*)$ with the same values of R_i, $i=1,\cdots,r$ by using the algorithm presented in Balakrishnan and Sandhu [15]. Then, we compute $\xi^*=(\lambda_1^*,\lambda_2^*,\theta_1^*,\theta_2^*,\alpha_1^*,\alpha_2^*,\beta_1^*,\beta_2^*)$. Repeat this procedure for $Nboot$ times to get $\hat{\xi}_1^*,\hat{\xi}_2^*,\cdots,\hat{\xi}_{Nboot}^*$ where $\hat{\xi}_i^*=(\lambda_{1i}^*,\lambda_{2i}^*,\theta_{1i}^*,\theta_{2i}^*,\alpha_{1i}^*,\alpha_{2i}^*,\beta_{1i}^*,\beta_{2i}^*)$, $i=1,2,\cdots,Nboot$. Next, we arrange $\hat{\xi}_i^*$ in ascending order and denote them by $\hat{\xi}_{(1)}^*,\hat{\xi}_{(2)}^*,\cdots,\hat{\xi}_{(Nboot)}^*$. Thus, the $(1-\delta)100\%$ approximate bootstrap-p CI for ξ is obtained as (L,U), where $L=\hat{\xi}^*_{Nboot(\frac{\delta}{2})}$ and $U=\hat{\xi}^*_{Nboot(1-\frac{\delta}{2})}$.

4.2 Boot-t CI

After obtaining $\hat{\xi}^*=(\lambda_1^*,\lambda_2^*,\theta_1^*,\theta_2^*,\alpha_1^*,\alpha_2^*,\beta_1^*,\beta_2^*)$ in Boot-p method, we compute the variance-covariance matrix for $\hat{\xi}^*$ by Eq. 18. Then, calculate the value of the statistic $T_{\xi}^*=\frac{\hat{\xi}^*-\hat{\xi}}{\sqrt{\widehat{var}(\hat{\xi}^*)}}$ and repeat this procedure for $Nboot$ times to get $T_{\xi_1}^*,\cdots,T_{\xi_{Nboot}}^*$. After that, we arrange them in ascending order and get $T_{\xi_{(1)}}^*,T_{\xi_{(2)}}^*,\cdots,T_{\xi_{(Nboot)}}^*$. Thus, the $(1-\delta)100\%$ approximate bootstrap-t confidence interval for ξ is obtained as (L,U), where

$$L=\hat{\xi}+\sqrt{var(\hat{\xi}^*_{(Nboot(\frac{\delta}{2}))})}*T^*_{\xi_{(Nboot(\frac{\delta}{2}))}}$$

and

$$U=\hat{\xi}+\sqrt{var(\hat{\xi}^*_{(Nboot(1-\frac{\delta}{2}))})}*T^*_{\xi_{(Nboot(1-\frac{\delta}{2}))}}$$

.

4.3 Boot-BC CI

Also obtaining $\hat{\xi}^* = (\lambda_1^*, \lambda_2^*, \theta_1^*, \theta_2^*, \alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*)$ as in Boot-p method, after that calculate the probability p_0 based on the ordered bootstrap distribution of $\hat{\xi}^*$

$$P_0 = p(\hat{\xi}^* < \hat{\xi}) = \frac{\hat{\xi}^* < \hat{\xi}}{N}, \ where\ i = 1, \cdots, Nboot. \quad (20)$$

Put χ and χ^{-1} denote to the cdf and inverse cdf of standard normal variable z, and then, the bias-correction constant z_0 is defined as follow:

$$z_0 = \chi^{-1}(P_0) = \chi^{-1}\Big(\frac{\hat{\xi}^* < \hat{\xi}}{N}\Big). \quad (21)$$

Where $p(\hat{\xi}^* < \hat{\xi}) = G(z_0)$ and $G(.)$ is the cdf of bootstrap distribution, hence the percentiles of the ordered bootstrap distribution of $\hat{\xi}$ are given as

$$L = \chi(2z_0 + z_{\delta/2}), \quad U = \chi(2z_0 + z_{1-\delta/2}). \quad (22)$$

There fore, the approximate Boot-BC $100(1-\delta)\%$ CI of $\hat{\xi}$ is given by

$$\Big[\hat{\xi}_{Boot-BC}\Big(\frac{\delta}{2}\Big), \hat{\xi}_{Boot-BC}\Big(1 - \frac{\delta}{2}\Big)\Big].$$

4.4 Boot-BCa CI

Furthermore, by calculating $\hat{\xi}^* = (\lambda_1^*, \lambda_2^*, \theta_1^*, \theta_2^*, \alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*)$ as in Boot-p method. Put $\chi(z) = \theta$ be the standard normal cdf where $z_\theta = \chi^{-1}(\theta)$ and the bias-correction constant z_0 is defined in Eq. 21, hence

$$\hat{\xi}^*_{Boot-BCa} = G^{-1}\Big[\chi\Big(z_0 + \frac{z_0 + z_\theta}{1 - a(z_0 + z_\theta)}\Big)\Big], \quad (23)$$

where a is indicated to the acceleration factor and it can be calculated by using a jackknife approach and can be expressed as

$$a = \frac{\sum_{i=1}^N (\hat{\xi} - \hat{\xi}_i)^3}{6\Big(\sum_{i=1}^N (\hat{\xi} - \hat{\xi}_i)^2\Big)^{3/2}}, \quad i = 1, 2, \cdots, Nboot. \quad (24)$$

Many authors discussed this method as, Efron and Tibshirani [16]. Then, the approximate Boot-BCa $100(1-\delta)\%$ CI of $\hat{\xi}$ is given by

$$\Big[\hat{\xi}_{Boot-BCa}\Big(\frac{\delta}{2}\Big), \hat{\xi}_{Boot-BCa}\Big(1 - \frac{\delta}{2}\Big)\Big].$$

5 Bayesian Inference

Bayes estimation is quite different from MLE and Bootstrap methods, because it takes into consideration both the information from observed sample data and the prior information. It can characterize the problems more rationally and reasonably. In this section, we obtain Bayes estimates and the corresponding credible intervals for the parameters $\lambda_1, \theta_1, \alpha_1, \beta_1, \lambda_2, \theta_2, \alpha_2, \beta_2$. To do this, we assume the prior distributions as independent Gamma distributions with different and known hyper parameters, i.e. $\lambda_1 \backsim Gamma(c_1, k_1)$, $\theta_1 \backsim Gamma(c_2, k_2)$, $\alpha_1 \backsim Gamma(c_3, k_3)$, $\beta_1 \backsim Gamma(c_4, k_4)$, $\lambda_2 \backsim Gamma(c_5, k_5)$, $\theta_2 \backsim Gamma(c_6, k_6)$, $\alpha_2 \backsim Gamma(c_7, k_7)$ and $\beta_2 \backsim Gamma(c_8, k_8)$. Hence, the prior distributions of $\eta_i = \lambda_1, \theta_1, \alpha_1, \beta_1, \lambda_2, \theta_2, \alpha_2$ or β_2 have the form

$$\pi_i(\eta_i) \propto \eta_i^{c_i - 1} e^{-k_i \eta_i}, \qquad c_i, k_i > 0,$$

where c_i, k_i and $i = 1, \cdots, 8$ the hyper-parameters which reflect the prior knowledge about the unknown parameters. Then, the joint posterior distribution of $\lambda_j, \theta_j, \alpha_j, \beta_j$, $j = 1, 2$ in given (Z, W, S) is
$\pi^*(\lambda_1, \theta_1, \alpha_1, \beta_1, \lambda_2, \theta_2, \alpha_2, \beta_2 | Z, W, S)$

$$\begin{aligned}
&\propto \lambda_1^{m_r + c_1 - 1} e^{-k_1 \lambda_1} \theta_1^{m_r + c_2 - 1} e^{-k_2 \theta_1} \alpha_1^{m_r \beta_1 + c_3 - 1} e^{-k_3 \alpha_1} \beta_1^{m_r + c_4 - 1} e^{-k_4 \beta_1} \\
&\times \lambda_2^{n_r + c_5 - 1} e^{-k_5 \lambda_2} \theta_2^{n_r + c_6 - 1} e^{-k_6 \theta_2} \alpha_2^{n_r \beta_2 + c_7 - 1} e^{-k_7 \alpha_2} \beta_2^{n_r + c_8 - 1} e^{-k_8 \beta_2} \\
&\times \prod_{i=1}^r (w_i^{z_i(-\beta_1 - 1)}) \prod_{i=1}^r (w_i^{(1-z_i)(-\beta_2 - 1)}) \exp\Big[-\theta_1 \sum_{i=1}^r \Big(\frac{\alpha_1}{w_i}\Big)^{\beta_1} z_i\Big] \\
&\times \exp\Big[-\theta_2 \sum_{i=1}^r \Big(\frac{\alpha_2}{w_i}\Big)^{\beta_2} (1 - z_i)\Big] \prod_{i=1}^r \Big\{1 - \exp\Big[-\Big(\frac{\alpha_1}{w_i}\Big)^{\beta_1}\Big]\Big\}^{-z_i(\theta_1 + 1)} \\
&\times \prod_{i=1}^r \Big\{1 - \exp\Big[-\Big(\frac{\alpha_2}{w_i}\Big)^{\beta_2}\Big]\Big\}^{-(1-z_i)(\theta_2 + 1)} \\
&\times \exp\Big(-\lambda_1 \sum_{i=1}^r (z_i + s_i)\Big(\Big\{\exp\Big[\Big(\frac{\alpha_1}{w_i}\Big)^{\beta_1}\Big] - 1\Big\}^{-\theta_1}\Big)\Big) \\
&\times \exp\Big(-\lambda_2 \sum_{i=1}^r (1 - z_i + t_i)\Big(\Big\{\exp\Big[\Big(\frac{\alpha_2}{w_i}\Big)^{\beta_2}\Big] - 1\Big\}^{-\theta_2}\Big)\Big). \quad (25)
\end{aligned}$$

Hence, the conditional posterior densities of $\lambda_1, \lambda_2, \theta_1, \theta_2, \alpha_1, \alpha_2, \beta_1$ and β_2 are as follows:

$$\begin{aligned}
\pi_1^*(\lambda_1) \propto \lambda_1^{m_r + c_1 - 1} \exp\Big\{-\lambda_1\Big(k_1 + \sum_{i=1}^r (z_i + s_i) \\
\times \Big(\Big\{\exp\Big[\Big(\frac{\alpha_1}{w_i}\Big)^{\beta_1}\Big] - 1\Big\}^{-\theta_1}\Big)\Big)\Big\}, \quad (26)
\end{aligned}$$

$$\begin{aligned}
\pi_2^*(\lambda_2) \propto \lambda_2^{n_r + c_5 - 1} \exp\Big\{-\lambda_2\Big(k_5 + \sum_{i=1}^r (1 - z_i + t_i) \\
\times \Big(\Big\{\exp\Big[\Big(\frac{\alpha_2}{w_i}\Big)^{\beta_2}\Big] - 1\Big\}^{-\theta_2}\Big)\Big)\Big\}, \quad (27)
\end{aligned}$$

$$\pi_3^*(\theta_1) \propto \theta_1^{m_r+c_2-1}\prod_{i=1}^{r}\Big\{1-\exp\Big[-\Big(\frac{\alpha_1}{w_i}\Big)^{\beta_1}\Big]\Big\}^{-z_i(\theta_1+1)}$$
$$\times\exp\Big[-\theta_1\Big(k_2+\sum_{i=1}^{r}\Big(\frac{\alpha_1}{w_i}\Big)^{\beta_1}z_i\Big)\Big]$$
$$\times\exp\Big(-\lambda_1\sum_{i=1}^{r}(z_i+s_i)\Big(\Big\{\exp\Big[\Big(\frac{\alpha_1}{w_i}\Big)^{\beta_1}\Big]-1\Big\}^{-\theta_1}\Big)\Big), \tag{28}$$

$$\pi_4^*(\theta_2) \propto \theta_2^{n_r+c_6-1}\prod_{i=1}^{r}\Big\{1-\exp\Big[-\Big(\frac{\alpha_2}{w_i}\Big)^{\beta_2}\Big]\Big\}^{-(1-z_i)(\theta_2+1)}$$
$$\times\exp\Big[-\theta_2\Big(k_6+\sum_{i=1}^{r}\Big(\frac{\alpha_2}{w_i}\Big)^{\beta_2}(1-z_i)\Big)\Big]$$
$$\times\exp\Big(-\lambda_2\sum_{i=1}^{r}(1-z_i+t_i)\Big(\Big\{\exp\Big[\Big(\frac{\alpha_2}{w_i}\Big)^{\beta_2}\Big]-1\Big\}^{-\theta_2}\Big)\Big), \tag{29}$$

$$\pi_5^*(\alpha_1) \propto \alpha_1^{m_r\beta_1+c_3-1}\prod_{i=1}^{r}\Big\{1-\exp\Big[-\Big(\frac{\alpha_1}{w_i}\Big)^{\beta_1}\Big]\Big\}^{-z_i(\theta_1+1)}$$
$$\times\exp\Big[-\alpha_1k_3-\theta_1\sum_{i=1}^{r}\Big(\frac{\alpha_1}{w_i}\Big)^{\beta_1}z_i\Big]$$
$$\times\exp\Big(-\lambda_1\sum_{i=1}^{r}(z_i+s_i)\Big(\Big\{\exp\Big[\Big(\frac{\alpha_1}{w_i}\Big)^{\beta_1}\Big]-1\Big\}^{-\theta_1}\Big)\Big), \tag{30}$$

$$\pi_6^*(\alpha_2) \propto \alpha_2^{n_r\beta_2+c_7-1}\prod_{i=1}^{r}\Big\{1-\exp\Big[-\Big(\frac{\alpha_2}{w_i}\Big)^{\beta_2}\Big]\Big\}^{-(1-z_i)(\theta_2+1)}$$
$$\times\exp\Big[-\alpha_2k_7-\theta_2\sum_{i=1}^{r}\Big(\frac{\alpha_2}{w_i}\Big)^{\beta_2}(1-z_i)\Big]$$
$$\times\exp\Big(-\lambda_2\sum_{i=1}^{r}(1-z_i+t_i)\Big(\Big\{\exp\Big[\Big(\frac{\alpha_2}{w_i}\Big)^{\beta_2}\Big]-1\Big\}^{-\theta_2}\Big)\Big), \tag{31}$$

$$\pi_7^*(\beta_1) \propto \beta_1^{m_r+c_4-1}\prod_{i=1}^{r}(w_i^{z_i(-\beta_1-1)})$$
$$\exp\Big[-\beta_1k_4-\theta_1\sum_{i=1}^{r}\Big(\frac{\alpha_1}{w_i}\Big)^{\beta_1}z_i\Big]$$
$$\times\prod_{i=1}^{r}\Big\{1-\exp\Big[-\Big(\frac{\alpha_1}{w_i}\Big)^{\beta_1}\Big]\Big\}^{-z_i(\theta_1+1)}$$
$$\times\exp\Big(-\lambda_1\sum_{i=1}^{r}(z_i+s_i)\Big(\Big\{\exp\Big[\Big(\frac{\alpha_1}{w_i}\Big)^{\beta_1}\Big]-1\Big\}^{-\theta_1}\Big)\Big), \tag{32}$$

and

$$\pi_8^*(\beta_2) \propto \beta_2^{n_r+c_8-1}\prod_{i=1}^{r}(w_i^{(1-z_i)(-\beta_2-1)})$$
$$\times\exp\Big[-\beta_2k_8-\theta_2\sum_{i=1}^{r}\Big(\frac{\alpha_2}{w_i}\Big)^{\beta_2}(1-z_i)\Big]$$
$$\times\prod_{i=1}^{r}\Big\{1-\exp\Big[-\Big(\frac{\alpha_2}{w_i}\Big)^{\beta_2}\Big]\Big\}^{-(1-z_i)(\theta_2+1)}$$
$$\times\exp\Big(-\lambda_2\sum_{i=1}^{r}(1-z_i+t_i)\Big(\Big\{\exp\Big[\Big(\frac{\alpha_2}{w_i}\Big)^{\beta_2}\Big]-1\Big\}^{-\theta_2}\Big)\Big). \tag{33}$$

