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Quadruple Coincidence Point Methodologies for Finding a Solution to a System of Integral Equations with a Directed Graph

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Abstract: The purpose of this manuscript is to present some quadruple coincidence point results for $\varphi-$Geraghty contraction mappings in metric spaces with a directed graph. In order to highlight the importance of the theoretical results, the existence and uniqueness of the solution to a system of integral equations are obtained.

Keywords: Quadruple coincidence point, directed graph, edge preserving, integral equations.

1 Introduction

After the emergence of Banach's theorem [1], the technique and the applications of fixed point became very important in diverse fields of mathematics, statistics, chemistry, computer science, biology, engineering, economics, game theory, theory of differential equations, theory of integral equations, theory of matrix equations, mathematical economics, etc. (see, e.g., [2,3,4,5]).

In 1987, the notion of a coupled fixed point is presented by Guo and Lakshmikantham [6]. Bhaskar and Lakshmikantham [7] established the concept of the mixed monotone property for given mappings.

Lakshimikantham and Ćirić [8] developed the results of [7] by defining the mixed $\pi-$monotone and using it to study the existence and uniqueness of solutions for boundary value problems in partially ordered metric spaces (POMSs, for short). Consequently, several coupled fixed point and coupled coincidence point results have appeared in the recent literature, for example, see [9,10, 11,12].

The effect of fixed points on graph theory in metric spaces was initiated by Jachymski [13]. Chifu and Petrusel [14] extended the results of [7] in a directed graph. Many researchers went to study this trend and some fixed point results in MSs endowed with a directed graph were obtained, see [15,16,17,18].

Recently, good work on coupled fixed points for mixed $\pi-$monotone mappings via Geraghty-type condition is presented by Kadelbur et al. [19].

Berinde and Borcut [20,21] were the first to present the idea of tripled fixed points as a generalization of coupled fixed points. They also contributed greatly for obtaining theorems that serve the field of fixed points in POMSs. A good number has worked in this direction, whether on the theoretical or the practical side, for further clarification, see [22,23,24,25,26].

Moreover, a valuable work that has been of great interest to readers is the idea of the quadruple fixed points, which was established by Karapinar [27]. Numerous applications have been listed by these points under appropriate conditions and satisfactory theoretical results have been deduced. For more details, see [28,29, 30].

Along with the results of Jachymski [13] and Karapinar [27], we present in this manuscript some quadruple coincidence point (QCP, for simplicity) results for $\varphi-$Geraghty contraction mappings in MSs endowed with a directed graph. Finally, the theoretical results are used to obtain the solution to a system of integral equations.

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2 Background and material

In this section, we present some notations and definitions which are useful for our work.

Assume that $(\mho,\Im)$ is a MS and ∇ is a diagonal of $\mho^2$. Assume also $\eth$ is a directed graph so that the set $W(\eth)$ of its vertices coincides with $\mho$ and $\nabla \subset \Gamma(\eth)$, where $\Gamma(\eth)$ denotes the set of the edges of the graph. Let $\eth$ has no parallel edges and therefore, we can define $\eth$ with the pair $(W(\eth),\Gamma(\eth))$.

Assume $\eth^{-1}$ is the graph obtained from $\eth$ by reversing the direction of edges. Hence,

$$\Gamma\left(\eth^{-1}\right)=\left\{(\xi,\varkappa)\in\mho^2:(\varkappa,\xi)\in\Gamma(\eth)\right\}.$$

Definition 1.[30] *Assume that $\daleth,\pi:\mho\to\mho$ are two mappings defined on a partially ordered set (POS) $(\mho,\preceq)$. $\daleth$ is called $\pi-$nondecreasing (resp., $\pi-$nonincreasing) if for each $\xi,\varkappa\in\mho$, $\pi\xi\preceq\pi\varkappa$ i.e., $\daleth\xi\preceq\daleth\varkappa$ (resp., $\daleth\varkappa\preceq\daleth\xi$).*

Note, if π is the identity mapping, then $\daleth$ is called nondecreasing (resp., nonincreasing).

Definition 2.[30] *Assume that $(\mho,\preceq)$ is a POS and $\Pi:\mho^4\to\mho$, $\pi:\mho\to\mho$ are two mappings. The mapping*

Π have a $\pi-$monotone property if Π is monotone $\pi-$nondecreasing in both of its arguments, that is, for each $\xi,\varkappa,\varpi,\eta\in\mho$, the assumptions below hold:

$$\begin{aligned}\xi_1,\xi_2 &\in \mho,\ \pi\xi_1\preceq\pi\xi_2\\ &\Longrightarrow \Pi(\xi_1,\varkappa,\varpi,\eta)\preceq\Pi(\xi_2,\varkappa,\varpi,\eta),\end{aligned}$$

$$\begin{aligned}\varkappa_1,\varkappa_2 &\in \mho,\ \pi\varkappa_1\preceq\pi\varkappa_2\\ &\Longrightarrow \Pi(\xi,\varkappa_1,\varpi,\eta)\preceq\Pi(\xi,\varkappa_2,\varpi,\eta),\end{aligned}$$

$$\begin{aligned}\varpi_1,\varpi_2 &\in \mho,\pi\varpi_1\preceq\pi\varpi_2\\ &\Longrightarrow \Pi(\xi,\varkappa,\varpi_1,\eta)\preceq\Pi(\xi,\varkappa,\varpi_2,\eta),\end{aligned}$$

and

$$\begin{aligned}\eta_1,\eta_2 &\in \mho,\ \pi\eta_1\preceq\pi\eta_2\\ &\Longrightarrow \Pi(\xi,\varkappa,\varpi,\eta_2)\preceq\Pi(\xi,\varkappa,\varpi,\eta_2).\end{aligned}$$

Clearly, if π is the identity map, then we say that Π has a monotone property.

Definition 3.[27] *Suppose that $\mho\neq\emptyset$ and $\Pi:\mho^4\to\mho$, $\pi:\mho\to\mho$ are two mappings. A point $(\xi,\varkappa,\varpi,\eta)\in\mho^4$ is called*

(Q_1)a quadruple fixed point of Π if

$$\begin{aligned}\xi &= \Pi(\xi,\varkappa,\varpi,\eta),\ \varkappa=\Pi(\varkappa,\varpi,\eta,\xi),\\ \varpi &= \Pi(\varpi,\eta,\xi,\varkappa)\ \text{and}\ \eta=\Pi(\eta,\xi,\varkappa,\varpi);\end{aligned}$$

(Q_2)a QCP of π and Π if

$$\begin{aligned}\pi\xi &= \Pi(\xi,\varkappa,\varpi,\eta),\ \pi\varkappa=\Pi(\varkappa,\varpi,\eta,\xi),\\ \pi\varpi &= \Pi(\varpi,\eta,\xi,\varkappa)\ \text{and}\ \pi\eta=\Pi(\eta,\xi,\varkappa,\varpi);\end{aligned}$$

(Q_3)a common quadruple fixed point (CQFP) of π and Π if

$$\begin{aligned}\xi &= \pi\xi=\Pi(\xi,\varkappa,\varpi,\eta),\\ \varkappa &= \pi\varkappa=\Pi(\varkappa,\varpi,\eta,\xi),\\ \varpi &= \pi\varpi=\Pi(\varpi,\eta,\xi,\varkappa),\\ \text{and}\ \eta &= \pi\eta=\Pi(\eta,\xi,\varkappa,\varpi).\end{aligned}$$

Definition 4.[28] *Assume that $(\mho,\Im)$ is a MS and $\Pi:\mho^4\to\mho$, $\pi:\mho\to\mho$ are two mappings. Π and π are called compatible mappings if*

$$\lim_{\beta\to\infty}\Im\begin{pmatrix}\pi\Pi(\xi_\beta,\varkappa_\beta,\varpi_\beta,\eta_\beta),\\ \Pi(\pi\xi_\beta,\pi\varkappa_\beta,\pi\varpi_\beta,\pi\eta_\beta)\end{pmatrix}=0,$$

$$\lim_{\beta\to\infty}\Im\begin{pmatrix}\pi\Pi(\varkappa_\beta,\varpi_\beta,\eta_\beta,\xi_\beta),\\ \Pi(\pi\varkappa_\beta,\pi\varpi_\beta,\pi\eta_\beta,\pi\xi_\beta)\end{pmatrix}=0,$$

$$\lim_{\beta\to\infty}\Im\begin{pmatrix}\pi\Pi(\varpi_\beta,\eta_\beta,\xi_\beta,\varkappa_\beta),\\ \Pi(\pi\varpi_\beta,\pi\eta_\beta,\pi\xi_\beta,\pi\varkappa_\beta)\end{pmatrix}=0,$$

and

$$\lim_{\beta\to\infty}\Im\begin{pmatrix}\pi\Pi(\eta_\beta,\xi_\beta,\varkappa_\beta,\varpi_\beta),\\ \Pi(\pi\eta_\beta,\pi\xi_\beta,\pi\varkappa_\beta,\pi\varpi_\beta)\end{pmatrix}=0,$$

whenever $\{\xi_\beta\}$, $\{\varkappa_\beta\}$, $\{\varpi_\beta\}$ and $\{\eta_\beta\}$ are sequences in $\mho$ so that

$$\begin{aligned}\lim_{\beta\to\infty}\Pi(\xi_\beta,\varkappa_\beta,\varpi_\beta,\eta_\beta) &= \lim_{\beta\to\infty}\pi\xi_\beta,\\ \lim_{\beta\to\infty}\Pi(\varkappa_\beta,\varpi_\beta,\eta_\beta,\xi_\beta) &= \lim_{\beta\to\infty}\pi\varkappa_\beta,\\ \lim_{\beta\to\infty}\Pi(\varpi_\beta,\eta_\beta,\xi_\beta,\varkappa_\beta) &= \lim_{\beta\to\infty}\pi\varpi_\beta,\\ \text{and}\ \lim_{\beta\to\infty}\Pi(\eta_\beta,\xi_\beta,\varkappa_\beta,\varpi_\beta) &= \lim_{\beta\to\infty}\pi\eta_\beta.\end{aligned}$$