From Eqs. 26, 27 the full conditional posterior density of λ_1, λ_2 are, respectively, $Gamma(m_r+c_1, k_1+\sum_{i=1}^{r}(z_i+s_i)\Big(\Big\{\exp\Big[\Big(\frac{\alpha_1}{w_i}\Big)^{\beta_1}\Big]-1\Big\}^{-\theta_1}\Big))$ and $Gamma(n_r+c_5, k_5+\sum_{i=1}^{r}(1-z_i+t_i)\Big(\Big\{\exp\Big[\Big(\frac{\alpha_2}{w_i}\Big)^{\beta_2}\Big]-1\Big\}^{-\theta_2}\Big))$. Thus, the samples of λ_1, λ_2 can be generated by using any gamma routine. From Eqs. 28– 33 the posterior of $\theta_1, \theta_2, \alpha_1, \alpha_2, \beta_1$ and β_2 do not present standard form, but the plots of them show that they are like to normal distribution, see Fig. (1). Therefore, we generate random samples from these distributions using the Metropolis-Hastings algorithm with the normal proposal distribution, see Tierney [17]. The Metropolis-Hastings algorithm within Gibbs sampling:

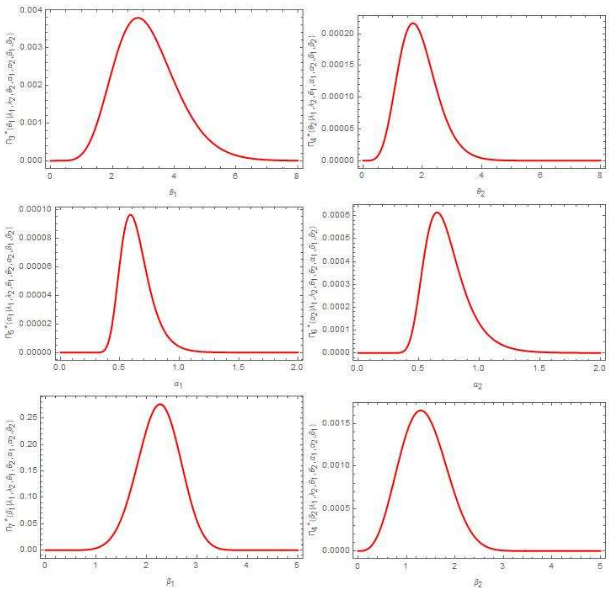

Fig. 1: Posterior density functions for $\theta_1, \theta_2, \alpha_1, \alpha_2, \beta_1$ and β_2

1.Start with $(\lambda_1^{(0)}, \lambda_2^{(0)}, \theta_1^{(0)}, \theta_2^{(0)}, \alpha_1^{(0)}, \alpha_2^{(0)}, \beta_1^{(0)}, \beta_2^{(0)})$.
2.Set $k=1$.

3.Generate $\lambda_1^{(k)}$ from $Gamma(m_r+c_1, k_1+\sum_{j=1}^{r}(z_j+s_j)\left(\exp\left[\left(\frac{\alpha_1^{(k-1)}}{w_j}\right)^{\beta_1^{(k-1)}}\right]-1\right)^{-\theta_1^{(k-1)}})$.

4.Generate $\lambda_2^{(k)}$ from $Gamma(n_r+c_5, k_5+\sum_{j=1}^{r}(1-z_j+t_j)\left(\exp\left[\left(\frac{\alpha_2^{(k-1)}}{w_j}\right)^{\beta_2^{(k-1)}}\right]-1\right)^{-\theta_2^{(k-1)}})$.

5.Using M-H,

(a)

(b)Generate θ_1^* from $N(\theta_1^{(k-1)}, var(\theta_1))$, θ_2^* from $N(\theta_2^{(k-1)}, var(\theta_2))$, α_1^* from $N(\alpha_1^{(k-1)}, var(\alpha_1))$, α_2^* from $N(\alpha_2^{(k-1)}, var(\alpha_2))$, β_1^* from $N(\beta_1^{(k-1)}, var(\beta_1))$ and β_2^* from $N(\beta_2^{(k-1)}, var(\beta_2))$.

(c)Evaluate the acceptance probabilities:

i.

ii. $\psi_1 = min[1, \frac{\pi_3^*(\theta_1^*|\lambda_1^{(k)},\lambda_2^{(k)},\theta_2^{(k-1)},\alpha_1^{(k-1)},\alpha_2^{(k-1)},\beta_1^{(k-1)},\beta_2^{(k-1)},\underline{w})}{\pi_3^*(\theta_1^{(k-1)}|\lambda_1^{(k)},\lambda_2^{(k)},\theta_2^{(k-1)},\alpha_1^{(k-1)},\alpha_2^{(k-1)},\beta_1^{(k-1)},\beta_2^{(k-1)},\underline{w})}]$.

iii. $\psi_2 = min[1, \frac{\pi_4^*(\theta_2^*|\lambda_1^{(k)},\lambda_2^{(k)},\theta_1^{(k)},\alpha_1^{(k-1)},\alpha_2^{(k-1)},\beta_1^{(k-1)},\beta_2^{(k-1)},\underline{w})}{\pi_4^*(\theta_2^{(k-1)}|\lambda_1^{(k)},\lambda_2^{(k)},\theta_1^{(k)},\alpha_1^{(k-1)},\alpha_2^{(k-1)},\beta_1^{(k-1)},\beta_2^{(k-1)},\underline{w})}]$.

iv. $\psi_3 = min[1, \frac{\pi_5^*(\alpha_1^*|\lambda_1^{(k)},\lambda_2^{(k)},\theta_1^{(k)},\theta_2^{(k)},\alpha_2^{(k-1)},\beta_1^{(k-1)},\beta_2^{(k-1)},\underline{w})}{\pi_5^*(\alpha_1^{(k-1)}|\lambda_1^{(k)},\lambda_2^{(k)},\theta_1^{(k)},\theta_2^{(k)},\alpha_2^{(k-1)},\beta_1^{(k-1)},\beta_2^{(k-1)},\underline{w})}]$.

v. $\psi_4 = min[1, \frac{\pi_6^*(\alpha_2^*|\lambda_1^{(k)},\lambda_2^{(k)},\theta_1^{(k)},\theta_2^{(k)},\alpha_1^{(k)},\beta_1^{(k-1)},\beta_2^{(k-1)},\underline{w})}{\pi_6^*(\alpha_2^{(k-1)}|\lambda_1^{(k)},\lambda_2^{(k)},\theta_1^{(k)},\theta_2^{(k)},\alpha_1^{(k)},\beta_1^{(k-1)},\beta_2^{(k-1)},\underline{w})}]$.

vi. $\psi_5 = min[1, \frac{\pi_7^*(\beta_1^*|\lambda_1^{(k)},\lambda_2^{(k)},\theta_1^{(k)},\theta_2^{(k)},\alpha_1^{(k)},\alpha_2^{(k)},\beta_2^{(k-1)},\underline{w})}{\pi_7^*(\beta_1^{(k-1)}|\lambda_1^{(k)},\lambda_2^{(k)},\theta_1^{(k)},\theta_2^{(k)},\alpha_1^{(k)},\alpha_2^{(k)},\beta_2^{(k-1)},\underline{w})}]$.

vii. $\psi_6 = min[1, \frac{\pi_8^*(\beta_2^*|\lambda_1^{(k)},\lambda_2^{(k)},\theta_1^{(k)},\theta_2^{(k)},\alpha_1^{(k)},\alpha_2^{(k)},\beta_1^{(k)},\underline{w})}{\pi_8^*(\beta_2^{(k-1)}|\lambda_1^{(k)},\lambda_2^{(k)},\theta_1^{(k)},\theta_2^{(k)},\alpha_1^{(k)},\alpha_2^{(k)},\beta_1^{(k)},\underline{w})}]$.

(d)Generate r_1, r_2, r_3, r_4, r_5 and r_6 from a Uniform distribution $(0,1)$.

(e)If $r_1 < \psi_1$, set $\theta_1^{(k)} = \theta_1^*$, otherwise $\theta_1^{(k)} = \theta_1^{(k-1)}$.

(f)If $r_2 < \psi_2$, set $\theta_2^{(k)} = \theta_2^*$, otherwise $\theta_2^{(k)} = \theta_2^{(k-1)}$.

(g)If $r_3 < \psi_3$, set $\alpha_1^{(k)} = \alpha_1^*$, otherwise $\alpha_1^{(k)} = \alpha_1^{(k-1)}$.

(h)If $r_4 < \psi_4$, set $\alpha_2^{(k)} = \alpha_2^*$, otherwise $\alpha_2^{(k)} = \alpha_2^{(k-1)}$.

(i)If $r_5 < \psi_5$, set $\beta_1^{(k)} = \beta_1^*$, otherwise $\beta_1^{(k)} = \beta_1^{(k-1)}$.

(j)If $r_6 < \psi_6$, set $\beta_2^{(k)} = \beta_2^*$, otherwise $\beta_2^{(k)} = \beta_2^{(k-1)}$.

6.Set $k = k+1$.

7.Repeat Steps 3–6 N times.

Hence to evaluate the convergence and to remove the affection of the selection of initial value, we remove the first M simulated values. Then the selected sample are $\lambda_1^{(k)}, \lambda_2^{(k)}, \theta_1^{(k)}, \theta_2^{(k)}, \alpha_1^{(k)}, \alpha_2^{(k)}, \beta_1^{(k)}$ and $\beta_2^{(k)}$, $k = M+1, ..., N$ for large N the Bayes estimates of $\xi = \lambda_1, \lambda_2, \theta_1, \theta_2, \alpha_1, \alpha_2, \beta_1$ or β_2 is given by

$$\hat{\xi}_{MC} = \frac{1}{N-M}\sum_{k=M+1}^{N} \xi^{(k)}. \quad (34)$$

To compute the credible intervals of $\lambda_1, \lambda_2, \theta_1, \theta_2, \alpha_1, \alpha_2, \beta_1$ and β_2 make $\lambda_1^{(k)}, \lambda_2^{(k)}, \theta_1^{(k)}, \theta_2^{(k)}, \alpha_1^{(k)}, \alpha_2^{(k)}, \beta_1^{(k)}$ and $\beta_2^{(k)}$, $k = 1, 2, ..., N$ in ascending order, hence $(1-\delta)100\%$ CI of them is given by

$$[\xi_{(N(\delta/2))}, \xi_{(N(1-\delta/2))}]. \quad (35)$$

6 Applications

In this case we analyze the strength data, which was originally reported by Badar and Priest (1982) and it represents the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 20 mm(Data Set 1) and10 mm(Data Set 2), with sample sizes $n = 74$ and $m = 63$, respectively. The data are presented in Tables 1 and 2. Based on the Kolmogorov-Smirnov (K-S) test, we checked the data fits the WFr model or not. For set 1, the calculated value of the K-S test is 0.068601 and the associated P-value equal 0.9282. For set 2, the calculated value of the K-S test is 0.068601 and the associated P-value equal 0.9282. So, the two sets fit the WFr model very well. Also, we have just plotted the data in Figure 2 for the first sample and in Figure 3 for the second sample. These figures show that WFD. can be a good model fitting this data.

Table 1: Data set 1 (gauge lengths of 20 mm).

1.312	1.314	1.479	1.552	1.700	1.803	1.861
1.865	1.944	1.958	1.966	1.997	2.006	2.021
2.027	2.055	2.063	2.098	2.140	2.179	2.224
2.240	2.253	2.270	2.272	2.274	2.301	2.301
2.359	2.382	2.382	2.426	2.434	2.435	2.478
2.490	2.511	2.514	2.535	2.554	2.566	2.570
2.586	2.629	2.633	2.642	2.648	2.684	2.697
2.726	2.770	2.773	2.800	2.809	2.818	2.821
2.848	2.880	2.809	2.818	2.821	2.848	2.880
2.954	3.012	3.067	3.084	3.090	3.096	3.128
3.233	3.433	3.585	3.585			

From the above data sets, a JP-II-C samples with the censoring scheme have been generated with $m = 74$ for the first sample and $n = 63$ for the second sample, hence, $N = m+n$ denoted the total sample size and when $r = 35$, $S = (17, 0_{16}, 17, 0_{16}, 17)$ and $T = (17, 0_{16}, 17, 0_{16}, 17)$, then $R = (34, 0_{16}, 34, 0_{16}, 34)$. The generated data sets are

Table 2: Data set 2 (gauge lengths of 10 mm).