Definition 5.[13] *Assume that $(\mho,\Im)$ is a complete MS and $\Gamma(\eth)$ is the set of the edges of the graph. The transitive property for $\Gamma(\eth)$ is holds if*

$$(\xi,a),(a,\varkappa)\in\Gamma(\eth)\ \text{implies}\ (\xi,\varkappa)\in\Gamma(\eth),\quad \text{for all}\ \xi,\varkappa,a\in\mho.$$

Definition 6.[19] *Suppose that $(\mho,\Im)$ is a complete MS and $\eth$ is a directed graph. A trio $(\mho,\Im,\eth)$ is called satisfies the property A, if for each sequence $(\xi_\beta)_{\beta\in\mathbb{N}}\subset\mho$ with $\xi_\beta\to\xi$, as $\beta\to\infty$ and $(\xi_\beta,\xi_{\beta+1})\in\Gamma(\eth)$, for $\beta\in\mathbb{N}$, we get $(\xi_\beta,\xi)\in\Gamma(\eth)$.*

3 Theoretical results

This part is devoted to present some QCP and CQFP results for $\varphi-$Geraghty contraction mappings in MSs endowed with directed graphs.

We indicate the set of all QCPs of the mappings $\Pi : \mho^4 \to \mho$ and $\pi : \mho \to \mho$ by QC(Π,π) so that

$$\mathrm{QC}(\Pi,\pi) = \left\{ \begin{array}{c} (\xi,\varkappa,\varpi,\eta) \in \mho^4 : \\ \Pi(\xi,\varkappa,\varpi,\eta) = \pi\xi, \\ \Pi(\varkappa,\varpi,\eta,\xi) = \pi\varkappa, \\ \Pi(\varpi,\eta,\xi,\varkappa) = \pi\varpi, \\ \Pi(\eta,\xi,\varkappa,\varpi) = \pi\eta \end{array} \right\}.$$

We start this part with the following notions:

Definition 7. *We say that the mappings $\Pi : \mho^4 \to \mho$ and $\pi : \mho \to \mho$ are $\pi-$edge preserving if*

$$\left[\left(\pi\xi,\pi\widetilde{\xi}\right),(\pi\varkappa,\pi\widetilde{\varkappa}),(\pi\varpi,\pi\widetilde{\varpi}),(\pi\eta,\pi\widetilde{\eta}) \in E(\eth)\right]$$
$$\Rightarrow \left[\left(\Pi(\xi,\varkappa,\varpi,\eta),\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\right),\right.$$
$$\left(\Pi(\varkappa,\varpi,\eta,\xi),\Pi\left(\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi}\right)\right),$$
$$\left(\Pi(\varpi,\eta,\xi,\varkappa),\Pi\left(\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa}\right)\right),$$
$$\left.\left(\Pi(\eta,\xi,\varkappa,\varpi),\Pi\left(\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi}\right)\right) \in \Gamma(\eth)\right].$$

Definition 8. *We say that the operator $\Pi : \mho^4 \to \mho$ is $\eth-$continuous if for each $(\xi^*,\varkappa^*,\varpi^*,\eta^*) \in \mho^4$ and for any sequence $(\beta_i)_i \in \mathbb{N}$ with*

$$\Pi\left(\xi_{\beta_i},\varkappa_{\beta_i},\varpi_{\beta_i},\eta_{\beta_i}\right) \to \xi^*,\ \Pi\left(\varkappa_{\beta_i},\varpi_{\beta_i},\eta_{\beta_i},\xi_{\beta_i}\right) \to \varkappa^*,$$
$$\Pi\left(\varpi_{\beta_i},\eta_{\beta_i},\xi_{\beta_i},\varkappa_{\beta_i}\right) \to \varpi^* \text{ and } \Pi\left(\eta_{\beta_i},\xi_{\beta_i},\varkappa_{\beta_i},\varpi_{\beta_i}\right) \to \eta^*,$$

as $i \to \infty$ and

$$\begin{array}{c} \left(\Pi\left(\xi_{\beta_i},\varkappa_{\beta_i},\varpi_{\beta_i},\eta_{\beta_i}\right),\Pi\left(\xi_{\beta_i+1},\varkappa_{\beta_i+1},\varpi_{\beta_i+1},\eta_{\beta_i+1}\right)\right), \\ \left(\Pi\left(\varkappa_{\beta_i},\varpi_{\beta_i},\eta_{\beta_i},\xi_{\beta_i}\right),\Pi\left(\varkappa_{\beta_i+1},\varpi_{\beta_i+1},\eta_{\beta_i+1},\xi_{\beta_i+1}\right)\right), \\ \left(\Pi\left(\varpi_{\beta_i},\eta_{\beta_i},\xi_{\beta_i},\varkappa_{\beta_i}\right),\Pi\left(\varpi_{\beta_i+1},\eta_{\beta_i+1},\xi_{\beta_i+1},\varkappa_{\beta_i+1}\right)\right), \\ \left(\Pi\left(\eta_{\beta_i},\xi_{\beta_i},\varkappa_{\beta_i},\varpi_{\beta_i}\right),\Pi\left(\eta_{\beta_i+1},\xi_{\beta_i+1},\varkappa_{\beta_i+1},\varpi_{\beta_i+1}\right)\right) \\ \in \Gamma(\eth), \end{array}$$

we have

$$\begin{array}{l} \Pi\begin{pmatrix} \Pi\left(\xi_{\beta_i},\varkappa_{\beta_i},\varpi_{\beta_i},\eta_{\beta_i}\right), \\ \Pi\left(\varkappa_{\beta_i},\varpi_{\beta_i},\eta_{\beta_i},\xi_{\beta_i}\right), \\ \Pi\left(\varpi_{\beta_i},\eta_{\beta_i},\xi_{\beta_i},\varkappa_{\beta_i}\right), \\ \Pi\left(\eta_{\beta_i},\xi_{\beta_i},\varkappa_{\beta_i},\varpi_{\beta_i}\right) \end{pmatrix} \to \Pi(\xi^*,\varkappa^*,\varpi^*,\eta^*) \\ \Pi\begin{pmatrix} \Pi\left(\varkappa_{\beta_i},\varpi_{\beta_i},\eta_{\beta_i},\xi_{\beta_i}\right), \\ \Pi\left(\varpi_{\beta_i},\eta_{\beta_i},\xi_{\beta_i},\varkappa_{\beta_i}\right), \\ \Pi\left(\eta_{\beta_i},\xi_{\beta_i},\varkappa_{\beta_i},\varpi_{\beta_i}\right), \\ \Pi\left(\xi_{\beta_i},\varkappa_{\beta_i},\varpi_{\beta_i},\eta_{\beta_i}\right) \end{pmatrix} \to \Pi(\varkappa^*,\varpi^*,\eta^*,\xi^*) \\ \Pi\begin{pmatrix} \Pi\left(\varpi_{\beta_i},\eta_{\beta_i},\xi_{\beta_i},\varkappa_{\beta_i}\right), \\ \Pi\left(\eta_{\beta_i},\xi_{\beta_i},\varkappa_{\beta_i},\varpi_{\beta_i}\right), \\ \Pi\left(\xi_{\beta_i},\varkappa_{\beta_i},\varpi_{\beta_i},\eta_{\beta_i}\right), \\ \Pi\left(\varkappa_{\beta_i},\varpi_{\beta_i},\eta_{\beta_i},\xi_{\beta_i}\right) \end{pmatrix} \to \Pi(\varpi^*,\eta^*,\xi^*,\varkappa^*) \\ \Pi\begin{pmatrix} \Pi\left(\eta_{\beta_i},\xi_{\beta_i},\varkappa_{\beta_i},\varpi_{\beta_i}\right), \\ \Pi\left(\xi_{\beta_i},\varkappa_{\beta_i},\varpi_{\beta_i},\eta_{\beta_i}\right), \\ \Pi\left(\varkappa_{\beta_i},\varpi_{\beta_i},\eta_{\beta_i},\xi_{\beta_i}\right), \\ \Pi\left(\varpi_{\beta_i},\eta_{\beta_i},\xi_{\beta_i},\varkappa_{\beta_i}\right) \end{pmatrix} \to \Pi(\eta^*,\xi^*,\varkappa^*,\varpi^*) \end{array},$$

as $i \to \infty$.

Assume that $(\mho,\Im)$ is a MS equipped with a directed graph $\eth$ verifying the standard conditions.

Consider the set $\left(\mho^4\right)_\pi^\Pi$ described by

$$\left(\mho^4\right)_\pi^\Pi = \left\{ \begin{array}{c} (\xi,\varkappa,\varpi,\eta) \in \mho^4 : \\ (\pi\xi,\Pi(\xi,\varkappa,\varpi,\eta)),(\pi\varkappa,\Pi(\varkappa,\varpi,\eta,\xi)), \\ (\pi\varpi,\Pi(\varpi,\eta,\xi,\varkappa)),(\pi\eta,\Pi(\eta,\xi,\varkappa,\varpi)) \\ \in \Gamma(\eth) \end{array} \right\}.$$

Also, consider Φ is the class of all functions $\varphi : [0,\infty) \to [0,\infty)$ so that the stipulations below hold:

(a_1) φ is non-decreasing;
(a_2) $\varphi(\sigma+\varsigma) \le \varphi(\sigma)+\varphi(\varsigma)$;
(a_3) φ is continuous;
(a_4) $\varphi(\varsigma) = 0 \Leftrightarrow \varsigma = 0$.

In addition, let Ψ be the class of all functions $\psi : [0,\infty)^4 \to [0,1)$ so that

(b_1) $\psi(\sigma,\varsigma,\tau,\rho) = \psi(\varsigma,\tau,\rho,\sigma) = \psi(\tau,\rho,\sigma,\varsigma) = \psi(\rho,\sigma,\varsigma,\tau)\ \forall\ \sigma,\varsigma,\tau,\rho \in [0,\infty)$;
(b_2) for any four sequences $\{\sigma_\beta\}$, $\{\varsigma_\beta\}$, $\{\tau_\beta\}$ and $\{\rho_\beta\}$ of positive real numbers,

$$\psi\left(\sigma_\beta,\varsigma_\beta,\tau_\beta,\rho_\beta\right) \to 1 \Rightarrow \sigma_\beta,\varsigma_\beta,\tau_\beta,\rho_\beta \to 0, \text{ as } \beta \to \infty.$$

Definition 9. *The mappings $\Pi : \mho^4 \to \mho$ and $\pi : \mho \to \mho$ are an $\psi-\varphi-$contraction if*

(c_1) Π and π are $\pi-$edge preserving;
(c_2) there exists $\psi \in \Psi$ and $\varphi \in \Phi$ so that for each $\xi,\varkappa,\varpi,\eta,\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta} \in \mho$ fulfilling $\left(\pi\xi,\pi\widetilde{\xi}\right),(\pi\varkappa,\pi\widetilde{\varkappa}),(\pi\varpi,\pi\widetilde{\varpi}),(\pi\eta,\pi\widetilde{\eta}) \in \Gamma(\eth)$,

$$\varphi\left(\Im\left(\Pi(\xi,\varkappa,\varpi,\eta),\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\right)\right)$$
$$\le \psi\begin{pmatrix} \Im\left(\pi\xi,\pi\widetilde{\xi}\right),\Im(\pi\varkappa,\pi\widetilde{\varkappa}), \\ \Im(\pi\varpi,\pi\widetilde{\varpi}),\Im(\pi\eta,\pi\widetilde{\eta}) \end{pmatrix}$$
$$\times\varphi\left(\aleph\left(\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\right)\right), \quad (1)$$

where

$$\aleph\left(\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\right)$$
$$= \max\left\{ \begin{array}{c} \Im\left(\pi\xi,\pi\widetilde{\xi}\right),\Im(\pi\varkappa,\pi\widetilde{\varkappa}), \\ \Im(\pi\varpi,\pi\widetilde{\varpi}),\Im(\pi\eta,\pi\widetilde{\eta}) \end{array} \right\}.$$

Now, our first main result is as follows:

Theorem 1. *Assume that $(\mho,\Im)$ is a complete MS equipped with a directed graph $\eth$. Assume also $\Pi : \mho^4 \to \mho$ and $\pi : \mho \to \mho$ are an $\psi-\varphi-$contraction so that the following postulates hold:*

(i) π is continuous and $\pi(\mho)$ is closed;
(ii) $\Pi\left(\mho^4\right) \subset \pi(\mho)$ and π and Π are compatible;
(iii) Π is $\eth-$continuous or the tripled $(\mho,\Im,\eth)$ verifies the property A;

(iv) $\Gamma(\eth)$ *justifies the transitive property.*

Then

$$\mathrm{QC}(\Pi,\pi)\neq\emptyset \text{ iff } \left(\mho^4\right)^{\Pi}_{\pi}\neq\emptyset.$$

Proof. Let the $\mathrm{QC}(\Pi,\pi)\neq\emptyset$ and $\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\in\mathrm{QC}(\Pi,\pi)$, we get

$$\left(\pi\widetilde{\xi},\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\right)=\left(\pi\widetilde{\xi},\pi\widetilde{\xi}\right),$$
$$\left(\pi\widetilde{\varkappa},\Pi\left(\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi}\right)\right)=\left(\pi\widetilde{\varkappa},\pi\widetilde{\varkappa}\right),$$
$$\left(\pi\widetilde{\varpi},\Pi\left(\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa}\right)\right)=\left(\pi\widetilde{\varpi},\pi\widetilde{\varpi}\right),$$

and $\left(\pi\widetilde{\eta},\Pi\left(\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi}\right)\right)=\left(\pi\widetilde{\eta},\pi\widetilde{\eta}\right)\in\nabla\subset\Gamma(\eth)$.

Therefore,

$$\left(\pi\widetilde{\xi},\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\right),\ \left(\pi\widetilde{\varkappa},\Pi\left(\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi}\right)\right),$$
$$\left(\pi\widetilde{\varpi},\Pi\left(\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa}\right)\right) \text{ and } \left(\pi\widetilde{\eta},\Pi\left(\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi}\right)\right)\in\Gamma(\eth),$$

this implies that $\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\in\left(\mho^4\right)^{\Pi}_{\pi}$ and hence $\left(\mho^4\right)^{\Pi}_{\pi}\neq\emptyset$.

Now, let $\left(\mho^4\right)^{\Pi}_{\pi}\neq\emptyset$. Assume that $\xi_0,\varkappa_0,\varpi_0,\eta_0\in\mho$ so that $(\xi_0,\varkappa_0,\varpi_0,\eta_0)\in\left(\mho^4\right)^{\Pi}_{\pi}$, we obtain

$$\begin{gathered}(\pi\xi_0,\Pi(\xi_0,\varkappa_0,\varpi_0,\eta_0)),\\(\pi\varkappa_0,\Pi(\varkappa_0,\varpi_0,\eta_0,\xi_0)),\\(\pi\varpi_0,\Pi(\varpi_0,\eta_0,\xi_0,\varkappa_0)),\\\text{and }(\pi\eta_0,\Pi(\eta_0,\xi_0,\varkappa_0,\varpi_0))\in\Gamma(\eth).\end{gathered}$$

Because $\Pi\left(\mho^4\right)\subset\pi(\mho)$, one can construct sequences $\{\xi_\beta\}$, $\{\varkappa_\beta\}$, $\{\varpi_\beta\}$ and $\{\eta_\beta\}$ in $\mho$ as the following:

$$\begin{aligned}\pi\xi_\beta&=\Pi\left(\xi_{\beta-1},\varkappa_{\beta-1},\varpi_{\beta-1},\eta_{\beta-1}\right),\\\pi\varkappa_\beta&=\Pi\left(\varkappa_{\beta-1},\varpi_{\beta-1},\eta_{\beta-1},\xi_{\beta-1}\right),\\\pi\varpi_\beta&=\Pi\left(\varpi_{\beta-1},\eta_{\beta-1},\xi_{\beta-1},\varkappa_{\beta-1}\right),\\\pi\eta_\beta&=\Pi\left(\eta_{\beta-1},\xi_{\beta-1},\varkappa_{\beta-1},\varpi_{\beta-1}\right),\ \text{for } \beta=1,2,\ldots\end{aligned}$$

If for some $\beta_0\in\mathbb{N}$, then

$$\begin{aligned}\pi\xi_{\beta_0}&=\pi\xi_{\beta_0-1},\ \pi\varkappa_{\beta_0}=\pi\varkappa_{\beta_0-1},\\\pi\varpi_{\beta_0}&=\pi\varpi_{\beta_0-1} \text{ and } \pi\eta_{\beta_0}=\pi\eta_{\beta_0-1}.\end{aligned}$$

Thus, $\left(\pi\xi_{\beta_0-1},\pi\varkappa_{\beta_0-1},\pi\varpi_{\beta_0-1},\pi\eta_{\beta_0-1}\right)$ is a QCP of π and Π. So, for each $\beta\in\mathbb{N}$, assume that

$$\pi\xi_\beta\neq\pi\xi_{\beta-1} \text{ or } \pi\varkappa_\beta\neq\pi\varkappa_{\beta-1}$$
$$\text{or } \pi\varpi_\beta\neq\pi\varpi_{\beta-1} \text{ or } \pi\eta_\beta\neq\pi\eta_{\beta-1}.$$