1.901	2.132	2.203	2.228	2.257	2.350	2.361
2.396	2.397	2.445	2.454	2.474	2.518	2.522
2.525	2.532	2.575	2.614	2.616	2.618	2.624
2.659	2.675	2.738	2.740	2.856	2.917	2.928
2.937	2.937	2.977	2.996	3.030	3.125	3.139
3.145	3.220	3.223	3.235	3.243	3.264	3.272
3.294	3.332	3.346	3.377	3.408	3.435	3.493
3.501	3.537	3.554	3.562	3.628	3.852	3.871
3.886	3.971	4.024	4.027	4.225	4.395	5.020

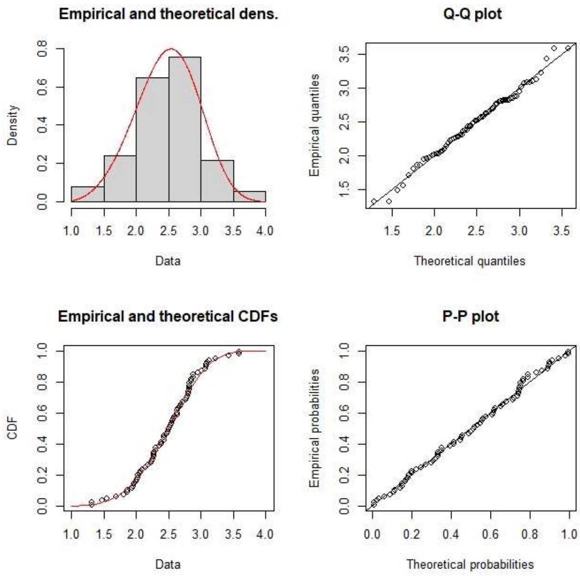

Fig. 2: Fitted first sample

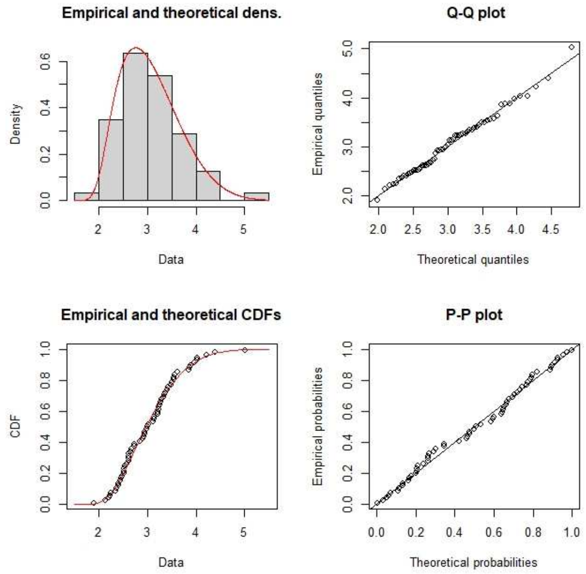

Fig. 3: Fitted second sample

w=(1.312, 1.314, 1.552, 1.7, 1.803, 1.861, 1.865, 1.901, 1.944, 1.958, 1.997,2.006, 2.021, 2.027, 2.055, 2.098, 2.132, 2.14, 2.253, 2.257,2.27, 2.272, 2.301, 2.361, 2.382, 2.382, 2.397, 2.434, 2.435, 2.474, 2.478, 2.49, 2.511, 2.535, 2.575) with Z=(1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 0). For bayesian estimation, MCMC method is used based on 10000 MCMC samples and discard the first 2000 values as 'burn-in', we used the informative priors with hyper-parameters are $c_1 = c_5 = 2$, $k_1 = k_5 = 2$, $c_i = 0.0001$ and $k_i = 0.01$ for $i = 2,3,4,6,7,8$, the bayesian estimates for $\lambda_i, \theta_i, \alpha_i$ and β_i, $i = 1,2$ are obtained under SE loss and LINEX loss functions. Based on the above information, we get

1.
2. Table 3 displays the point estimates based on MLEs, Bootstrap and MCMC methods for $\lambda_i, \theta_i, \alpha_i$ and β_i, $i = 1,2$.

Table 3: Different point estimates of $\lambda_1, \lambda_2, \theta_1, \theta_2, \alpha_1, \alpha_2, \beta_1, \beta_2$.

(.)	MLE	Boot-p	Boot-t	MCMC		
				SE	LINEX	
					q=-0.5	q=0.5
λ_1	0.0391	0.0512	0.0234	0.0158	0.0176	0.0176
λ_2	1.3538	1.5401	1.5367	1.4105	1.4880	1.3564
θ_1	1.6832	1.4015	1.6374	1.6844	1.6830	1.6829
θ_2	1.0483	0.8727	0.9587	1.1346	1.1391	1.1369
α_1	1.4278	1.5377	1.4951	1.3421	1.3453	1.3448
α_2	2.9321	2.7104	3.0012	2.9416	2.9396	2.9300
β_1	3.2148	2.9201	2.9842	3.1341	3.1376	3.1366
β_2	3.7508	3.4707	3.5390	3.6426	3.6460	3.6434

3. Tables (4)–(7) display the results of 95% ACIs, using MLE, Boot-p CI, Boot-t CI, Boot-BC CI, Boot-BCa CI and MCMC for these parameter.]

Table 4: 95% CI for λ_1 and λ_2.

Method	λ_1			λ_2		
	Lower	Upper	Length	Lower	Upper	Length
MLE	0.0001	0.2892	0.2891	0.0010	2.5922	2.5912
Boot-p	0.0282	0.0771	0.0489	1.1596	2.1046	0.9450
Boot-t	0.0221	0.1374	0.1153	0.9365	2.5319	1.5954
Boot-BC	0.0312	0.0642	0.0330	0.9046	2.0608	1.1562
Boot-BCa	0.0318	0.0642	0.0324	0.9023	2.0465	1.1442
MCMC	0.0105	0.0223	0.0118	0.5091	2.5187	2.0096

4. The comparing between the approximation of the expected values of the number of failures from the first production line (A.E.M_r) and the exact number of failures (S.E.M_r) when $r = 17,27,37,47$ and 57 is discussed in Table (8), where $p = P(Z = 1) = P(X < Y)$ and $q = 1 - p$.

Table 5: 95% CI for θ_1 and θ_2.

Method	θ_1			θ_2		
	Lower	Upper	Length	Lower	Upper	Length
ACI	0.0001	1.6470	1.6469	0.0001	1.9825	1.9824
BPCI	1.2949	1.4700	0.1751	0.6364	1.1162	0.4798
BTCI	1.2109	2.1001	0.8892	0.3309	1.5314	1.2005
BBCCI	1.4295	1.4730	0.0435	0.7564	1.1206	0.3642
BBCaCI	1.4324	1.4730	0.0406	0.7313	1.1206	0.3893
CRI	1.6655	1.7127	0.0471	1.0082	1.2518	0.2436

Table 6: 95% CI for α_1 and α_2.

Method	α_1			α_2		
	Lower	Upper	Length	Lower	Upper	Length
MLE	0.3760	2.4797	2.1037	0.0001	2.1991	2.1990
Boot-p	1.5202	1.5617	0.0415	2.6134	2.8233	0.2099
Boot-t	1.0297	1.7998	0.7701	2.5124	3.4128	0.9004
Boot-BC	1.5455	1.6087	0.0632	2.7728	2.8255	0.0527
Boot-BCa	1.5454	1.6087	0.0633	2.7728	2.8255	0.0527
MCMC	1.2879	1.3964	0.1085	2.8124	3.0921	0.2797

Table 7: 95% CI for β_1 and β_2.

Method	β_1			β_2		
	Lower	Upper	Length	Lower	Upper	Length
MLE	0.0001	3.2959	3.2958	2.5321	4.1615	1.6294
Boot-p	2.8131	2.9900	0.1769	2.9301	3.9546	1.0245
Boot-t	2.3529	3.0194	0.6665	2.9185	3.8791	0.9606
Boot-BC	2.9491	2.9909	0.0418	3.7222	3.9788	0.2566
Boot-BCa	2.9478	2.9909	0.0431	3.7222	3.9788	0.2566
MCMC	3.0656	3.1995	0.1338	3.5373	3.7502	0.2129

Table 8: The comparison between A.E.M_r and S.E.M_r for real data, $(m,n) = (74,63)$.

SC	r	$(R_1,\cdots,R_r)$	$p=0.24$		$p=0.76$	
			A.E.M_r	S.E.M_r	A.E.M_r	S.E.M_r
1	17	$(120,0_{(16)})$	9.3767	9.552	9.3767	9.535
2	27	$(110,0_{(26)})$	14.364	14.828	14.7859	15.189
3	37	$(100,0_{(36)})$	19.7859	20.377	20.1695	20.373
4	47	$(90,0_{(46)})$	25.2076	25.686	25.5522	25.651
5	57	$(80,0_{(56)})$	30.6291	31.047	30.9351	31.027

7 Simulation Study

A simulation was performed to compare the performance of the different methods discussed in this paper. Suppose various sample sizes for the two populations as $m,n = 10,20,30,40,50,60$ and various values of JP-II-CS $r = 5,10,15,20,30,40,50,60,70,80$. Also, set the parameters $(\lambda_1,\lambda_2,\theta_1,\theta_2,\alpha_1,\alpha_2,$ $\beta_1,\beta_2) = (0.5,0.6,2.5,2.69,0.69,0.8,1.57,1.8)$. The MSEs, lentghs of 95% CIs and the corresponding coverage probability (CP) for the parameters $\lambda_1,\lambda_2,\theta_1,\theta_2,$ $\alpha_1,\alpha_2,\beta_1$ and β_2 have been evaluated using MLEs and MCMC with 10000 observations under SE and LINEX loss functions. This process is repeated 1000 times and the results of the mean values of MSE, lengths and CP, are displayed in Tables (10)–(17).

Moreover, in this section a simulation study was conducted to compute the expected number of failures from the first production line (S.E.M_r) and also compute the approximated expected number of failures (A.E.M_r). We assumed various sample sizes for the two populations as $m,n = 5,10,15,20,25,30,40;50$ and various choices of JP-II-CS $r = 5,10,20,30,40$. 1000 samples from the two WFD populations have been generated under the same truth values of these parameters, the results are presented in Table (9). The calculations in Table (9) are computed under the following assumptions: $p = P(X_1 < X_2)$ where X_1, X_2 are the life time of the first production line units and the second production line units, respectively, in which X_1 is selected from $WFD(0.5,2.5,0.69,1.57)$ and X_2 from $WFD(0.6,2.69,0.8,1.8)$, once more X_1 selected from $WFD(0.6,2.69,0.8,1.8)$ and X_2 from $WFD(0.5,2.5,0.69,1.57)$, we set the hyper-parameter $c_i = 1$, $i = 1,\cdots,8$, $k_1 = k_5 = 2$, $k_j = 1.3$, j=2,3,4,6,7,8. We calculate the A.E.M_r according to Parsi and Bairamov [18]as follows:

$$\mu_{Z_1} = E(Z_1) = \frac{m}{mp+nq}p,$$

$$\mu_{\acute{R}_1} = \frac{r_1(m - \frac{mp}{mp+nq})}{N-1},$$

$$\mu_{Z_2} = \left\{\left(m - \mu_{Z_1} - \mu_{\acute{R}_1}\right)\right\} / \left\{\left(m - \mu_{Z_1} - \mu_{\acute{R}_1}\right)p + \left(n - (1-\mu_{Z_1}) - (r_1 - \mu_{\acute{R}_1})\right)q\right\}p,$$

$$\mu_{\acute{R}_2} = \frac{r_2(m - \Sigma_{j=1}^{2}\mu_{Z_j} - \mu_{\acute{R}_1})}{N - \mu_{\acute{R}_1} - 2},$$

$$\mu_{Z_i} = \left\{m - \sum_{j=1}^{i-1}\mu_{Z_i} - \sum_{j=1}^{i-1}\mu_{\acute{R}_j}\right\} / \left\{\left(m - \sum_{j=1}^{i-1}\mu_{Z_i} - \sum_{j=1}^{i-1}\mu_{\acute{R}_j}\right)p + \left(n - \sum_{j=1}^{i-1}(1-\mu_{Z_j}) - \sum_{j=1}^{j-1}(r_j - \mu_{\acute{R}_j})\right)q\right\}p,$$

and

$$\mu_{\acute{R}_i} = \frac{r_i(m - \Sigma_{j=1}^{i}\mu_{Z_j} - \Sigma_{j=1}^{i-1}\mu_{\acute{R}_j})}{N - \Sigma_{j=1}^{i-1}\mu_{\acute{R}_j} - i}, \qquad ,i = 1,\cdots,r.$$

Therefore, the approximate value for the expected value of the number of failures from X is

$$(A.E.M_r) = E(M_r) = \sum_{i=1}^{r}\mu_{Z_i}.$$

Table 9: The comparison between A.E.M_r and S.E.M_r for simulated data.

SC	(m,n)	r	$(R_1,\cdots,R_r)$	$p=0.39$		$p=0.61$	
				A.E.M_r	S.E.M_r	A.E.M_r	S.E.M_r
1	$(10,10)$	5	$(0_{(4)},15)$	2.0074	3.216	2.9926	3.271
2	$(15,5)$			3.34693	4.276	4.09004	4.271
3	$(5,15)$			0.909958	1.862	1.65307	1.828
4	$(10,10)$		$(15,0_{(4)})$	2.31262	2.68	2.68738	2.715
5	$(15,5)$			3.62895	3.896	3.91135	3.852
6	$(5,15)$			1.08865	1.431	1.37105	1.475
7	$(10,10)$		$(6,0,4,0,5)$	2.14595	3.129	3.05521	3.055
8	$(15,5)$			3.47258	4.179	4.1392	4.192
9	$(5,15)$			1.0273	1.754	1.75828	1.714
10	$(10,10)$	10	$(0_{(9)},10)$	4.1998	5.811	5.8002	5.829
11	$(15,5)$			6.87799	8.067	8.07487	8.135
12	$(5,15)$			1.92513	3.085	3.12201	3.09
13	$(10,10)$		$(10,0_{(9)})$	4.85991	5.205	5.14009	5.114
14	$(15,5)$			7.42025	7.607	7.64982	7.645
15	$(5,15)$			2.35018	20631	2.57975	2.657
16	$(20,20)$	20	$(0_{(19)},20)$	8.42817	11.677	11.5718	11.589
17	$(25,15)$			11.0003	13.934	13.9453	13.954
18	$(15,25)$			6.05471	8.998	8.99965	8.896
19	$(20,20)$		$(20,0_{(19)})$	9.87969	10.098	10.1203	10.161
20	$(25,15)$			12.405	12.632	12.6422	12.684
21	$(15,25)$			7.35776	7.588	7.59503	7.694
22	$(30,30)$	30	$(0_{(29)},30)$	12.6563	17.45	17.3437	17.315
23	$(40,20)$			17.8698	22.222	22.0306	22.234
24	$(20,40)$			7.96942	12.044	12.1302	12.143
25	$(30,30)$		$(30,0_{(29)})$	14.8902	15.105	15.1098	15.313
26	$(40,20)$			19.9205	20.152	20.1388	20.085
27	$(20,40)$			9.86121	10.222	10.0795	10.208
28	$(40,40)$	40	$(0_{(39)},40)$	16.8844	23.105	23.1156	23.214
29	$(50,30)$			22.0295	28.097	27.8667	28.129
30	$(30,50)$			12.1333	17.877	17.9705	17.997
31	$(40,40)$		$(40,0_{(29)})$	19.8971	20.143	20.1029	20.055
32	$(50,30)$			24.9176	25.102	25.125	25.125
33	$(30,50)$			14.875	15.16	15.0824	15.211

Table 10: MSE, length and coverage probability (CP) of estimates for the parameter λ_1.