Since

$$\begin{aligned}(\pi\xi_0,\Pi(\xi_0,\varkappa_0,\varpi_0,\eta_0))&=(\pi\xi_0,\pi\xi_1),\\(\pi\varkappa_0,\Pi(\varkappa_0,\varpi_0,\eta_0,\xi_0))&=(\pi\varkappa_0,\pi\varkappa_1),\\(\pi\varpi_0,\Pi(\varpi_0,\eta_0,\xi_0,\varkappa_0))&=(\pi\varpi_0,\pi\varpi_1),\\(\pi\eta_0,\Pi(\eta_0,\xi_0,\varkappa_0,\varpi_0))&=(\pi\eta_0,\pi\eta_1)\in\Gamma(\eth),\end{aligned}$$

and Π and π are $\pi-$edge preserving, one can write

$$\begin{aligned}(\Pi(\xi_0,\varkappa_0,\varpi_0,\eta_0),\Pi(\xi_1,\varkappa_1,\varpi_1,\eta_1))&=(\pi\xi_1,\pi\xi_2),\\(\Pi(\varkappa_0,\varpi_0,\eta_0,\xi_0),\Pi(\varkappa_1,\varpi_1,\eta_1,\xi_1))&=(\pi\varkappa_1,\pi\varkappa_2),\\(\Pi(\varpi_0,\eta_0,\xi_0,\varkappa_0),\Pi(\varpi_1,\eta_1,\xi_1,\varkappa_1))&=(\pi\varpi_1,\pi\varpi_2),\\(\Pi(\eta_0,\xi_0,\varkappa_0,\varpi_0),\Pi(\eta_1,\xi_1,\varkappa_1,\varpi_1))&=(\pi\eta_1,\pi\eta_2)\\&\in\Gamma(\eth).\end{aligned}$$

By induction, we get

$$\begin{gathered}(\pi\xi_{\beta-1},\pi\xi_\beta),\ (\pi\varkappa_{\beta-1},\pi\varkappa_\beta),\ (\pi\varpi_{\beta-1},\pi\varpi_\beta),\\(\pi\eta_{\beta-1},\pi\eta_\beta)\in\Gamma(\eth),\ \forall\,\beta\in\mathbb{N}.\end{gathered}$$

Applying *(1), we* conclude that

$$\begin{aligned}&\varphi\left(\Im\left(\pi\xi_\beta,\pi\xi_{\beta+1}\right)\right)\\&=\varphi\left(\Im\left(\begin{array}{c}\Pi\left(\xi_{\beta-1},\varkappa_{\beta-1},\varpi_{\beta-1},\eta_{\beta-1}\right),\\\Pi\left(\xi_\beta,\varkappa_\beta,\varpi_\beta,\eta_\beta\right)\end{array}\right)\right)\\&\leq\psi\left(\begin{array}{c}\Im\left(\pi\xi_{\beta-1},\pi\xi_\beta\right),\Im\left(\pi\varkappa_{\beta-1},\pi\varkappa_\beta\right),\\\Im\left(\pi\varpi_{\beta-1},\pi\varpi_\beta\right),\Im\left(\pi\eta_{\beta-1},\pi\eta_\beta\right)\end{array}\right)\varphi(\aleph),\end{aligned}\tag{2}$$

similarly

$$\begin{aligned}&\varphi\left(\Im\left(\pi\varkappa_\beta,\pi\varkappa_{\beta+1}\right)\right)\\&=\varphi\left(\Im\left(\begin{array}{c}\Pi\left(\varkappa_{\beta-1},\varpi_{\beta-1},\eta_{\beta-1},\xi_{\beta-1}\right),\\\Pi\left(\varkappa_\beta,\varpi_\beta,\eta_\beta,\xi_\beta\right)\end{array}\right)\right)\\&\leq\psi\left(\begin{array}{c}\Im\left(\pi\varkappa_{\beta-1},\pi\varkappa_\beta\right),\Im\left(\pi\varpi_{\beta-1},\pi\varpi_\beta\right),\\\Im\left(\pi\eta_{\beta-1},\pi\eta_\beta\right),\Im\left(\pi\xi_{\beta-1},\pi\xi_\beta\right)\end{array}\right)\varphi(\aleph)\\&=\psi\left(\begin{array}{c}\Im\left(\pi\xi_{\beta-1},\pi\xi_\beta\right),\Im\left(\pi\varkappa_{\beta-1},\pi\varkappa_\beta\right),\\\Im\left(\pi\varpi_{\beta-1},\pi\varpi_\beta\right),\Im\left(\pi\eta_{\beta-1},\pi\eta_\beta\right)\end{array}\right)\varphi(\aleph),\end{aligned}\tag{3}$$

$$\begin{aligned}&\varphi\left(\Im\left(\pi\varpi_\beta,\pi\varpi_{\beta+1}\right)\right)\\&=\varphi\left(\Im\left(\begin{array}{c}\Pi\left(\varpi_{\beta-1},\eta_{\beta-1},\xi_{\beta-1},\varkappa_{\beta-1}\right),\\\Pi\left(\varpi_\beta,\eta_\beta,\xi_\beta,\varkappa_\beta\right)\end{array}\right)\right)\\&\leq\psi\left(\begin{array}{c}\Im\left(\pi\varpi_{\beta-1},\pi\varpi_\beta\right),\Im\left(\pi\eta_{\beta-1},\pi\eta_\beta\right),\\\Im\left(\pi\xi_{\beta-1},\pi\xi_\beta\right),\Im\left(\pi\varkappa_{\beta-1},\pi\varkappa_\beta\right)\end{array}\right)\varphi(\aleph)\\&=\psi\left(\begin{array}{c}\Im\left(\pi\xi_{\beta-1},\pi\xi_\beta\right),\Im\left(\pi\varkappa_{\beta-1},\pi\varkappa_\beta\right),\\\Im\left(\pi\varpi_{\beta-1},\pi\varpi_\beta\right),\Im\left(\pi\eta_{\beta-1},\pi\eta_\beta\right)\end{array}\right)\varphi(\aleph),\end{aligned}\tag{4}$$

and

$$\begin{aligned}&\varphi\left(\Im\left(\pi\eta_\beta,\pi\eta_{\beta+1}\right)\right)\\&=\varphi\left(\Im\left(\begin{array}{c}\Pi\left(\eta_{\beta-1},\xi_{\beta-1},\varkappa_{\beta-1},\varpi_{\beta-1}\right),\\\Pi\left(\eta_\beta,\xi_\beta,\varkappa_\beta,\varpi_\beta\right)\end{array}\right)\right)\\&\leq\psi\left(\begin{array}{c}\Im\left(\pi\eta_{\beta-1},\pi\eta_\beta\right),\Im\left(\pi\xi_{\beta-1},\pi\xi_\beta\right),\\\Im\left(\pi\varkappa_{\beta-1},\pi\varkappa_\beta\right),\Im\left(\pi\varpi_{\beta-1},\pi\varpi_\beta\right)\end{array}\right)\varphi(\aleph)\\&=\psi\left(\begin{array}{c}\Im\left(\pi\xi_{\beta-1},\pi\xi_\beta\right),\Im\left(\pi\varkappa_{\beta-1},\pi\varkappa_\beta\right),\\\Im\left(\pi\varpi_{\beta-1},\pi\varpi_\beta\right),\Im\left(\pi\eta_{\beta-1},\pi\eta_\beta\right)\end{array}\right)\varphi(\aleph),\end{aligned}\tag{5}$$

for each $\beta\in\mathbb{N}$, where

$$\aleph=\aleph\left(\begin{array}{c}\pi\xi_{\beta-1},\pi\xi_\beta,\pi\varkappa_{\beta-1},\pi\varkappa_\beta,\\\pi\varpi_{\beta-1},\pi\varpi_\beta,\pi\eta_{\beta-1},\pi\eta_\beta\end{array}\right).$$

Form (2)-(5), we obtain

$$\begin{aligned}
&\varphi\left(\aleph\begin{pmatrix}\pi\xi_\beta,\pi\xi_{\beta+1},\pi\varkappa_\beta,\pi\varkappa_{\beta+1},\\ \pi\varpi_\beta,\pi\varpi_{\beta+1},\pi\eta_\beta,\pi\eta_{\beta+1}\end{pmatrix}\right)\\
&=\varphi\left(\max\left\{\begin{matrix}\Im\left(\pi\xi_\beta,\pi\xi_{\beta+1}\right),\Im\left(\pi\varkappa_\beta,\pi\varkappa_{\beta+1}\right),\\ \Im\left(\pi\varpi_\beta,\pi\varpi_{\beta+1}\right),\Im\left(\pi\eta_\beta,\pi\eta_{\beta+1}\right)\end{matrix}\right\}\right)\\
&\leq\psi\begin{pmatrix}\Im\left(\pi\xi_{\beta-1},\pi\xi_\beta\right),\Im\left(\pi\varkappa_{\beta-1},\pi\varkappa_\beta\right),\\ \Im\left(\pi\varpi_{\beta-1},\pi\varpi_\beta\right),\Im\left(\pi\eta_{\beta-1},\pi\eta_\beta\right)\end{pmatrix}\\
&\quad\times\varphi\left(\aleph\begin{pmatrix}\pi\xi_{\beta-1},\pi\xi_\beta,\pi\varkappa_{\beta-1},\pi\varkappa_\beta,\\ \pi\varpi_{\beta-1},\pi\varpi_\beta,\pi\eta_{\beta-1},\pi\eta_\beta\end{pmatrix}\right)\\
&<\varphi\left(\aleph\begin{pmatrix}\pi\xi_{\beta-1},\pi\xi_\beta,\pi\varkappa_{\beta-1},\pi\varkappa_\beta,\\ \pi\varpi_{\beta-1},\pi\varpi_\beta,\pi\eta_{\beta-1},\pi\eta_\beta\end{pmatrix}\right),\ \forall\,\beta\in\mathbb{N}. \qquad (6)
\end{aligned}$$

It follows form (6) that

$$\begin{aligned}
&\varphi\left(\aleph\begin{pmatrix}\pi\xi_\beta,\pi\xi_{\beta+1},\pi\varkappa_\beta,\pi\varkappa_{\beta+1},\\ \pi\varpi_\beta,\pi\varpi_{\beta+1},\pi\eta_\beta,\pi\eta_{\beta+1}\end{pmatrix}\right)\\
&<\varphi\left(\aleph\begin{pmatrix}\pi\xi_{\beta-1},\pi\xi_\beta,\pi\varkappa_{\beta-1},\pi\varkappa_\beta,\\ \pi\varpi_{\beta-1},\pi\varpi_\beta,\pi\eta_{\beta-1},\pi\eta_\beta\end{pmatrix}\right).
\end{aligned}$$

The properties of φ leads to

$$\begin{aligned}
&\aleph\begin{pmatrix}\pi\xi_\beta,\pi\xi_{\beta+1},\pi\varkappa_\beta,\pi\varkappa_{\beta+1},\\ \pi\varpi_\beta,\pi\varpi_{\beta+1},\pi\eta_\beta,\pi\eta_{\beta+1}\end{pmatrix}\\
&<\aleph\begin{pmatrix}\pi\xi_{\beta-1},\pi\xi_\beta,\pi\varkappa_{\beta-1},\pi\varkappa_\beta,\\ \pi\varpi_{\beta-1},\pi\varpi_\beta,\pi\eta_{\beta-1},\pi\eta_\beta\end{pmatrix}.
\end{aligned}$$

Then the sequence

$$\Im_\beta=\aleph\begin{pmatrix}\pi\xi_{\beta-1},\pi\xi_\beta,\pi\varkappa_{\beta-1},\pi\varkappa_\beta,\\ \pi\varpi_{\beta-1},\pi\varpi_\beta,\pi\eta_{\beta-1},\pi\eta_\beta\end{pmatrix}$$

is decreasing. It follows that $\Im_\beta\to\Im$ as $\beta\to\infty$ for some $\Im\geq 0$.

Now, we prove that $\Im=0$. Suppose to the contrary, that is $\Im>0$, then from (6), one can get

$$\begin{aligned}
&\frac{\varphi\left(\aleph\begin{pmatrix}\pi\xi_\beta,\pi\xi_{\beta+1},\pi\varkappa_\beta,\pi\varkappa_{\beta+1},\\ \pi\varpi_\beta,\pi\varpi_{\beta+1},\pi\eta_\beta,\pi\eta_{\beta+1}\end{pmatrix}\right)}{\varphi\left(\aleph\begin{pmatrix}\pi\xi_{\beta-1},\pi\xi_\beta,\pi\varkappa_{\beta-1},\pi\varkappa_\beta,\\ \pi\varpi_{\beta-1},\pi\varpi_\beta,\pi\eta_{\beta-1},\pi\eta_\beta\end{pmatrix}\right)}\\
&\leq\psi\begin{pmatrix}\Im\left(\pi\xi_{\beta-1},\pi\xi_\beta\right),\Im\left(\pi\varkappa_{\beta-1},\pi\varkappa_\beta\right),\\ \Im\left(\pi\varpi_{\beta-1},\pi\varpi_\beta\right),\Im\left(\pi\eta_{\beta-1},\pi\eta_\beta\right)\end{pmatrix}<1.
\end{aligned}$$

Passing $\beta\to\infty$, we have

$$\psi\begin{pmatrix}\Im\left(\pi\xi_{\beta-1},\pi\xi_\beta\right),\Im\left(\pi\varkappa_{\beta-1},\pi\varkappa_\beta\right),\\ \Im\left(\pi\varpi_{\beta-1},\pi\varpi_\beta\right),\Im\left(\pi\eta_{\beta-1},\pi\eta_\beta\right)\end{pmatrix}\to 1.$$

Since $\varphi\in\Phi$, we obtain

$\Im\left(\pi\xi_{\beta-1},\pi\xi_\beta\right)\to 0$, $\Im\left(\pi\varkappa_{\beta-1},\pi\varkappa_\beta\right)\to 0$,
$\Im\left(\pi\varpi_{\beta-1},\pi\varpi_\beta\right)\to 0$ and $\Im\left(\pi\eta_{\beta-1},\pi\eta_\beta\right)\to 0$,
as $\beta\to\infty$, therefore

$$\lim_{\beta\to\infty}\Im_\beta=\lim_{\beta\to\infty}\aleph\begin{pmatrix}\pi\xi_{\beta-1},\pi\xi_\beta,\\ \pi\varkappa_{\beta-1},\pi\varkappa_\beta,\\ \pi\varpi_{\beta-1},\pi\varpi_\beta,\\ \pi\eta_{\beta-1},\pi\eta_\beta\end{pmatrix}=0, \qquad (7)$$

which is inconsistent with the assumption $\Im>0$. Hence, we have

$$\Im_\beta=\aleph\begin{pmatrix}\pi\xi_{\beta-1},\pi\xi_\beta,\pi\varkappa_{\beta-1},\pi\varkappa_\beta,\\ \pi\varpi_{\beta-1},\pi\varpi_\beta,\pi\eta_{\beta-1},\pi\eta_\beta\end{pmatrix}\to 0,$$

as $\beta\to\infty$.

Now, we prove that $\{\pi\xi_\beta\}$, $\{\pi\varkappa_\beta\}$, $\{\pi\varpi_\beta\}$ and $\{\pi\eta_\beta\}$ are Cauchy sequences. Suppose on the contrary that at least one of $\{\pi\xi_\beta\}$, $\{\pi\varkappa_\beta\}$, $\{\pi\varpi_\beta\}$ and $\{\pi\eta_\beta\}$ is not a Cauchy sequence. Thus there exists an $\varepsilon>0$ for which we can get subsequences $\{\pi\xi_{\beta_k}\}$, $\{\pi\xi_{\wp_k}\}$ of $\{\pi\xi_\beta\}$, $\{\pi\varkappa_{\beta_k}\}$, $\{\pi\varkappa_{\wp_k}\}$ of $\{\pi\varkappa_\beta\}$, $\{\pi\varpi_{\beta_k}\}$, $\{\pi\varpi_{\wp_k}\}$ of $\{\pi\varpi_\beta\}$ and $\{\pi\eta_{\beta_k}\}$, $\{\pi\eta_{\wp_k}\}$ of $\{\pi\eta_\beta\}$ with $\beta_k>\wp_k\geq k$ so that

$$\aleph\begin{pmatrix}\pi\xi_{\beta_k},\pi\xi_{\wp_k},\pi\varkappa_{\beta_k},\pi\varkappa_{\wp_k},\\ \pi\varpi_{\beta_k},\pi\varpi_{\wp_k},\pi\eta_{\beta_k},\pi\eta_{\wp_k}\end{pmatrix}\geq\varepsilon, \qquad (8)$$

and

$$\aleph\begin{pmatrix}\pi\xi_{\beta_k-1},\pi\xi_{\wp_k},\pi\varkappa_{\beta_k-1},\pi\varkappa_{\wp_k},\\ \pi\varpi_{\beta_k-1},\pi\varpi_{\wp_k},\pi\eta_{\beta_k-1},\pi\eta_{\wp_k}\end{pmatrix}<\varepsilon. \qquad (9)$$

By (8), (9) and triangle inequality, we get

$$\begin{aligned}
\varepsilon\leq\vartheta_k&=\aleph\begin{pmatrix}\pi\xi_{\beta_k},\pi\xi_{\wp_k},\pi\varkappa_{\beta_k},\pi\varkappa_{\wp_k},\\ \pi\varpi_{\beta_k},\pi\varpi_{\wp_k},\pi\eta_{\beta_k},\pi\eta_{\wp_k}\end{pmatrix}\\
&\leq\aleph\begin{pmatrix}\pi\xi_{\beta_k},\pi\xi_{\beta_k-1},\pi\varkappa_{\beta_k},\pi\varkappa_{\beta_k-1},\\ \pi\varpi_{\beta_k},\pi\varpi_{\beta_k-1},\pi\eta_{\beta_k},\pi\eta_{\beta_k-1}\end{pmatrix}\\
&\quad+\aleph\begin{pmatrix}\pi\xi_{\beta_k-1},\pi\xi_{\wp_k},\pi\varkappa_{\beta_k-1},\pi\varkappa_{\wp_k},\\ \pi\varpi_{\beta_k-1},\pi\varpi_{\wp_k},\pi\eta_{\beta_k-1},\pi\eta_{\wp_k}\end{pmatrix}\\
&<\aleph\begin{pmatrix}\pi\xi_{\beta_k},\pi\xi_{\beta_k-1},\pi\varkappa_{\beta_k},\pi\varkappa_{\beta_k-1},\\ \pi\varpi_{\beta_k},\pi\varpi_{\beta_k-1},\pi\eta_{\beta_k},\pi\eta_{\beta_k-1}\end{pmatrix}+\varepsilon.
\end{aligned}$$