(m,n)	r	$(R_1,\cdots,R_r)$	MLE			Bootstrap		MCMC				
			MSE	Length	CP	Boot-p	Boot-t	SE	LINEX		Length	CP
									q=-0.5	q=0.5		
(10,10)	5	$(0_{(4)},15)$	0.0704	2.2969	0.991	0.0702	0.698	0.0662	0.0701	0.0643	1.6018	0.996
		$(0_{(3)},2,13)$	0.0640	2.3476	0.995	0.0635	0.0634	0.0637	0.0642	0.0632	1.6692	0.997
	10	$(0_{(9)},10)$	0.0681	2.0297	0.994	0.0677	0.0675	0.0652	0.0660	0.0649	1.4073	0.996
		$(10,0_{(9)})$	0.0636	2.7830	0.992	0.0638	0.0635	0.0648	0.0658	0.0648	1.4802	0.996
	15	$(0_{(14)},5)$	0.0679	2.8709	0.992	0.0675	0.0675	0.0658	0.0661	0.0648	1.3428	0.997
		$(5,0_{(14)})$	0.0651	2.2499	0.993	0.0650	0.0650	0.0650	0.0657	0.0649	1.3445	0.996
(10,20)	10	$(0_{(9)},20)$	0.0693	2.0538	0.994	0.0692	0.0690	0.0688	0.0883	0.0875	1.5900	0.995
		$(20,0_{(9)})$	0.0670	2.0020	0.995	0.0670	0.0667	0.0671	0.0675	0.0669	1.5732	0.997
	15	$(0_{(14)},15)$	0.0681	2.8209	0.994	0.0680	0.0675	0.0678	0.0781	0.0676	1.5024	0.996
		$(15,0_{(14)})$	0.0667	2.0288	0.995	0.0667	0.0666	0.0665	0.0669	0.0665	1.4907	0.996
	20	$(0_{(19)},10)$	0.0681	2.8931	0.993	0.0680	0.0674	0.0669	0.0673	0.0669	1.5380	0.997
		$(10,0_{(19)})$	0.0663	2.5563	0.995	0.0661	0.0658	0.0649	0.0659	0.0648	1.4899	0.996
(20,10)	10	$(0_{(9)},20)$	0.0681	2.8930	0.992	0.0679	0.0679	0.0680	0.0681	0.0673	1.6100	0.996
		$(20,0_{(9)})$	0.0669	2.0641	0.994	0.0670	0.0669	0.0671	0.0674	0.0669	1.4380	0.997
	15	$(0_{(14)},15)$	0.0680	2.1200	0.993	0.0675	0.0674	0.0668	0.0670	0.0665	1.5073	0.996
		$(15,0_{(14)})$	0.0673	1.8399	0.994	0.0671	0.0670	0.0665	0.0670	0.0663	1.4599	0.996
	20	$(0_{(19)},10)$	0.0674	2.6823	0.990	0.0673	0.0671	0.0669	0.0675	0.0669	1.5003	0.997
		$(10,0_{(19)})$	0.0665	2.6502	0.995	0.0665	0.0663	0.0658	0.0660	0.0658	1.4822	0.997
(20,30)	20	$(0_{(19)},30)$	0.0592	1.9924	0.993	0.0591	0.0589	0.0590	0.0595	0.0587	1.5043	0.996
		$(30,0_{(19)})$	0.0586	1.8356	0.993	0.0585	0.0585	0.0585	0.0589	0.0583	1.5128	0.996
	30	$(0_{(29)},20)$	0.0590	2.0020	0.995	0.0589	0.0586	0.0585	0.0588	0.0584	1.4588	0.997
		$(20,0_{(29)})$	0.0587	2.5321	0.994	0.0587	0.0586	0.0588	0.0589	0.0584	1.4935	0.997
	40	$(0_{(39)},10)$	0.0580	2.6812	0.994	0.0579	0.0576	0.0573	0.0578	0.0572	1.5385	0.998
		$(10,0_{(39)})$	0.0577	2.4348	0.995	0.0577	0.0575	0.0575	0.0576	0.0570	1.5113	0.998
(30,20)	20	$(0_{(19)},30)$	0.0585	1.9934	0.991	0.0584	0.0582	0.0581	0.0581	0.0579	1.4421	0.997
		$(30,0_{(19)})$	0.0582	2.0485	0.992	0.0581	0.0579	0.0576	0.0579	0.0575	1.4445	0.997
	30	$(0_{(29)},20)$	0.0579	2.3902	0.994	0.0579	0.0578	0.0579	0.0583	0.0576	1.5146	0.997
		$(20,0_{(29)})$	0.0568	2.0989	0.994	0.0566	0.0565	0.0563	0.0569	0.0562	1.5095	0.996
	40	$(0_{(39)},10)$	0.0573	1.9301	0.994	0.0572	0.0570	0.0569	0.0572	0.0569	1.3962	0.996
		$(10,0_{(39)})$	0.0571	2.0790	0.995	0.0571	0.0570	0.0570	0.0577	0.0569	1.4715	0.997
(40,50)	40	$(0_{(39)},50)$	0.0529	2.0152	0.994	0.0528	0.0525	0.0521	0.0528	0.0520	1.4834	0.996
		$(50,0_{(39)})$	0.0525	1.9150	0.995	0.0524	0.0523	0.0520	0.0524	0.0518	1.3890	0.997
	50	$(0_{(49)},40)$	0.0526	2.3630	0.995	0.0525	0.0523	0.0525	0.0527	0.0520	1.5004	0.997
		$(40,0_{(49)})$	0.0520	2.3584	0.993	0.0519	0.0517	0.0523	0.0525	0.0515	1.4626	0.997
	60	$(0_{(59)},30)$	0.0517	1.9585	0.996	0.0515	0.0512	0.0512	0.0519	0.0510	1.5020	0.998
		$(30,0_{(59)})$	0.0516	1.9333	0.996	0.0515	0.0514	0.0512	0.0515	0.0509	1.4917	0.998
(50,40)	40	$(0_{(39)},50)$	0.0518	1.8980	0.995	0.0516	0.0515	0.0509	0.0517	0.0508	1.4805	0.996
		$(50,0_{(39)})$	0.0509	2.1523	0.995	0.0508	0.0506	0.0505	0.0508	0.0505	1.5014	0.996
	50	$(0_{(49)},40)$	0.0515	1.8735	0.994	0.0513	0.0512	0.0508	0.0512	0.0505	1.3942	0.997
		$(40,0_{(49)})$	0.0508	1.9141	0.993	0.0508	0.0507	0.0510	0.0510	0.0501	1.5002	0.998
	60	$(0_{(59)},30)$	0.0510	1.9471	0.994	0.0509	0.0506	0.0506	0.0507	0.0500	1.4719	0.998
		$(30,0_{(59)})$	0.0499	2.0735	0.995	0.0498	0.0498	0.0495	0.0501	0.0494	1.4936	0.998
(50,60)	60	$(0_{(59)},50)$	0.0352	2.3852	0.995	0.0352	0.0350	0.0350	0.0350	0.0341	1.4450	0.997
		$(50,0_{(59)})$	0.0345	2.3238	0.995	0.0343	0.0340	0.0339	0.0339	0.0330	1.4980	0.997
	70	$(0_{(69)},40)$	0.0326	2.5073	0.994	0.0325	0.0323	0.0324	0.0328	0.0321	1.3900	0.997
		$(40,0_{(69)})$	0.0323	2.0123	0.995	0.0322	0.0320	0.0320	0.0223	0.0319	1.4266	0.996
	80	$(0_{(79)},30)$	0.0315	2.0538	0.994	0.0314	0.0312	0.0311	0.0312	0.0310	1.3972	0.998
		$(30,0_{(79)})$	0.0300	2.4587	0.994	0.0299	0.0297	0.0297	0.0299	0.0296	1.4562	0.996
(60,50)	60	$(0_{(59)},50)$	0.0324	1.9879	0.995	0.0324	0.0323	0.0322	0.0326	0.0321	1.4630	0.996
		$(50,0_{(59)})$	0.0315	2.5257	0.993	0.0312	0.0310	0.0307	0.0307	0.0300	1.4444	0.997
	70	$(0_{(69)},40)$	0.0294	2.0458	0.995	0.0293	0.0293	0.0289	0.0295	0.0286	1.4595	0.996
		$(40,0_{(69)})$	0.0282	1.9368	0.994	0.0281	0.0280	0.0280	0.0286	0.0275	1.5040	0.998
	80	$(0_{(79)},30)$	0.0250	2.3169	0.995	0.0248	0.0247	0.0255	0.0255	0.0249	1.4985	0.998
		$(30,0_{(79)})$	0.0246	2.3401	0.995	0.0245	0.0243	0.0240	0.0247	0.0239	1.4789	0.998

Table 11: MSE, length and coverage probability (CP) of estimates for the parameter λ_2.

(m,n)	r	$(R_1,\cdots,R_r)$	MLE			Bootstrap		MCMC				
			MSE	Length	CP	Boot-p	Boot-t	SE	LINEX		Length	CP
									q=-0.5	q=0.5		
(10,10)	5	$(0_{(4)},15)$	0.0643	2.3567	0.995	0.0635	0.0632	0.0562	0.0575	0.0560	1.8264	0.996
		$(0_{(3)},2,13)$	0.0642	2.0389	0.994	0.0635	0.0631	0.0560	0.0575	0.0558	1.7962	0.996
	10	$(0_{(9)},10)$	0.0642	2.6120	0.993	0.0625	0.0624	0.0563	0.0579	0.0563	1.8340	0.997
		$(10,0_{(9)})$	0.0631	2.1110	0.994	0.0617	0.0610	0.0559	0.0572	0.0555	1.6937	0.996
	15	$(0_{(14)},5)$	0.0637	2.3720	0.993	0.0618	0.0615	0.0560	0.0561	0.0554	1.6592	0.996
		$(5,0_{(14)})$	0.0630	2.1254	0.992	0.0614	0.0611	0.0551	0.0560	0.0549	1.5395	0.997
(10,20)	10	$(0_{(9)},20)$	0.0607	2.6342	0.993	0.0586	0.0579	0.0546	0.0547	0.0546	1.7019	0.996
		$(20,0_{(9)})$	0.0605	2.0268	0.995	0.0589	0.0585	0.0541	0.0565	0.0530	1.2570	0.996
	15	$(0_{(14)},15)$	0.0599	2.0720	0.994	0.0578	0.0575	0.0534	0.0556	0.0534	1.6119	0.993
		$(15,0_{(14)})$	0.0598	2.3347	0.994	0.0578	0.0572	0.0562	0.0581	0.0559	1.5437	0.995
	20	$(0_{(19)},10)$	0.0593	2.3430	0.993	0.0572	0.0570	0.0531	0.0545	0.0524	1.6236	0.995
		$(10,0_{(19)})$	0.0590	2.3420	0.992	0.0560	0.0559	0.0523	0.0540	0.0520	1.7163	0.996
(20,10)	10	$(0_{(9)},20)$	0.0605	2.6230	0.992	0.0569	0.0565	0.0543	0.0565	0.0541	1.5179	0.996
		$(20,0_{(9)})$	0.0601	2.0828	0.995	0.0579	0.0573	0.0543	0.0551	0.0538	1.5675	0.996
	15	$(0_{(14)},15)$	0.0594	2.6510	0.994	0.0586	0.0585	0.0539	0.0549	0.0531	1.6786	0.997
		$(15,0_{(14)})$	0.0593	2.2993	0.996	0.0587	0.0582	0.0530	0.0548	0.0525	1.4629	0.996
	20	$(0_{(19)},10)$	0.0593	2.6530	0.995	0.0592	0.0585	0.0528	0.0540	0.0527	1.6834	0.997
		$(10,0_{(19)})$	0.0590	2.3720	0.994	0.0578	0.0573	0.0528	0.0553	0.0526	1.6096	0.996
(20,30)	20	$(0_{(19)},30)$	0.0564	2.2725	0.993	0.0549	0.0546	0.0493	0.0515	0.0485	1.7891	0.996
		$(30,0_{(19)})$	0.0556	2.3250	0.993	0.0558	0.0553	0.0485	0.0504	0.0478	1.6261	0.997
	30	$(0_{(29)},20)$	0.0559	2.4120	0.992	0.0556	0.0551	0.0487	0.0492	0.0483	1.5777	0.997
		$(20,0_{(29)})$	0.0548	2.4114	0.995	0.0531	0.0525	0.0481	0.0496	0.0479	1.5110	0.997
	40	$(0_{(39)},10)$	0.0551	2.5523	0.992	0.0529	0.0526	0.0498	0.0501	0.0495	1.4930	0.998
		$(10,0_{(39)})$	0.0534	2.0600	0.993	0.0523	0.0522	0.0452	0.0476	0.0438	1.4246	0.996
(30,20)	20	$(0_{(19)},30)$	0.0523	2.3870	0.995	0.0519	0.0510	0.0487	0.0491	0.0476	1.3751	0.997
		$(30,0_{(19)})$	0.0525	2.3443	0.995	0.0498	0.0495	0.0481	0.0482	0.0481	1.7544	0.997
	30	$(0_{(29)},20)$	0.0517	2.0630	0.995	0.0497	0.0496	0.0459	0.0466	0.0455	1.7084	0.996
		$(20,0_{(29)})$	0.0512	2.6640	0.994	0.0501	0.0497	0.0487	0.0506	0.0481	1.6086	0.997
	40	$(0_{(39)},10)$	0.0500	2.4540	0.994	0.0463	0.0468	0.0414	0.0415	0.0414	1.5582	0.996
		$(10,0_{(39)})$	0.0497	2.4470	0.995	0.0458	0.0458	0.0421	0.0429	0.0419	1.5378	0.997
(40,50)	40	$(0_{(39)},50)$	0.0453	2.5832	0.992	0.0447	0.0435	0.0415	0.0424	0.0412	1.5603	0.996
		$(50,0_{(39)})$	0.0452	1.8272	0.994	0.0443	0.0439	0.0300	0.0324	0.0300	1.5035	0.996
	50	$(0_{(49)},40)$	0.0422	2.3487	0.993	0.0404	0.0398	0.0391	0.0396	0.0389	1.5019	0.997
		$(40,0_{(49)})$	0.0416	1.9118	0.994	0.0417	0.0412	0.0387	0.0393	0.0385	1.4110	0.997
	60	$(0_{(59)},30)$	0.0407	1.9138	0.995	0.0391	0.0388	0.0359	0.0360	0.0359	1.4424	0.998
		$(30,0_{(59)})$	0.0407	1.8510	0.995	0.0385	0.0384	0.0352	0.0359	0.0349	1.4366	0.997
(50,40)	40	$(0_{(39)},50)$	0.0439	2.2550	0.993	0.0429	0.0425	0.0337	0.0345	0.0335	1.6478	0.996
		$(50,0_{(39)})$	0.0430	2.0536	0.995	0.0429	0.0423	0.0328	0.0339	0.0325	1.5179	0.997
	50	$(0_{(49)},40)$	0.0419	1.9478	0.995	0.0420	0.0412	0.0331	0.0342	0.0327	1.5625	0.996
		$(40,0_{(49)})$	0.0415	2.6914	0.994	0.0409	0.0399	0.0320	0.0335	0.0314	1.4612	0.998
	60	$(0_{(59)},30)$	0.0405	2.3927	0.995	0.0387	0.0383	0.0323	0.0329	0.0322	1.4267	0.997
		$(30,0_{(59)})$	0.0399	1.9225	0.994	0.0385	0.0381	0.0318	0.0324	0.0315	1.4708	0.996
(50,60)	60	$(0_{(59)},50)$	0.0271	2.1200	0.995	0.0275	0.0269	0.0221	0.0236	0.0219	1.4371	0.997
		$(50,0_{(59)})$	0.0265	2.5217	0.994	0.0264	0.0261	0.0215	0.0229	0.0213	1.4440	0.997
	70	$(0_{(69)},40)$	0.0268	2.2397	0.993	0.0271	0.0267	0.0219	0.0228	0.0216	1.4255	0.996
		$(40,0_{(69)})$	0.0253	2.3265	0.992	0.0242	0.0245	0.0211	0.0224	0.0210	1.4075	0.997
	80	$(0_{(79)},30)$	0.0225	2.3083	0.994	0.0224	0.0221	0.0212	0.0213	0.0212	1.3864	0.996
		$(30,0_{(79)})$	0.0211	1.9492	0.995	0.0209	0.0204	0.0195	0.0209	0.0195	1.3910	0.996
(60,50)	60	$(0_{(59)},50)$	0.0267	1.9652	0.994	0.0253	0.0249	0.0217	0.0223	0.0212	1.4279	0.997
		$(50,0_{(59)})$	0.0264	1.9196	0.993	0.0262	0.0259	0.0209	0.0218	0.0209	1.5081	0.996
	70	$(0_{(69)},40)$	0.0249	1.8714	0.994	0.0235	0.0229	0.0208	0.0211	0.0207	1.4032	0.997
		$(40,0_{(69)})$	0.0241	1.9831	0.995	0.0238	0.0233	0.0201	0.0210	0.0198	1.4300	0.996
	80	$(0_{(79)},30)$	0.0223	1.9856	0.995	0.0222	0.0217	0.0189	0.0194	0.0188	1.4123	0.998
		$(30,0_{(79)})$	0.0221	1.8661	0.995	0.0214	0.0215	0.0185	0.0193	0.0179	1.4171	0.998

Table 12: MSE, length and coverage probability (CP) of estimates for the parameter θ_1.

(m,n)	r	$(R_1,\cdots,R_r)$	MLE			Bootstrap		MCMC				
			MSE	Length	CP	Boot-p	Boot-t	SE	LINEX		Length	CP
									q=-0.5	q=0.5		
(10,10)	5	$(0_{(4)},15)$	0.0959	2.9738	0.994	0.0958	0.0953	0.0948	0.0954	0.0945	1.9111	0.996
		$(0_{(3)},2,13)$	0.0925	2.4215	0.992	0.0922	0.0919	0.0816	0.0823	0.0815	1.8955	0.996
	10	$(0_{(9)},10)$	0.0924	2.4510	0.993	0.0921	0.0917	0.0874	0.0885	0.0871	2.0075	0.997
		$(10,0_{(9)})$	0.0917	2.5387	0.992	0.0915	0.0908	0.0875	0.0882	0.0869	2.0074	0.997
	15	$(0_{(14)},5)$	0.0918	2.8343	0.994	0.0918	0.0911	0.0873	0.0875	0.0869	2.0036	0.995
		$(5,0_{(14)})$	0.0895	2.4505	0.994	0.0891	0.0887	0.0859	0.0863	0.0854	2.0021	0.995
(10,20)	10	$(0_{(9)},20)$	0.0832	2.5501	0.993	0.0830	0.0824	0.0813	0.0825	0.0812	2.0009	0.995
		$(20,0_{(9)})$	0.0832	2.8226	0.992	0.0831	0.0823	0.0809	0.0815	0.0807	1.9474	0.996
	15	$(0_{(14)},15)$	0.0829	2.6931	0.994	0.0830	0.0825	0.0815	0.0821	0.0812	1.9551	0.996
		$(15,0_{(14)})$	0.0821	2.5853	0.991	0.0822	0.0817	0.0795	0.0800	0.0794	1.9538	0.996
	20	$(0_{(19)},10)$	0.0809	2.6598	0.994	0.0806	0.0801	0.0789	0.0795	0.0785	1.9241	0.995
		$(10,0_{(19)})$	0.0798	2.7542	0.993	0.0785	0.0783	0.0765	0.0777	0.0764	1.9308	0.997
(20,10)	10	$(0_{(9)},20)$	0.0821	2.5449	0.992	0.0815	0.0808	0.0793	0.0799	0.0792	2.0061	0.997
		$(20,0_{(9)})$	0.0809	2.8716	0.995	0.0805	0.0805	0.0790	0.0796	0.0789	1.9402	0.996
	15	$(0_{(14)},15)$	0.0813	2.5391	0.993	0.0812	0.0805	0.0889	0.0892	0.0885	1.9278	0.996
		$(15,0_{(14)})$	0.0801	2.8027	0.992	0.0801	0.0800	0.0885	0.0889	0.0885	1.9465	0.997
	20	$(0_{(19)},10)$	0.0795	2.5257	0.994	0.0794	0.0791	0.0769	0.0775	0.0763	2.0116	0.996
		$(10,0_{(19)})$	0.0789	2.6522	0.993	0.0788	0.0785	0.0762	0.0763	0.0762	2.0403	0.996
(20,30)	20	$(0_{(19)},30)$	0.0715	2.5402	0.992	0.0714	0.0710	0.0640	0.0643	0.0641	1.9864	0.995
		$(30,0_{(19)})$	0.0708	2.4593	0.992	0.0705	0.0704	0.0638	0.0642	0.0634	2.0033	0.995
	30	$(0_{(29)},20)$	0.0709	2.5455	0.992	0.0709	0.0705	0.0632	0.0639	0.0631	1.9265	0.995
		$(20,0_{(29)})$	0.0694	2.6350	0.994	0.0695	0.0691	0.0627	0.0628	0.0627	1.9312	0.996
	40	$(0_{(39)},10)$	0.0691	2.4400	0.995	0.0690	0.0685	0.0616	0.0611	0.0609	1.9451	0.996
		$(10,0_{(39)})$	0.0687	2.5209	0.993	0.0688	0.0685	0.0613	0.0619	0.0613	2.0019	0.995
(30,20)	20	$(0_{(19)},30)$	0.0701	2.4361	0.992	0.0699	0.0698	0.0637	0.0642	0.0635	1.9549	0.995
		$(30,0_{(19)})$	0.0698	2.4317	0.992	0.0699	0.0695	0.0634	0.0639	0.0634	1.9465	0.996
	30	$(0_{(29)},20)$	0.0698	2.7970	0.993	0.0698	0.0696	0.0636	0.0639	0.0635	1.9326	0.995
		$(20,0_{(29)})$	0.0685	2.6538	0.994	0.0684	0.0681	0.0623	0.0631	0.0622	1.9312	0.997
	40	$(0_{(39)},10)$	0.0684	2.6203	0.993	0.0679	0.0678	0.0629	0.0631	0.0625	1.9212	0.996
		$(10,0_{(39)})$	0.0679	2.4425	0.992	0.0678	0.0675	0.0609	0.0612	0.0609	1.9212	0.995
(40,50)	40	$(0_{(39)},50)$	0.0544	2.4519	0.994	0.0544	0.0541	0.0498	0.0506	0.0497	1.9388	0.997
		$(50,0_{(39)})$	0.0538	2.6691	0.992	0.0537	0.0532	0.0492	0.0399	0.0490	1.9128	0.996
	50	$(0_{(49)},40)$	0.0529	2.5463	0.994	0.0525	0.0522	0.0491	0.0498	0.0487	1.9390	0.996
		$(40,0_{(49)})$	0.0527	2.7491	0.994	0.0526	0.0526	0.0493	0.0497	0.0485	2.0016	0.998
	60	$(0_{(59)},30)$	0.0508	2.7970	0.995	0.0509	0.0503	0.0479	0.0486	0.0477	1.9348	0.996
		$(30,0_{(59)})$	0.0495	2.3066	0.991	0.0501	0.0494	0.0460	0.0469	0.0459	1.9211	0.997
(50,40)	40	$(0_{(39)},50)$	0.0533	2.4610	0.992	0.0529	0.0528	0.0485	0.0491	0.0484	1.9494	0.997
		$(50,0_{(39)})$	0.0527	2.5680	0.995	0.0524	0.0524	0.0462	0.0468	0.0462	1.9236	0.997
	50	$(0_{(49)},40)$	0.0514	2.4860	0.995	0.0509	0.0508	0.0460	0.0465	0.0461	1.9251	0.996
		$(40,0_{(49)})$	0.0519	2.4397	0.993	0.0518	0.0514	0.0459	0.0463	0.0457	1.9133	0.995
	60	$(0_{(59)},30)$	0.0503	2.4013	00993	0.0503	0.0503	0.0444	0.0448	0.0444	1.9119	0.997
		$(30,0_{(59)})$	0.0491	2.3952	0.993	0.0490	0.0487	0.0437	0.0437	0.0437	1.9315	0.996
(50,60)	60	$(0_{(59)},50)$	0.0335	2.3528	0.993	0.0333	0.0332	0.0282	0.0283	0.0282	1.9018	0.996
		$(50,0_{(59)})$	0.0322	2.4957	0.992	0.0322	0.0319	0.0274	0.0282	0.0271	1.9216	0.995
	70	$(0_{(69)},40)$	0.0328	2.4989	0.993	0.0327	0.0325	0.0280	0.0286	0.0278	1.8909	0.997
		$(40,0_{(69)})$	0.0316	2.4451	0.993	0.0315	0.0315	0.0266	0.0267	0.0266	1.8905	0.997
	80	$(0_{(79)},30)$	0.0304	2.3907	0.994	0.0301	0.0298	0.0259	0.0259	0.0255	1.9311	0.996
		$(30,0_{(79)})$	0.0297	2.6137	0.994	0.0297	0.0295	0.0243	0.0249	0.0238	1.9118	0.997
(60,50)	60	$(0_{(59)},50)$	0.0327	2.3821	0.995	0.0325	0.0324	0.0275	0.0281	0.0274	1.8714	0.996
		$(50,0_{(59)})$	0.0311	2.3922	0.994	0.0312	0.0311	0.0268	0.0271	0.0268	1.9606	0.997
	70	$(0_{(69)},40)$	0.0319	2.5454	0.993	0.0318	0.0315	0.0253	0.0255	0.0252	1.9612	0.997
		$(40,0_{(69)})$	0.0307	2.3911	0.994	0.0306	0.0302	0.0249	0.0249	0.0245	1.9737	0.998
	80	$(0_{(79)},30)$	0.0299	2.3564	0.994	0.0297	0.0296	0.0221	0.0223	0.0220	1.9157	0.998
		$(30,0_{(79)})$	0.0271	2.3492	0.995	0.0270	0.0768	0.0210	0.0218	0.0209	1.9011	0.998

Table 13: MSE, length and coverage probability (CP) of estimates for the parameter θ_2.