Passing limit as $k\to\infty$, we can write

$$\vartheta_k=\aleph\begin{pmatrix}\pi\xi_{\beta_k},\pi\xi_{\wp_k},\pi\varkappa_{\beta_k},\pi\varkappa_{\wp_k},\\ \pi\varpi_{\beta_k},\pi\varpi_{\wp_k},\pi\eta_{\beta_k},\pi\eta_{\wp_k}\end{pmatrix}\to\varepsilon. \qquad (10)$$

Since

$$\begin{gathered}\left(\pi\xi_{\beta-1},\pi\xi_\beta\right),\ \left(\pi\varkappa_{\beta-1},\pi\varkappa_\beta\right),\left(\pi\varpi_{\beta-1},\pi\varpi_\beta\right),\\ \left(\pi\eta_{\beta-1},\pi\eta_\beta\right)\in\Gamma\left(\eth\right),\forall\,\beta\in\mathbb{N},\end{gathered}$$

and $\Gamma(\eth)$ justifies the transitive property, we have

$$\begin{aligned}
\varphi(\vartheta_k) &= \varphi\left(\aleph\left(\begin{matrix}\pi\xi_{\beta_k},\pi\xi_{\wp_k},\pi\varkappa_{\beta_k},\pi\varkappa_{\wp_k},\\ \pi\varpi_{\beta_k},\pi\varpi_{\wp_k},\pi\eta_{\beta_k},\pi\eta_{\wp_k}\end{matrix}\right)\right)\\
&\leq \varphi\left(\aleph\left(\begin{matrix}\pi\xi_{\beta_k},\pi\xi_{\beta_k+1},\pi\varkappa_{\beta_k},\pi\varkappa_{\beta_k+1},\\ \pi\varpi_{\beta_k},\pi\varpi_{\beta_k+1},\pi\eta_{\beta_k},\pi\eta_{\beta_k+1}\end{matrix}\right)\right)\\
&\quad+\varphi\left(\aleph\left(\begin{matrix}\pi\xi_{\beta_k+1},\pi\xi_{\wp_k+1},\pi\varkappa_{\beta_k+1},\pi\varkappa_{\wp_k+1},\\ \pi\varpi_{\beta_k+1},\pi\varpi_{\wp_k+1},\pi\eta_{\beta_k+1},\pi\eta_{\wp_k+1}\end{matrix}\right)\right)\\
&\quad+\varphi\left(\aleph\left(\begin{matrix}\pi\xi_{\wp_k+1},\pi\xi_{\wp_k},\pi\varkappa_{\wp_k+1},\pi\varkappa_{\wp_k},\\ \pi\varpi_{\wp_k+1},\pi\varpi_{\wp_k},\pi\eta_{\wp_k+1},\pi\eta_{\wp_k}\end{matrix}\right)\right)\\
&= \varphi\left(\aleph\left(\begin{matrix}\pi\xi_{\beta_k},\pi\xi_{\beta_k+1},\pi\varkappa_{\beta_k},\pi\varkappa_{\beta_k+1},\\ \pi\varpi_{\beta_k},\pi\varpi_{\beta_k+1},\pi\eta_{\beta_k},\pi\eta_{\beta_k+1}\end{matrix}\right)\right)\\
&\quad+\varphi\left(\aleph\left(\begin{matrix}\pi\xi_{\wp_k+1},\pi\xi_{\wp_k},\pi\varkappa_{\wp_k+1},\pi\varkappa_{\wp_k},\\ \pi\varpi_{\wp_k+1},\pi\varpi_{\wp_k},\pi\eta_{\wp_k+1},\pi\eta_{\wp_k}\end{matrix}\right)\right)\\
&\quad+\varphi\left(\aleph\left(\begin{matrix}\pi\xi_{\beta_k+1},\pi\xi_{\wp_k+1},\pi\varkappa_{\beta_k+1},\pi\varkappa_{\wp_k+1},\\ \pi\varpi_{\beta_k+1},\pi\varpi_{\wp_k+1},\pi\eta_{\beta_k+1},\pi\eta_{\wp_k+1}\end{matrix}\right)\right)\\
&\leq \varphi\left(\aleph\left(\begin{matrix}\pi\xi_{\beta_k},\pi\xi_{\beta_k+1},\pi\varkappa_{\beta_k},\pi\varkappa_{\beta_k+1},\\ \pi\varpi_{\beta_k},\pi\varpi_{\beta_k+1},\pi\eta_{\beta_k},\pi\eta_{\beta_k+1}\end{matrix}\right)\right)\\
&\quad+\varphi\left(\aleph\left(\begin{matrix}\pi\xi_{\wp_k+1},\pi\xi_{\wp_k},\pi\varkappa_{\wp_k+1},\pi\varkappa_{\wp_k},\\ \pi\varpi_{\wp_k+1},\pi\varpi_{\wp_k},\pi\eta_{\wp_k+1},\pi\eta_{\wp_k}\end{matrix}\right)\right)\\
&\quad+\psi\left(\begin{matrix}\Im\left(\pi\xi_{\beta_k},\pi\xi_{\wp_k}\right),\Im\left(\pi\varkappa_{\beta_k},\pi\varkappa_{\wp_k}\right),\\ \Im\left(\pi\varpi_{\beta_k},\pi\varpi_{\wp_k}\right),\Im\left(\pi\eta_{\beta_k},\pi\eta_{\wp_k}\right)\end{matrix}\right)\\
&\quad\times\varphi\left(\aleph\left(\begin{matrix}\pi\xi_{\beta_k},\pi\xi_{\wp_k},\pi\varkappa_{\beta_k},\pi\varkappa_{\wp_k},\\ \pi\varpi_{\beta_k},\pi\varpi_{\wp_k},\pi\eta_{\beta_k},\pi\eta_{\wp_k}\end{matrix}\right)\right)\\
&= \varphi\left(\Im_{\beta_k+1}\right)+\varphi\left(\Im_{\wp_k+1}\right)\\
&\quad+\psi\left(\begin{matrix}\Im\left(\pi\xi_{\beta_k},\pi\xi_{\wp_k}\right),\Im\left(\pi\varkappa_{\beta_k},\pi\varkappa_{\wp_k}\right),\\ \Im\left(\pi\varpi_{\beta_k},\pi\varpi_{\wp_k}\right),\Im\left(\pi\eta_{\beta_k},\pi\eta_{\wp_k}\right)\end{matrix}\right)\varphi(\vartheta_k)\\
&< \varphi\left(\Im_{\beta_k+1}\right)+\varphi\left(\Im_{\wp_k+1}\right)+\varphi(\vartheta_k).
\end{aligned}$$

Hence, we obtain

$$\varphi(\vartheta_k) < \varphi\left(\Im_{\beta_k+1}\right)+\varphi\left(\Im_{\wp_k+1}\right)+\varphi(\vartheta_k).$$

Letting $k\to\infty$, using (7), (10) and the properties of φ, we can write

$$\psi\left(\begin{matrix}\Im\left(\pi\xi_{\beta_k},\pi\xi_{\wp_k}\right),\Im\left(\pi\varkappa_{\beta_k},\pi\varkappa_{\wp_k}\right),\\ \Im\left(\pi\varpi_{\beta_k},\pi\varpi_{\wp_k}\right),\Im\left(\pi\eta_{\beta_k},\pi\eta_{\wp_k}\right)\end{matrix}\right)\to 1.$$

The properties of ψ leads to

$$\begin{aligned}
&\Im\left(\pi\xi_{\beta_k},\pi\xi_{\wp_k}\right)\to 0,\ \Im\left(\pi\varkappa_{\beta_k},\pi\varkappa_{\wp_k}\right)\to 0,\\
&\Im\left(\pi\varpi_{\beta_k},\pi\varpi_{\wp_k}\right)\to 0 \text{ and } \Im\left(\pi\eta_{\beta_k},\pi\eta_{\wp_k}\right)\to 0,
\end{aligned}$$

as $k\to\infty$. Also, we have

$$\lim_{k\to\infty}\vartheta_k=\lim_{k\to\infty}\aleph\left(\begin{matrix}\pi\xi_{\beta_k},\pi\xi_{\wp_k},\\ \pi\varkappa_{\beta_k},\pi\varkappa_{\wp_k},\\ \pi\varpi_{\beta_k},\pi\varpi_{\wp_k},\\ \pi\eta_{\beta_k},\pi\eta_{\wp_k}\end{matrix}\right)=0,$$

which contradicts $\varepsilon>0$.

Hence, we get $\{\pi\xi_\beta\}$, $\{\pi\varkappa_\beta\}$, $\{\pi\varpi_\beta\}$ and $\{\pi\eta_\beta\}$ are Cauchy sequences. Since $(\mho,\Im)$ is a complete and $\pi(\mho)$ is a closed subset of $\mho$, there are $\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\in\pi(\mho)$ so that

$$\begin{aligned}
\lim_{\beta\to\infty}\pi\xi_\beta &= \lim_{\beta\to\infty}\Pi\left(\xi_\beta,\varkappa_\beta,\varpi_\beta,\eta_\beta\right)=\widetilde{\xi},\\
\lim_{\beta\to\infty}\pi\varkappa_\beta &= \lim_{\beta\to\infty}\Pi\left(\varkappa_\beta,\varpi_\beta,\eta_\beta,\xi_\beta,\right)=\widetilde{\varkappa},\\
\lim_{\beta\to\infty}\pi\varpi_\beta &= \lim_{\beta\to\infty}\Pi\left(\varpi_\beta,\eta_\beta,\xi_\beta,\varkappa_\beta\right)=\widetilde{\varpi},\\
\text{and }\lim_{\beta\to\infty}\pi\eta_\beta &= \lim_{\beta\to\infty}\Pi\left(\eta_\beta,\xi_\beta,\varkappa_\beta,\varpi_\beta\right)=\widetilde{\eta}.
\end{aligned}$$

Form assumption (ii) of our theorem, we can summarize

$$\begin{aligned}
&\lim_{\beta\to\infty}\Im\left(\begin{matrix}\pi\Pi\left(\xi_\beta,\varkappa_\beta,\varpi_\beta,\eta_\beta\right),\\ \Pi\left(\pi\xi_\beta,\pi\varkappa_\beta,\pi\varpi_\beta,\pi\eta_\beta\right)\end{matrix}\right)=0,\\
&\lim_{\beta\to\infty}\Im\left(\begin{matrix}\pi\Pi\left(\varkappa_\beta,\varpi_\beta,\eta_\beta,\xi_\beta\right),\\ \Pi\left(\pi\varkappa_\beta,\pi\varpi_\beta,\pi\eta_\beta,\pi\xi_\beta\right)\end{matrix}\right)=0,\\
&\lim_{\beta\to\infty}\Im\left(\begin{matrix}\pi\Pi\left(\varpi_\beta,\eta_\beta,\xi_\beta,\varkappa_\beta\right),\\ \Pi\left(\pi\varpi_\beta,\pi\eta_\beta,\pi\xi_\beta,\pi\varkappa_\beta\right)\end{matrix}\right)=0,\\
&\text{and }\lim_{\beta\to\infty}\Im\left(\begin{matrix}\pi\Pi\left(\eta_\beta,\xi_\beta,\varkappa_\beta,\varpi_\beta\right),\\ \Pi\left(\pi\eta_\beta,\pi\xi_\beta,\pi\varkappa_\beta,\pi\varpi_\beta\right)\end{matrix}\right)=0. \qquad (11)
\end{aligned}$$

Now, we discuss the two stipulations which listed in (iii).

(S_1) Let Π be $\eth-$continuous. Based on the triangle inequality, we get

$$\begin{aligned}
&\Im\left(\pi\widetilde{\xi},\Pi\left(\pi\xi_\beta,\pi\varkappa_\beta,\pi\varpi_\beta,\pi\eta_\beta\right)\right)\\
&\leq \Im\left(\pi\widetilde{\xi},\pi\Pi\left(\xi_\beta,\varkappa_\beta,\varpi_\beta,\eta_\beta\right)\right)\\
&\quad+\Im\left(\begin{matrix}\pi\Pi\left(\xi_\beta,\varkappa_\beta,\varpi_\beta,\eta_\beta\right),\\ \Pi\left(\pi\xi_\beta,\pi\varkappa_\beta,\pi\varpi_\beta,\pi\eta_\beta\right)\end{matrix}\right).
\end{aligned}$$

When $\beta\to\infty$, by using (11) and the continuity of π and since Π is $\eth-$continuous, we have

$$\Im\left(\pi\widetilde{\xi},\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\right)=0\Longleftrightarrow\pi\widetilde{\xi}=\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right).$$

With the same scenario, one can write

$$\Im\left(\pi\widetilde{\varkappa},\Pi\left(\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi}\right)\right)=0\Longleftrightarrow\pi\widetilde{\varkappa}=\Pi\left(\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi},\right),$$

$$\Im\left(\pi\widetilde{\varpi},\Pi\left(\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa}\right)\right)=0\Longleftrightarrow\pi\widetilde{\varpi}=\Pi\left(\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa}\right),$$

and

$$\Im\left(\pi\widetilde{\eta},\Pi\left(\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi}\right)\right)=0\Longleftrightarrow\pi\widetilde{\eta}=\Pi\left(\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi}\right).$$

Thus $\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)$ is a QCP of the mappings Π and π. Hence, $\mathrm{QC}(\Pi,\pi)\neq\emptyset$.

(S_2) Assume that the triple $(\mho,\Im,\eth)$ satisfies the property A. Therefore

$$\begin{aligned}
&\pi\xi=\widetilde{\xi},\ \pi\varkappa=\widetilde{\varkappa},\ \pi\varpi=\widetilde{\varpi},\\
&\text{and } \pi\eta=\widetilde{\eta}\ \text{ for some } \xi,\varkappa,\varpi,\eta\in\mho,
\end{aligned}$$

and we get

$\Im(\xi_\beta,\xi)$, $\Im(\varkappa_\beta,\varkappa)$, $\Im(\varpi_\beta,\varpi)$,and $\Im(\eta_\beta,\eta)$ $\in \Gamma(\eth)$, $\forall\beta\in\mathbb{N}$.

From *(1)*, one can obtain

$$\begin{aligned}
&\varphi\begin{pmatrix}\Im(\pi\xi,\Pi(\xi,\varkappa,\varpi,\eta))+\Im(\pi\varkappa,\Pi(\varkappa,\varpi,\eta,\xi))\\+\Im(\pi\varpi,\Pi(\varpi,\eta,\xi,\varkappa))+\Im(\pi\eta,\Pi(\eta,\xi,\varkappa,\varpi))\end{pmatrix}\\
&\leq\varphi\begin{pmatrix}\Im(\pi\xi,\pi\xi_{\beta+1})+\Im(\pi\xi_{\beta+1},\Pi(\xi,\varkappa,\varpi,\eta))\\+\Im(\pi\varkappa,\pi\varkappa_{\beta+1})+\Im(\pi\varkappa_{\beta+1},\Pi(\varkappa,\varpi,\eta,\xi))\\+\Im(\pi\varpi,\pi\varpi_{\beta+1})+\Im(\pi\varpi_{\beta+1},\Pi(\varpi,\eta,\xi,\varkappa))\\+\Im(\pi\eta,\pi\eta_{\beta+1})+\Im(\pi\eta_{\beta+1},\Pi(\eta,\xi,\varkappa,\varpi))\end{pmatrix}\\
&\leq\varphi(\Im(\Pi(\xi_\beta,\varkappa_\beta,\varpi_\beta,\eta_\beta),\Pi(\xi,\varkappa,\varpi,\eta)))\\
&\quad+\varphi(\Im(\Pi(\varkappa_\beta,\varpi_\beta,\eta_\beta,\xi_\beta,),\Pi(\varkappa,\varpi,\eta,\xi)))\\
&\quad+\varphi(\Im(\Pi(\varpi_\beta,\eta_\beta,\xi_\beta,\varkappa_\beta),\Pi(\varpi,\eta,\xi,\varkappa)))\\
&\quad+\varphi(\Im(\Pi(\eta_\beta,\xi_\beta,\varkappa_\beta,\varpi_\beta),\Pi(\eta,\xi,\varkappa,\varpi)))\\
&\quad+\varphi(\Im(\pi\xi,\pi\xi_{\beta+1}))+\varphi(\Im(\pi\varkappa,\pi\varkappa_{\beta+1}))\\
&\quad+\varphi(\Im(\pi\varpi,\pi\varpi_{\beta+1}))+\varphi(\Im(\pi\eta,\pi\eta_{\beta+1}))\\
&\leq 4\psi\begin{pmatrix}\Im(\pi\xi_\beta,\pi\xi),\Im(\pi\varkappa_\beta,\pi\varkappa),\\\Im(\pi\varpi_\beta,\pi\varpi),\Im(\pi\eta_\beta,\pi\eta)\end{pmatrix}\\
&\quad\times\varphi(\aleph(\pi\xi_\beta,\pi\xi,\pi\varkappa_\beta,\pi\varkappa,\pi\varpi_\beta,\pi\varpi,\pi\eta_\beta,\pi\eta))\\
&\quad+\varphi(\Im(\pi\xi,\pi\xi_{\beta+1}))+\varphi(\Im(\pi\varkappa,\pi\varkappa_{\beta+1}))\\
&\quad+\varphi(\Im(\pi\varpi,\pi\varpi_{\beta+1}))+\varphi(\Im(\pi\eta,\pi\eta_{\beta+1}))\\
&\to 0,\text{ as }\beta\to\infty.
\end{aligned}$$

Therefore

$$\varphi\begin{pmatrix}\Im(\pi\xi,\Pi(\xi,\varkappa,\varpi,\eta))\\+\Im(\pi\varkappa,\Pi(\varkappa,\varpi,\eta,\xi))\\+\Im(\pi\varpi,\Pi(\varpi,\eta,\xi,\varkappa))\\+\Im(\pi\eta,\Pi(\eta,\xi,\varkappa,\varpi))\end{pmatrix}=0.$$

The properties of φ implies that

$$\begin{gathered}\Im(\pi\xi,\Pi(\xi,\varkappa,\varpi,\eta))\\+\Im(\pi\varkappa,\Pi(\varkappa,\varpi,\eta,\xi))\\+\Im(\pi\varpi,\Pi(\varpi,\eta,\xi,\varkappa))\\+\Im(\pi\eta,\Pi(\eta,\xi,\varkappa,\varpi))=0.\end{gathered}$$

Hence

$$\pi\xi=\Pi(\xi,\varkappa,\varpi,\eta),\ \pi\varkappa=\Pi(\varkappa,\varpi,\eta,\xi),$$
$$\pi\varpi=\Pi(\varpi,\eta,\xi,\varkappa)\text{ and }\pi\eta=\Pi(\eta,\xi,\varkappa,\varpi).$$

This finishes the proof.