(m,n)	r	$(R_1,\cdots,R_r)$	MLE			Bootstrap		MCMC				
			MSE	Length	CP	Boot-p	Boot-t	SE	LINEX		Length	CP
									q=-0.5	q=0.5		
(10,10)	5	$(0_{(4)},15)$	0.0663	2.6630	0.992	0.0662	0.0660	0.0624	0.0624	0.0621	1.6123	0.995
		$(0_{(3)},2,13)$	0.0655	2.5549	0.993	0.0653	0.0653	0.0621	0.0621	0.0618	1.5098	0.995
	10	$(0_{(9)},10)$	0.0656	2.5638	0.992	0.0655	0.0652	0.0618	0.0621	0.0616	1.6190	0.996
		$(10,0_{(9)})$	0.0649	2.5456	0.992	0.0649	0.0645	0.0615	0.0626	0.0612	1.6169	0.995
	15	$(0_{(14)},5)$	0.0626	2.5133	0.993	0.0625	0.0622	0.0598	0.0605	0.0597	1.7352	0.995
		$(5,0_{(14)})$	0.0618	2.5460	0.993	0.0619	0.0615	0.0577	0.0585	0.0575	1.6821	0.996
(10,20)	10	$(0_{(9)},20)$	0.0636	2.4763	0.992	0.0635	0.0632	0.0598	0.0599	0.0593	1.6359	0.996
		$(20,0_{(9)})$	0.0621	2.4739	0.992	0.0620	0.0618	0.0582	0.0587	0.0582	1.7150	0.995
	15	$(0_{(14)},15)$	0.0618	2.5276	0.993	0.0618	0.0615	0.0583	0.0584	0.0578	1.7165	0.996
		$(15,0_{(14)})$	0.0605	2.6530	0.994	0.0604	0.0602	0.0572	0.0579	0.0570	1.6156	0.997
	20	$(0_{(19)},10)$	0.0609	2.5358	0.993	0.0609	0.0608	0.0575	0.0578	0.0574	1.6151	0.996
		$(10,0_{(19)})$	0.0593	2.4832	0.994	0.0594	0.0593	0.0561	0.0565	0.0560	1.6391	0.996
(20,10)	10	$(0_{(9)},20)$	0.0627	2.5943	0.992	0.0625	0.0624	0.0577	0.0579	0.0576	1.6561	0.995
		$(20,0_{(9)})$	0.0619	2.6514	0.992	0.0618	0.0614	0.0563	0.0567	0.0562	1.7024	0.995
	15	$(0_{(14)},15)$	0.0622	2.5427	0.992	0.0622	0.0620	0.0576	0.0576	0.0574	1.6386	0.996
		$(15,0_{(14)})$	0.0614	2.6730	0.992	0.0612	0.0611	0.0562	0.0569	0.0561	1.7342	0.996
	20	$(0_{(19)},10)$	0.0607	2.6012	0.993	0.0505	0.0505	0.0556	0.0559	0.0556	1.8359	0.997
		$(10,0_{(19)})$	0.0595	2.5257	0.994	0.0595	0.0593	0.0545	0.0551	0.0543	1.6980	0.997
(20,30)	20	$(0_{(19)},30)$	0.0512	2.4708	0.993	0.0512	0.0511	0.0474	0.0479	0.0473	1.6372	0.996
		$(30,0_{(19)})$	0.0507	2.4860	0.993	0.0506	0.0504	0.0461	0.0469	0.0460	1.6538	0.996
	30	$(0_{(29)},20)$	0.0509	2.4368	0.993	0.0508	0.0506	0.0455	0.0460	0.0454	1.6754	0.995
		$(20,0_{(29)})$	0.0493	2.5064	0.994	0.0495	0.0490	0.0447	0.0450	0.0445	1.7625	0.996
	40	$(0_{(39)},10)$	0.0492	2.5037	0.993	0.0490	0.0489	0.0443	0.0452	0.0440	1.8026	0.997
		$(10,0_{(39)})$	0.0483	2.4576	0.994	0.0482	0.0482	0.0432	0.0439	0.0432	1.7036	0.997
(30,20)	20	$(0_{(19)},30)$	0.0497	2.4363	0.993	0.0495	0.0494	0.0466	0.0468	0.0462	1.6435	0.996
		$(30,0_{(19)})$	0.0491	2.5658	0.994	0.0490	0.0488	0.0465	0.0472	0.0463	1.6752	0.996
	30	$(0_{(29)},20)$	0.0494	2.5263	0.994	0.0491	0.0490	0.0469	0.0473	0.0468	1.7636	0.997
		$(20,0_{(29)})$	0.0485	2.5483	0.994	0.0484	0.0481	0.0441	0.0445	0.0441	1.7036	0.996
	40	$(0_{(39)},10)$	0.0490	2.4990	0.993	0.0488	0.0485	0.0438	0.0444	0.0435	1.6435	0.996
		$(10,0_{(39)})$	0.0478	2.5026	0.994	0.0480	0.0478	0.0426	0.0428	0.0426	1.7630	0.997
(40,50)	40	$(0_{(39)},50)$	0.0372	2.4374	0.992	0.0389	0.0388	0.0319	0.0325	0.0318	1.6938	0.997
		$(50,0_{(39)})$	0.0358	2.4867	0.993	0.0359	0.0356	0.0318	0.0324	0.0315	1.6535	0.996
	50	$(0_{(49)},40)$	0.0353	2.5363	0.993	0.0352	0.0352	0.0307	0.0314	0.0306	1.6287	0.997
		$(40,0_{(49)})$	0.0339	2.4367	0.993	0.0340	0.0335	0.0293	0.0300	0.0292	1.7325	0.998
	60	$(0_{(59)},30)$	0.0335	2.4187	0.994	0.0334	0.0332	0.0285	0.0286	0.0285	1.7156	0.996
		$(30,0_{(59)})$	0.0328	2.3261	0.994	0.0328	0.0325	0.0274	0.0279	0.0273	1.6264	0.998
(50,40)	40	$(0_{(39)},50)$	0.0363	2.5163	0.993	0.0362	0.0360	0.0294	0.0298	0.0293	1.7386	0.996
		$(50,0_{(39)})$	0.0345	2.4352	0.993	0.0344	0.0341	0.0289	0.0291	0.0289	1.6195	0.996
	50	$(0_{(49)},40)$	0.0346	2.5615	0.992	0.0345	0.0342	0.0288	0.0296	0.0285	1.6378	0.996
		$(40,0_{(49)})$	0.0329	2.4865	0.994	0.0328	0.0325	0.0281	0.0289	0.0281	1.6760	0.997
	60	$(0_{(59)},30)$	0.0317	2.4619	0.993	0.0315	0.0315	0.0268	0.0269	0.0265	1.6959	0.997
		$(30,0_{(59)})$	0.0304	2.3546	0.994	0.0305	0.0302	0.0247	0.0253	0.0247	1.7409	0.997
(50,60)	60	$(0_{(59)},50)$	0.0230	2.5887	0.993	0.0229	0.0225	0.0183	0.0189	0.0183	1.6878	0.996
		$(50,0_{(59)})$	0.0223	2.5092	0.994	0.0221	0.0220	0.0182	0.0185	0.0181	1.6357	0.996
	70	$(0_{(69)},40)$	0.0208	2.5327	0.994	0.0207	0.0205	0.0160	0.0167	0.0159	1.6038	0.997
		$(40,0_{(69)})$	0.0192	2.4596	0.993	0.0191	0.0189	0.0154	0.0163	0.0155	1.6453	0.996
	80	$(0_{(79)},30)$	0.0191	2.5325	0.993	0.0190	0.0190	0.0149	0.0158	0.0146	1.6458	0.997
		$(30,0_{(79)})$	0.0187	2.4535	0.995	0.0185	0.0184	0.0130	0.0136	0.0131	1.6354	0.997
(60,50)	60	$(0_{(59)},50)$	0.0222	2.5464	0.994	0.0220	0.0215	0.0172	0.0175	0.0170	1.6367	0.996
		$(50,0_{(59)})$	0.0218	2.5792	0.994	0.0215	0.0214	0.0159	0.0164	0.0158	1.6359	0.997
	70	$(0_{(69)},40)$	0.0210	2.3925	0.993	0.0207	0.0206	0.0150	0.0158	0.0149	1.6261	0.997
		$(40,0_{(69)})$	0.0203	2.5780	0.994	0.0203	0.0201	0.0143	0.0146	0.0143	1.6146	0.997
	80	$(0_{(79)},30)$	0.0194	2.4424	0.995	0.0191	0.0190	0.0128	0.0129	0.0125	1.6953	0.998
		$(30,0_{(79)})$	0.0177	2.5093	0.995	0.0175	0.0274	0.0107	0.0108	0.0107	1.6534	0.998

Table 14: MSE, length and coverage probability (CP) of estimates for the parameter α_1.

(m,n)	r	$(R_1,\cdots,R_r)$	MLE			Bootstrap		MCMC				
			MSE	Length	CP	Boot-p	Boot-t	SE	LINEX		Length	CP
									q=-0.5	q=0.5		
(10,10)	5	$(0_{(4)},15)$	0.0079	2.0584	0.992	0.0078	0.0078	0.0071	0.0078	0.0069	1.9809	0.995
		$(0_{(3)},2,13)$	0.0076	2.4038	0.992	0.0075	0.0073	0.0065	0.0074	0.0063	1.9651	0.996
	10	$(0_{(9)},10)$	0.0077	2.4307	0.993	0.0075	0.0074	0.0066	0.0071	0.0063	1.9367	0.996
		$(10,0_{(9)})$	0.0071	2.3603	0.992	0.0070	0.0068	0.0066	0.0069	0.0066	1.7387	0.997
	15	$(0_{(14)},5)$	0.0073	2.4185	0.994	0.0072	0.0070	0.0069	0.0072	0.0068	1.8256	0.997
		$(5,0_{(14)})$	0.0073	2.6534	0.992	0.0071	0.0070	0.0062	0.0063	0.0059	1.8279	0.996
(10,20)	10	$(0_{(9)},20)$	0.0072	2.8700	0.995	0.0072	0.0071	0.0067	0.0074	0.0065	1.9276	0.996
		$(20,0_{(9)})$	0.0066	2.0929	0.994	0.0065	0.0063	0.0059	0.0060	0.0050	2.0039	0.996
	15	$(0_{(14)},15)$	0.0065	2.1004	0.994	0.0065	0.0059	0.0057	0.0063	0.0056	1.7629	0.997
		$(15,0_{(14)})$	0.0059	1.9287	0.995	0.0058	0.0058	0.0052	0.0054	0.0052	1.8392	0.997
	20	$(0_{(19)},10)$	0.0061	2.3513	0.994	0.0059	0.0058	0.0051	0.0059	0.0050	1.9195	0.997
		$(10,0_{(19)})$	0.0057	2.4368	0.995	0.0056	0.0054	0.0058	0.0061	0.0056	1.8252	0.996
(20,10)	10	$(0_{(9)},20)$	0.0059	2.2030	0.995	0.0058	0.0055	0.0049	0.0058	0.0047	1.9456	0.996
		$(20,0_{(9)})$	0.0054	2.3133	0.991	0.0054	0.0052	0.0045	0.0046	0.0045	1.6123	0.995
	15	$(0_{(14)},15)$	0.0056	2.3349	0.994	0.0055	0.0054	0.0055	0.0058	0.0054	1.9356	0.996
		$(15,0_{(14)})$	0.0055	2.5600	0.993	0.0054	0.0053	0.0051	0.0059	0.0050	1.8311	0.997
	20	$(0_{(19)},10)$	0.0051	2.3595	0.995	0.0050	0.0049	0.0046	0.0053	0.0045	1.8122	0.996
		$(10,0_{(19)})$	0.0048	2.5584	0.994	0.0047	0.0045	0.0042	0.0047	0.0040	1.8293	0.997
(20,30)	20	$(0_{(19)},30)$	0.0057	2.4375	0.994	0.0055	0.0055	0.0052	0.0057	0.0051	1.7609	0.996
		$(30,0_{(19)})$	0.0056	2.5352	0.993	0.0055	0.0053	0.0052	0.0055	0.0052	1.8264	0.995
	30	$(0_{(29)},20)$	0.0049	2.4178	0.995	0.0048	0.0045	0.0046	0.0049	0.0045	1.8207	0.997
		$(20,0_{(29)})$	0.0053	2.3979	0.993	0.0052	0.0050	0.0047	0.0051	0.0045	1.9266	0.996
	40	$(0_{(39)},10)$	0.0050	1.8145	0.993	0.0050	0.0049	0.0041	0.0048	0.0040	1.7255	0.996
		$(10,0_{(39)})$	0.0046	2.5288	0.995	0.0045	0.0043	0.0041	0.0043	0.0041	1.7174	0.996
(30,20)	20	$(0_{(19)},30)$	0.0049	1.9026	0.993	0.0048	0.0046	0.0041	0.0043	0.0040	1.8487	0.996
		$(30,0_{(19)})$	0.0051	1.9525	0.994	0.0050	0.0047	0.0045	0.0049	0.0042	1.8298	0.997
	30	$(0_{(29)},20)$	0.0044	1.9861	0.992	0.0045	0.0042	0.0036	0.0041	0.0035	1.9188	0.998
		$(20,0_{(29)})$	0.0040	1.9364	0.995	0.0038	0.0038	0.0035	0.0038	0.0034	1.8263	0.997
	40	$(0_{(39)},10)$	0.0043	1.8619	0.994	0.0042	0.0040	0.0041	0.0048	0.0040	1.6139	0.998
		$(10,0_{(39)})$	0.0038	1.9538	0.993	0.0037	0.0035	0.0035	0.0039	0.0035	1.7194	0.996
(40,50)	40	$(0_{(39)},50)$	0.0048	2.0783	0.993	0.0048	0.0045	0.0045	0.0048	0.0045	1.5247	0.996
		$(50,0_{(39)})$	0.0041	2.2233	0.994	0.0040	0.0040	0.0040	0.0044	0.0039	1.1101	0.997
	50	$(0_{(49)},40)$	0.0047	2.1796	0.993	0.0045	0.0044	0.0045	0.0051	0.0043	1.3024	0.997
		$(40,0_{(49)})$	0.0045	1.9877	0.993	0.0044	0.0042	0.0042	0.0046	0.0041	1.2115	0.996
	60	$(0_{(59)},30)$	0.0040	2.0926	0.995	0.0039	0.0039	0.0048	0.0049	0.0048	1.4021	0.998
		$(30,0_{(59)})$	0.0037	2.2514	0.995	0.0038	0.0035	0.0034	0.0037	0.0034	1.2127	0.996
(50,40)	40	$(0_{(39)},50)$	0.0042	2.0460	0.995	0.0041	0.0038	0.0039	0.0040	0.0038	1.5274	0.997
		$(50,0_{(39)})$	0.0042	2.1915	0.994	0.0041	0.0039	0.0036	0.0038	0.0035	1.4182	0.997
	50	$(0_{(49)},40)$	0.0039	1.8014	0.994	0.0038	0.0038	0.0041	0.0038	0.0058	1.3187	0.996
		$(40,0_{(49)})$	0.0036	1.8341	0.994	0.0035	0.0032	0.0032	0.0035	0.0032	1.2121	0.986
	60	$(0_{(59)},30)$	0.0035	1.7884	0.995	0.0034	0.0032	0.0030	0.0036	0.0029	1.4121	0.997
		$(30,0_{(59)})$	0.0029	2.0427	0.993	0.0028	0.0026	0.0027	0.0028	0.0025	1.6133	0.996
(50,60)	60	$(0_{(59)},50)$	0.0035	1.9805	0.993	0.0034	0.0032	0.0033	0.0035	0.0032	1.4135	0.995
		$(50,0_{(59)})$	0.0031	1.7183	0.993	0.0030	0.0028	0.0029	0.0030	0.0029	1.3131	0.997
	70	$(0_{(69)},40)$	0.0034	1.9345	0.993	0.0035	0.0032	0.0031	0.0034	0.0030	1.2102	0.995
		$(40,0_{(69)})$	0.0026	1.8672	0.994	0.0025	0.0024	0.0025	0.0029	0.0022	1.2075	0.997
	80	$(0_{(79)},30)$	0.0025	1.7512	0.995	0.0025	0.0023	0.0021	0.0023	0.0021	1.2102	0.996
		$(30,0_{(79)})$	0.0019	1.7236	0.993	0.0020	0.0018	0.0016	0.0016	0.0016	1.2011	0.998
(60,50)	60	$(0_{(59)},50)$	0.0033	1.8280	0.994	0.0032	0.0030	0.0027	0.0030	0.0026	1.2098	0.997
		$(50,0_{(59)})$	0.0032	1.9445	0.994	0.0030	0.0031	0.0031	0.0031	0.0030	1.3008	0.996
	70	$(0_{(69)},40)$	0.0029	1.6041	0.995	0.0028	0.0025	0.0025	0.0028	0.0024	1.2114	0.996
		$(40,0_{(69)})$	0.0024	1.6285	0.994	0.0025	0.0022	0.0022	0.0022	0.0022	1.1084	0.997
	80	$(0_{(79)},30)$	0.0023	1.7084	0.993	0.0022	0.0019	0.0023	0.0025	0.0022	1.2097	0.996
		$(30,0_{(79)})$	0.0016	1.6577	0.995	0.0015	0.0015	0.0015	0.0016	0.0015	1.1131	0.998

Table 15: MSE, length and coverage probability (CP) of estimates for the parameter α_2.

(m,n)	r	$(R_1,\cdots,R_r)$	MLE			Bootstrap		MCMC				
			MSE	Length	CP	Boot-p	Boot-t	SE	LINEX		Length	CP
									q=-0.5	q=0.5		
(10,10)	5	$(0_{(4)},15)$	0.0098	1.8246	0.992	0.0095	0.094	0.0092	0.0098	0.0091	1.5175	0.997
		$(0_{(3)},2,13)$	0.0087	1.7412	0.992	0.0086	0.0084	0.0073	0.0076	0.0073	1.3178	0.996
	10	$(0_{(9)},10)$	0.0093	1.5955	0.993	0.0092	0.0090	0.0089	0.0092	0.0085	1.5215	0.996
		$(10,0_{(9)})$	0.0088	1.5294	0.992	0.0087	0.0076	0.0077	0.0084	0.0076	1.2185	0.995
	15	$(0_{(14)},5)$	0.0084	1.3462	0.993	0.0083	0.0082	0.0081	0.0084	0.0080	1.2129	0.996
		$(5,0_{(14)})$	0.0079	1.5646	0.993	0.0078	0.0076	0.0076	0.0078	0.0075	0.9936	0.995
(10,20)	10	$(0_{(9)},20)$	0.0085	1.3850	0.994	0.0084	0.0082	0.0082	0.0085	0.0081	1.0161	0.995
		$(20,0_{(9)})$	0.0081	1.6646	0.994	0.0080	0.0079	0.0076	0.0076	0.0076	1.0187	0.996
	15	$(0_{(14)},15)$	0.0084	1.7321	0.993	0.0083	0.0082	0.0077	0.0081	0.0077	1.0218	0.995
		$(15,0_{(14)})$	0.0079	1.6785	0.992	0.0078	0.0076	0.0073	0.0076	0.0072	1.0152	0.997
	20	$(0_{(19)},10)$	0.0078	1.6200	0.992	0.0076	0.0075	0.0073	0.0075	0.0073	1.0179	0.996
		$(10,0_{(19)})$	0.0072	1.5172	0.993	0.0072	0.0071	0.0069	0.0069	0.0069	1.0117	0.996
(20,10)	10	$(0_{(9)},20)$	0.0078	1.6497	0.992	0.0075	0.0074	0.0074	0.0074	0.0073	1.0089	0.995
		$(20,0_{(9)})$	0.0074	1.6740	0.992	0.0073	0.0070	0.0069	0.0073	0.0068	1.0141	0.995
	15	$(0_{(14)},15)$	0.0075	1.5101	0.993	0.0073	0.0072	0.0072	0.0072	0.0074	1.0123	0.996
		$(15,0_{(14)})$	0.0069	1.3606	0.993	0.0068	0.0067	0.0066	0.0067	0.0065	1.0142	0.996
	20	$(0_{(19)},10)$	0.0071	1.4185	0.994	0.0070	0.0068	0.0067	0.0069	0.0066	1.0207	0.997
		$(10,0_{(19)})$	0.0065	1.3406	0.994	0.0064	0.0062	0.0062	0.0063	0.0060	1.0118	0.996
(20,30)	20	$(0_{(19)},30)$	0.0068	1.3320	0.993	0.0067	0.0065	0.0068	0.0069	0.0065	1.0106	0.995
		$(30,0_{(19)})$	0.0064	1.4165	0.992	0.0063	0.0063	0.0062	0.0065	0.0061	1.0175	0.995
	30	$(0_{(29)},20)$	0.0063	1.4576	0.993	0.0062	0.0060	0.0058	0.0058	0.0059	1.0145	0.996
		$(20,0_{(29)})$	0.0059	1.8101	0.993	0.0058	0.0056	0.0055	0.0057	0.0054	1.0099	0.996
	40	$(0_{(39)},10)$	0.0056	1.4695	0.994	0.0055	0.0052	0.0052	0.0055	0.0052	1.0090	0.995
		$(10,0_{(39)})$	0.0057	1.4322	0.994	0.0055	0.0054	0.0050	0.0053	0.0048	1.0093	0.997
(30,20)	20	$(0_{(19)},30)$	0.0064	1.7222	0.993	0.0063	0.0062	0.0054	0.0058	0.0054	1.0105	0.996
		$(30,0_{(19)})$	0.0058	1.5180	0.995	0.0057	0.0055	0.0057	0.0057	0.0054	1.0143	0.996
	30	$(0_{(29)},20)$	0.0059	1.4379	0.993	0.0058	0.0056	0.0056	0.0057	0.0056	1.0155	0.997
		$(20,0_{(29)})$	0.0051	1.3076	0.993	0.0050	0.0049	0.0053	0.0054	0.0053	1.0083	0.996
	40	$(0_{(39)},10)$	0.0049	1.6764	0.994	0.0046	0.0045	0.0045	0.0048	0.0044	1.0114	0.995
		$(10,0_{(39)})$	0.0047	1.5027	0.994	0.0046	0.0045	0.0043	0.0043	0.0042	1.0119	0.996
(40,50)	40	$(0_{(39)},50)$	0.0048	1.3217	0.994	0.0046	0.0045	0.0044	0.0045	0.0044	1.0136	0.996
		$(50,0_{(39)})$	0.0042	1.5356	0.994	0.0041	0.0040	0.0037	0.0041	0.0036	1.0047	0.997
	50	$(0_{(49)},40)$	0.0033	1.5097	0.992	0.0032	0.0030	0.0030	0.0033	0.0029	1.0146	0.995
		$(40,0_{(49)})$	0.0032	1.7027	0.992	0.0032	0.0031	0.0026	0.0028	0.0025	1.0095	0.996
	60	$(0_{(59)},30)$	0.0025	1.4867	0.993	0.0024	0.0023	0.0019	0.0021	0.0019	1.0086	0.995
		$(30,0_{(59)})$	0.0023	1.4988	0.993	0.0022	0.0020	0.0021	0.0025	0.0020	1.0108	0.995
(50,40)	40	$(0_{(39)},50)$	0.0044	1.4885	0.993	0.0042	0.0041	0.0040	0.0045	0.0039	1.0122	0.996
		$(50,0_{(39)})$	0.0041	1.5013	0.994	0.0040	0.0040	0.0039	0.0040	0.0038	1.0094	0.997
	50	$(0_{(49)},40)$	0.0035	1.3479	0.995	0.0034	0.0032	0.0030	0.0031	0.0030	1.0125	0.997
		$(40,0_{(49)})$	0.0032	1.6308	0.995	0.0031	0.0030	0.0029	0.0030	0.0029	1.0102	0.997
	60	$(0_{(59)},30)$	0.0024	1.4256	0.994	0.0024	0.0023	0.0023	0.0026	0.0022	1.0094	0.996
		$(30,0_{(59)})$	0.0026	1.7233	0.994	0.0025	0.0022	0.0023	0.0027	0.0022	1.0077	0.998
(50,60)	60	$(0_{(59)},50)$	0.0037	1.5195	0.994	0.0035	0.0035	0.0028	0.0029	0.0028	1.0128	0.997
		$(50,0_{(59)})$	0.0033	1.3246	0.995	0.0032	0.0030	0.0029	0.0031	0.0029	1.0102	0.996
	70	$(0_{(69)},40)$	0.0028	1.4853	0.995	0.0027	0.0025	0.0025	0.0026	0.0024	1.0167	0.998
		$(40,0_{(69)})$	0.0030	1.3823	0.995	0.0029	0.0027	0.0026	0.0029	0.0026	1.0082	0.997
	80	$(0_{(79)},30)$	0.0025	1.2549	0.994	0.0024	0.0023	0.0023	0.0025	0.0023	1.0094	0.996
		$(30,0_{(79)})$	0.0023	1.6562	0.994	0.0024	0.0022	0.0017	0.0019	0.0017	1.0074	0.997
(60,50)	60	$(0_{(59)},50)$	0.0035	1.2995	0.995	0.0034	0.0032	0.0027	0.0031	0.0026	1.0134	0.996
		$(50,0_{(59)})$	0.0033	1.6828	0.995	0.0032	0.0033	0.0029	0.0029	0.0029	1.0075	0.997
	70	$(0_{(69)},40)$	0.0028	1.6965	0.993	0.0026	0.0025	0.0022	0.0023	0.0022	1.0119	0.997
		$(40,0_{(69)})$	0.0024	1.6392	0.995	0.0023	0.0022	0.0018	0.0019	0.0017	1.0079	0.996
	80	$(0_{(79)},30)$	0.0023	1.6045	0.994	0.0023	0.0021	0.0018	0.0021	0.0018	1.0136	0.998
		$(30,0_{(79)})$	0.0019	1.3772	0.995	0.0019	0.0018	0.0017	0.0018	0.0017	1.0067	0.997

Table 16: MSE, length and coverage probability (CP) of estimates for the parameter β_1.

(m,n)	r	$(R_1,\cdots,R_r)$	MLE			Bootstrap		MCMC				
			MSE	Length	CP	Boot-p	Boot-t	SE	LINEX		Length	CP
									q=-0.5	q=0.5		
(10,10)	5	$(0_{(4)},15)$	0.0725	2.1047	0.991	0.0723	0.0718	0.0678	0.0682	0.0675	1.6113	0.995
		$(0_{(3)},2,13)$	0.0692	2.0186	0.991	0.0692	0.0690	0.0645	0.0654	0.0642	1.6377	0.995
	10	$(0_{(9)},10)$	0.0682	1.9890	0.992	0.0680	0.0675	0.0648	0.0661	0.0646	1.5395	0.985
		$(10,0_{(9)})$	0.0675	1.9820	0.993	0.0673	0.0669	0.0616	0.0619	0.0613	1.3256	0.996
	15	$(0_{(14)},5)$	0.0675	2.1514	0.993	0.0670	0.0668	0.0609	0.0614	0.0605	1.4209	0.997
		$(5,0_{(14)})$	0.0664	2.2853	0.991	0.0665	0.0662	0.0601	0.0615	0.0600	1.5142	0.996
(10,20)	10	$(0_{(9)},20)$	0.0678	2.3260	0.991	0.0672	0.0669	0.0646	0.0646	0.0638	1.6042	0.996
		$(20,0_{(9)})$	0.0652	2.0929	0.992	0.0650	0.0649	0.0619	0.0625	0.0619	1.6039	0.997
	15	$(0_{(14)},15)$	0.0624	2.3105	0.991	0.0623	0.0621	0.0593	0.0599	0.0592	1.5281	0.995
		$(15,0_{(14)})$	0.0619	1.9360	0.991	0.0619	0.0618	0.0576	0.0584	0.0567	1.4842	0.996
	20	$(0_{(19)},10)$	0.0625	2.2066	0.994	0.0624	0.0620	0.0615	0.0616	0.0612	1.4918	0.997
		$(10,0_{(19)})$	0.0619	2.3536	0.993	0.0618	0.0615	0.0592	0.0595	0.0591	1.7232	0.996
(20,10)	10	$(0_{(9)},20)$	0.0663	2.1200	0.992	0.0662	0.0660	0.0648	0.0648	0.0645	1.3736	0.997
		$(20,0_{(9)})$	0.0636	2.3660	0.992	0.0635	0.0629	0.0616	0.0617	0.0616	1.7326	0.996
	15	$(0_{(14)},15)$	0.0633	2.1738	0.992	0.0632	0.0629	0.0594	0.0598	0.0593	1.5126	0.995
		$(15,0_{(14)})$	0.0629	2.2530	0.993	0.0628	0.0625	0.0588	0.0591	0.0586	1.5974	0.995
	20	$(0_{(19)},10)$	0.0633	2.0100	0.993	0.0631	0.0630	0.0619	0.0621	0.0618	1.6078	0.995
		$(10,0_{(19)})$	0.0615	2.1870	0.992	0.0612	0.0611	0.0587	0.0591	0.0585	1.4837	0.995
(20,30)	20	$(0_{(19)},30)$	0.0569	2.0640	0.993	0.0568	0.0565	0.0520	0.0526	0.0219	1.6043	0.996
		$(30,0_{(19)})$	0.0563	2.1995	0.993	0.0562	0.0560	0.0518	0.0518	0.0517	1.4148	0.997
	30	$(0_{(29)},20)$	0.0551	2.0794	0.992	0.0550	0.0548	0.0530	0.0548	0.0519	1.5198	0.995
		$(20,0_{(29)})$	0.0536	1.8043	0.991	0.0535	0.0532	0.0497	0.0510	0.0495	1.3352	0.996
	40	$(0_{(39)},10)$	0.0548	1.7990	0.993	0.0546	0.0545	0.0527	0.0538	0.0526	1.5229	0.996
		$(10,0_{(39)})$	0.0525	1.9227	0.994	0.0524	0.0522	0.0521	0.0527	0.0515	1.5154	0.997
(30,20)	20	$(0_{(19)},30)$	0.0563	2.0910	0.992	0.0561	0.0560	0.0534	0.0542	0.0532	1.4219	0.996
		$(30,0_{(19)})$	0.0559	2.1450	0.991	0.0558	0.0555	0.0515	0.0518	0.0512	1.4268	0.996
	30	$(0_{(29)},20)$	0.0555	2.1500	0.991	0.0552	0.0549	0.0509	0.0515	0.0508	1.3255	0.995
		$(20,0_{(29)})$	0.0544	2.1604	0.993	0.0545	0.0542	0.0505	0.0508	0.0504	1.4278	0.997
	40	$(0_{(39)},10)$	0.0525	1.8878	0.993	0.0523	0.0523	0.0493	0.0499	0.0491	1.4162	0.996
		$(10,0_{(39)})$	0.0518	1.9310	0.992	0.0517	0.0517	0.0489	0.0498	0.0488	1.4120	0.996
(40,50)	40	$(0_{(39)},50)$	0.0458	2.0296	0.992	0.0457	0.0455	0.0441	0.0445	0.0438	1.4216	0.998
		$(50,0_{(39)})$	0.0454	2.1072	0.993	0.0452	0.0451	0.0429	0.0437	0.0421	1.4215	0.995
	50	$(0_{(49)},40)$	0.0432	2.0693	0.994	0.0431	0.0428	0.0402	0.0407	0.0402	1.4558	0.996
		$(40,0_{(49)})$	0.0424	1.9731	0.991	0.0425	0.0421	0.0388	0.0389	0.0387	1.3134	0.996
	60	$(0_{(59)},30)$	0.0430	1.8310	0.993	0.0429	0.0426	0.0392	0.0395	0.0391	1.4489	0.995
		$(30,0_{(59)})$	0.0422	1.9471	0.994	0.0420	0.0420	0.0389	0.0392	0.0387	1.3818	0.997
(50,40)	40	$(0_{(39)},50)$	0.0447	2.0960	0.993	0.0445	0.0444	0.0412	0.0418	0.0411	1.5314	0.997
		$(50,0_{(39)})$	0.0438	2.1908	0.993	0.0437	0.0435	0.0405	0.0407	0.0404	1.4764	0.996
	50	$(0_{(49)},40)$	0.0447	2.1445	0.992	0.0446	0.0442	0.0408	0.0413	0.0407	1.3244	0.997
		$(40,0_{(49)})$	0.0426	1.9681	0.992	0.0425	0.0424	0.0393	0.0398	0.0390	1.4319	0.998
	60	$(0_{(59)},30)$	0.0419	2.0837	0.994	0.0418	0.0416	0.0386	0.0390	0.0395	1.4121	0.998
		$(30,0_{(59)})$	0.0403	1.9953	0.993	0.0402	0.0402	0.0369	0.0375	0.0368	1.3908	0.997
(50,60)	60	$(0_{(59)},50)$	0.0306	1.9731	0.994	0.0305	0.0301	0.0287	0.0294	0.0285	1.4089	0.994
		$(50,0_{(59)})$	0.0283	1.9009	0.994	0.0282	0.0280	0.0257	0.0262	0.0256	1.3944	0.996
	70	$(0_{(69)},40)$	0.0256	1.9629	0.995	0.0255	0.0252	0.0220	0.0227	0.0220	1.4067	0.997
		$(40,0_{(69)})$	0.0229	1.9507	0.993	0.0228	0.0226	0.0201	0.0201	0.0198	1.3947	0.997
	80	$(0_{(79)},30)$	0.0225	1.9336	0.995	0.0224	0.0224	0.0192	0.0193	0.0192	1.3826	0.996
		$(30,0_{(79)})$	0.0198	1.8515	0.995	0.0200	0.0196	0.0140	0.0141	0.0139	1.4097	0.996
(60,50)	60	$(0_{(59)},50)$	0.0277	1.9709	0.993	0.0275	0.0272	0.0232	0.0232	0.0231	1.3874	0.996
		$(50,0_{(59)})$	0.0263	1.8731	0.994	0.0263	0.0260	0.0224	0.0229	0.0222	1.4404	0.997
	70	$(0_{(69)},40)$	0.0252	1.9489	0.994	0.0251	0.0250	0.0232	0.0234	0.0231	1.5108	0.996
		$(40,0_{(69)})$	0.0238	1.9250	0.995	0.0235	0.0235	0.0196	0.0196	0.0195	1.5048	0.997
	80	$(0_{(79)},30)$	0.0208	1.8892	0.994	0.0207	0.0205	0.0193	0.0198	0.0192	1.4061	0.997
		$(30,0_{(79)})$	0.0193	1.8274	0.995	0.0192	0.0190	0.0148	0.0154	0.0146	1.3267	0.998

Table 17: MSE, length and coverage probability (CP) of estimates for the parameter β_2.

(m,n)	r	$(R_1,\cdots,R_r)$	MLE			Bootstrap		MCMC				
			MSE	Length	CP	Boot-p	Boot-t	SE	LINEX		Length	CP
									q=-0.5	q=0.5		
(10,10)	5	$(0_{(4)},15)$	0.0811	1.9423	0.992	0.0810	0.0808	0.0783	0.0785	0.0782	1.8069	0.995
		$(0_{(3)},2,13)$	0.0731	2.3304	0.994	0.0730	0.0726	0.0705	0.0714	0.0702	1.8060	0.996
	10	$(0_{(9)},10)$	0.0692	2.2759	0.994	0.0691	0.0691	0.0638	0.0641	0.0638	1.8068	0.996
		$(10,0_{(9)})$	0.0642	2.1810	0.995	0.0642	0.0642	0.0595	0.0598	0.0594	1.6157	0.997
	15	$(0_{(14)},5)$	0.0628	2.1501	0.993	0.0626	0.0625	0.0581	0.0589	0.0580	1.5093	0.996
		$(5,0_{(14)})$	0.0620	2.2319	0.994	0.0620	0.0617	0.0578	0.0586	0.0575	1.4085	0.995
(10,20)	10	$(0_{(9)},20)$	0.0684	2.1979	0.994	0.0683	0.0682	0.0626	0.0632	0.0625	1.5055	0.985
		$(20,0_{(9)})$	0.0677	2.0996	0.993	0.0675	0.0674	0.0635	0.0641	0.0632	1.3095	0.995
	15	$(0_{(14)},15)$	0.0671	2.0684	0.992	0.0669	0.0666	0.0619	0.0626	0.0618	1.3082	0.996
		$(15,0_{(14)})$	0.0655	1.8480	0.992	0.0652	0.0651	0.0614	0.0616	0.0614	1.3166	0.997
	20	$(0_{(19)},10)$	0.0641	1.9473	0.991	0.0640	0.0635	0.0592	0.0595	0.0592	1.4084	0.996
		$(10,0_{(19)})$	0.0622	1.9687	0.991	0.0621	0.0618	0.0591	0.0597	0.0590	1.3462	0.996
(20,10)	10	$(0_{(9)},20)$	0.0683	1.6987	0.994	0.0382	0.0381	0.0623	0.0629	0.0622	1.5032	0.997
		$(20,0_{(9)})$	0.0675	2.1975	0.993	0.0672	0.0670	0.0624	0.0628	0.0620	1.3172	0.995
	15	$(0_{(14)},15)$	0.0668	1.7181	0.993	0.0665	0.0664	0.0609	0.0615	0.0609	1.4046	0.996
		$(15,0_{(14)})$	0.0649	2.2300	0.992	0.0648	0.0646	0.0580	0.0591	0.0582	1.3104	0.995
	20	$(0_{(19)},10)$	0.0636	2.2842	0.993	0.0635	0.0632	0.0571	0.0578	0.0570	1.3065	0.995
		$(10,0_{(19)})$	0.0620	1.8411	0.993	0.0620	0.0615	0.0596	0.0597	0.0595	2.0065	0.996
(20,30)	20	$(0_{(19)},30)$	0.0646	1.9059	0.992	0.0645	0.0642	0.0616	0.0619	0.0615	2.0048	0.995
		$(30,0_{(19)})$	0.0632	1.7077	0.994	0.0630	0.0628	0.0595	0.0599	0.0594	1.5070	0.997
	30	$(0_{(29)},20)$	0.0625	2.1192	0.994	0.0624	0.0623	0.0597	0.0599	0.0595	1.4582	0.998
		$(20,0_{(29)})$	0.0617	2.3582	0.993	0.0615	0.0614	0.0692	0.0691	0.0692	1.6955	0.997
	40	$(0_{(39)},10)$	0.0622	2.0658	0.994	0.0622	0.0621	0.0587	0.0589	0.0585	1.3545	0.996
		$(10,0_{(39)})$	0.0619	1.9356	0.993	0.0618	0.0615	0.0580	0.0588	0.0579	1.5049	0.996
(30,20)	20	$(0_{(19)},30)$	0.0645	2.1773	0.993	0.0642	0.0640	0.0607	0.0609	0.0607	1.3243	0.995
		$(30,0_{(19)})$	0.0641	1.8619	0.993	0.0640	0.0637	0.0596	0.0598	0.0595	1.5678	0.995
	30	$(0_{(29)},20)$	0.0634	2.1970	0.994	0.0632	0.0630	0.0577	0.0584	0.0576	1.4663	0.997
		$(20,0_{(29)})$	0.0625	1.7274	0.992	0.0624	0.0622	0.0682	0.0687	0.0680	1.4043	0.996
	40	$(0_{(39)},10)$	0.0632	1.9401	0.991	0.0630	0.0629	0.0585	0.0586	0.0582	1.3945	0.996
		$(10,0_{(39)})$	0.0620	1.9183	0.992	0.0620	0.0617	0.0581	0.0588	0.0580	1.3876	0.997
(40,50)	40	$(0_{(39)},50)$	0.0633	2.1184	0.993	0.0632	0.0630	0.0598	0.0605	0.0598	1.5005	0.995
		$(50,0_{(39)})$	0.0625	1.9511	0.994	0.0624	0.0622	0.0596	0.0599	0.0595	1.4065	0.996
	50	$(0_{(49)},40)$	0.0582	2.0405	0.994	0.0581	0.0580	0.0543	0.0547	0.0542	1.4002	0.997
		$(40,0_{(49)})$	0.0579	1.8977	0.994	0.0576	0.0574	0.0538	0.0545	0.0536	1.5042	0.997
	60	$(0_{(59)},30)$	0.0583	2.1740	0.993	0.0582	0.0581	0.0542	0.0549	0.0541	1.3936	0.998
		$(30,0_{(59)})$	0.0582	1.7952	0.994	0.0580	0.0579	0.0543	0.0552	0.0543	1.4847	0.997
(50,40)	40	$(0_{(39)},50)$	0.0616	1.8569	0.994	0.0615	0.0612	0.0587	0.0587	0.0587	1.5652	0.996
		$(50,0_{(39)})$	0.0616	1.7903	0.992	0.0516	0.0615	0.0575	0.0583	0.0572	1.2053	0.997
	50	$(0_{(49)},40)$	0.0609	1.9445	0.994	0.0606	0.0604	0.0569	0.0578	0.0566	1.4144	0.997
		$(40,0_{(49)})$	0.0594	1.8843	0.993	0.0592	0.0590	0.0533	0.0533	0.0531	1.5005	0.996
	60	$(0_{(59)},30)$	0.0585	1.9759	0.994	0.0583	0.0582	0.0519	0.0526	0.0514	1.4532	0.994
		$(30,0_{(59)})$	0.0576	1.9857	0.995	0.0572	0.0571	0.0506	0.0512	0.0505	1.4233	0.997
(50,60)	60	$(0_{(59)},50)$	0.0537	2.0559	0.994	0.0534	0.0532	0.0483	0.0485	0.0483	1.3842	0.997
		$(50,0_{(59)})$	0.0461	1.7983	0.994	0.0460	0.0457	0.0437	0.0438	0.0436	1.5001	0.996
	70	$(0_{(69)},40)$	0.0454	1.8621	0.995	0.0452	0.0453	0.0412	0.0415	0.0410	1.3746	0.996
		$(40,0_{(69)})$	0.0453	1.9929	0.995	0.0452	0.0450	0.0401	0.0405	0.0401	1.4033	0.997
	80	$(0_{(79)},30)$	0.0400	1.7481	0.995	0.0401	0.0398	0.0353	0.0356	0.0351	1.3835	0.998
		$(30,0_{(79)})$	0.0391	1.8406	0.993	0.0390	0.0390	0.0343	0.0351	0.0342	1.3833	0.998
(60,50)	60	$(0_{(59)},50)$	0.0467	1.8892	0.996	0.0464	0.0462	0.0419	0.0419	0.0418	1.4036	0.997
		$(50,0_{(59)})$	0.0458	1.8870	0.994	0.0455	0.0453	0.0398	0.0404	0.0397	1.4273	0.995
	70	$(0_{(69)},40)$	0.0434	1.9589	0.993	0.0432	0.0431	0.0382	0.0385	0.0381	1.6009	0.996
		$(40,0_{(69)})$	0.0418	1.8829	0.995	0.0415	0.0414	0.0379	0.0386	0.0375	1.4023	0.997
	80	$(0_{(79)},30)$	0.0407	1.8799	0.996	0.0405	0.0400	0.0358	0.0366	0.0357	1.3840	0.998
		$(30,0_{(79)})$	0.0386	1.7291	0.995	0.0385	0.0382	0.0333	0.0339	0.0332	1.2981	0.997

8 Conclusion

In this paper, the estimations of the unknown parameters for two Weibull- Fréchet populations based on the JP-II-CS are studied under MLEs, Bootstrap CIs and Bayesian estimates. The Bayes estimates have been computed with respect to two loss functions. Also, we considered real life data set for illustrative purpose. Finally, an elaborate simulation study was conducted for different sample sizes (m,n) and different values of r for the comparison of the proposed estimates. From the results, we observe the following:

1.
2. It is observed that from Tables 4–7 the Boot-BC and Boot-BCa are better than Boot-t and Boot-p, in the sense of having smallest lengths.
3. From Tables 8, 9 the values of A.E.M_r are smaller than the values of S.E.M_r in all schemes.
4. It is evident that from Tables 10–17 the MSEs of MCMC are smaller than the MSEs of MLE, also, CP of MCMC are larger than CP of MLE. Then, the performance of the Bayes estimation of the parameters $\lambda_i, \theta_i, \alpha_i$ and β_i, $i = 1,2$ is better than the MLE estimation.
5. It is clear that from Tables 10–17 when m,n and r increase the MSEs and the lengths decrease.
6. It is evident that from Tables 10–17 the MCMC CIs give more accurate results than the ACIs since the lengths of the MCMC CIs are less than the lengths of ACIs, for various sample sizes.
7. From Tables 10–17 when $m > n$ for equal values of r, the MSEs decrease.
8. The Bayes estimates under LINEX with $a = 0.5$ provides better estimates in the sense of having smaller MSEs from Tables 10–17.
9. It's clear that the bootstrap-t is better than MLE and bootstrap-p from Tables 10–17.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Ethar M. Shokr is assistant lecturer of Mathematical Statistics at Mathematics Department, Faculty of Science, Mansoura University, Egypt. She received a Master's From Faculty of Science, Mansoura University, Egypt in 2018. Her main research interests are Combinatorial Problems, Statistical inferences and Distribution theory.

Rashad M. EL-Sagheer is a lecturer of Mathematical Statistics at Mathematics Department Faculty of Science AL-Azhar University Cairo Egypt. Received his B. A. Degree in Mathematics from Faculty of Science AL-Azhar University Cairo Egypt in 2004. He received Master's and PhD degrees in Mathematical Statistics from Faculty of Science Sohag University Egypt in 2011 and 2014 respectively. His areas of research where he has several publications in the international journals and conferences include: Statistical inference, Theory of estimation, Bayesian inference, Order statistics, Records, Theory of reliability, censored data, Life testing and distribution theory, Statistical computing and reliability analysis. He published and Co-authored more than 55 papers in reputed international journals. He supervised for M. Sc. and PhD students.

Beih S. El-Desouky is presently employed as a professor of Mathematical statistics and combinatorics in Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt. He received his PhD in Mathematical statistics in 1984 from Assiut University, Egypt. His research interests include: Theory of reliability, ordered data, characterization, statistical inference, distribution theory, discriminant analysis and classes of life distributions. Moreover combinatorial problems and their applications in probability. He published and Co-authored more than 100 papers in reputed international journals. He supervised more than 60 M. Sc. thesis and more than 30 Ph. D. thesis.