Corollary 1. *Suppose that $(\mho,\Im,\preceq)$ is a partially ordered complete MS and assume that $\Pi:\mho^4\to\mho$ satisfies the monotone $\pi-$nondecreasing property and $\pi:\mho\to\mho$ is continuous. Let the assumptions below hold:*

(i)there are $\xi_0,\varkappa_0,\varpi_0,\eta_0\in\mho$ so that

$$\begin{aligned}\pi\xi_0&\preceq\Pi(\xi_0,\varkappa_0,\varpi_0,\eta_0),\\\pi\varkappa_0&\preceq\Pi(\varkappa_0,\varpi_0,\eta_0,\xi_0),\\\pi\varpi_0&\preceq\Pi(\varpi_0,\eta_0,\xi_0,\varkappa_0),\\\text{and }\pi\eta_0&\preceq\Pi(\eta_0,\xi_0,\varkappa_0,\varpi_0);\end{aligned}$$

(ii)there exists $\psi\in\Psi$ and $\varphi\in\Phi$ so that for each $\xi,\varkappa,\varpi,\eta,\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi}\in\mho$, we have

$$\left(\pi\xi\preceq\pi\widetilde{\xi},\ \pi\varkappa\preceq\pi\widetilde{\varkappa},\ \pi\varpi\preceq\pi\widetilde{\varpi},\ \pi\eta\preceq\pi\widetilde{\eta}\right)$$
$$\text{or }\left(\pi\widetilde{\xi}\preceq\pi\xi,\ \pi\widetilde{\varkappa}\preceq\pi\varkappa,\ \pi\widetilde{\varpi}\preceq\pi\varpi,\ \pi\widetilde{\eta}\preceq\pi\eta\right)$$

and

$$\begin{aligned}&\varphi\left(\Im\left(\Pi(\xi,\varkappa,\varpi,\eta),\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\right)\right)\\&\leq\psi\begin{pmatrix}\Im\left(\pi\xi,\pi\widetilde{\xi}\right),\Im(\pi\varkappa,\pi\widetilde{\varkappa}),\\\Im(\pi\varpi,\pi\widetilde{\varpi}),\Im(\pi\eta,\pi\widetilde{\eta})\end{pmatrix}\\&\quad\times\varphi\left(\aleph\left(\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\right)\right),\end{aligned}$$

where

$$\begin{aligned}&\aleph\left(\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\right)\\&=\max\left\{\begin{array}{c}\Im\left(\pi\xi,\pi\widetilde{\xi}\right),\Im(\pi\varkappa,\pi\widetilde{\varkappa}),\\\Im(\pi\varpi,\pi\widetilde{\varpi}),\Im(\pi\eta,\pi\widetilde{\eta})\end{array}\right\}.\end{aligned}$$

$(iii)(S_1)$ Π is continuous or,
(S_2) if $\{\xi_\beta\}$ is an increasing sequence in $\mho$ and $\xi_\beta\to\xi$ as $\beta\to\infty$, then $\xi_\beta\preceq\xi$ for all β.

Then Π has a QCP.

Proof. The proof follows immediately from Theorem *1* if we take $\Gamma(\eth)=\{(\xi,\varkappa)\in\mho^2:\xi\preceq\varkappa\}$.

Now, we shall denote the CQFPs by CQF(Π,π) so that

$$\mathrm{CQF}(\Pi,\pi)=\left\{\begin{array}{c}(\xi,\varkappa,\varpi,\eta)\in\mho^4:\\\Pi(\xi,\varkappa,\varpi,\eta)=\pi\xi=\xi,\\\Pi(\varkappa,\varpi,\eta,\xi)=\pi\varkappa=\varkappa,\\\Pi(\varpi,\eta,\xi,\varkappa)=\pi\varpi=\varpi,\\\Pi(\eta,\xi,\varkappa,\varpi)=\pi\eta=\eta\end{array}\right\}.$$

The second main theorem of our results is as follows:

Theorem 2. *In addition to the postulates of Theorem 1, assume that*

$(v)$for any two elements $(\xi,\varkappa,\varpi,\eta),\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\in\mho^4$ there is $(\xi^*,\varkappa^*,\varpi^*,\eta^*)\in\mho^4$ so that

$$\begin{gathered}(\pi\xi,\pi\xi^*),\left(\pi\widetilde{\xi},\pi\xi^*\right),(\pi\varkappa,\pi\varkappa^*),(\pi\widetilde{\varkappa},\pi\varkappa^*),\\(\pi\varpi,\pi\varpi^*),(\pi\widetilde{\varpi},\pi\varpi^*),(\pi\eta,\pi\eta^*),(\pi\widetilde{\eta},\pi\eta^*)\\\in\Gamma(\eth).\end{gathered}$$

Then

$$\mathrm{CQF}(\Pi,\pi)\neq\emptyset\text{ iff }\left(\mho^4\right)_\pi^\Pi\neq\emptyset.$$

Proof. Theorem *1* leads to there exists a QCP $(\xi,\varkappa,\varpi,\eta)\in\mho^4$, i.e.,

$$\pi\xi=\Pi(\xi,\varkappa,\varpi,\eta),\ \pi\varkappa=\Pi(\varkappa,\varpi,\eta,\xi),$$
$$\pi\varpi=\Pi(\varpi,\eta,\xi,\varkappa)\text{ and }\pi\eta=\Pi(\eta,\xi,\varkappa,\varpi).$$

Let there is another QCP $\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\in\mho^4$, that is

$$\pi\widetilde{\xi}=\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right),\ \pi\widetilde{\varkappa}=\Pi\left(\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi}\right),$$
$$\pi\widetilde{\varpi}=\Pi\left(\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa}\right)\text{ and }\pi\widetilde{\eta}=\Pi\left(\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi}\right).$$

Assumption (v) implies that there is $(\xi^*,\varkappa^*,\varpi^*,\eta^*)\in\mho^4$ so that

$$(\pi\xi,\pi\xi^*),\left(\pi\widetilde{\xi},\pi\xi^*\right),(\pi\varkappa,\pi\varkappa^*),(\pi\widetilde{\varkappa},\pi\varkappa^*),$$
$$(\pi\varpi,\pi\varpi^*),(\pi\widetilde{\varpi},\pi\varpi^*),(\pi\eta,\pi\eta^*),(\pi\widetilde{\eta},\pi\eta^*)\in\Gamma(\eth).$$

Putting $\xi_0^*=\xi^*$, $\varkappa_0^*=\varkappa^*$, $\varpi_0^*=\varpi^*$, $\eta_0^*=\eta^*$ and with the same manner to proof of Theorem 1, take sequences $\left\{\xi_\beta^*\right\}$, $\left\{\varkappa_\beta^*\right\}$, $\left\{\varpi_\beta^*\right\}$ and $\left\{\eta_\beta^*\right\}$ in $\mho$ verifying

$$\pi\xi_\beta^*=\Pi\left(\xi_{\beta-1}^*,\varkappa_{\beta-1}^*,\varpi_{\beta-1}^*,\eta_{\beta-1}^*\right),$$
$$\pi\varkappa_\beta^*=\Pi\left(\varkappa_{\beta-1}^*,\varpi_{\beta-1}^*,\eta_{\beta-1}^*,\xi_{\beta-1}^*\right),$$
$$\pi\varpi_\beta^*=\Pi\left(\varpi_{\beta-1}^*,\eta_{\beta-1}^*,\xi_{\beta-1}^*,\varkappa_{\beta-1}^*\right),$$
$$\text{and }\pi\eta_\beta=\Pi\left(\eta_{\beta-1}^*,\xi_{\beta-1}^*,\varkappa_{\beta-1}^*,\varpi_{\beta-1}^*\right),\text{ for }\beta\in\mathbb{N}.$$

Beginning from $\xi_0=\xi$, $\varkappa_0=\varkappa$, $\varpi_0=\varpi$, $\eta_0=\eta$ and $\widetilde{\xi}_0=\widetilde{\xi}$, $\widetilde{\varkappa}_0=\widetilde{\varkappa}$, $\widetilde{\varpi}_0=\widetilde{\varpi}$, $\widetilde{\eta}_0=\widetilde{\eta}$, take sequences $\{\xi_\beta\}$, $\{\varkappa_\beta\}$, $\{\varpi_\beta\}$, $\{\eta_\beta\}$ and $\left\{\widetilde{\xi}_\beta\right\}$, $\{\widetilde{\varkappa}_\beta\}$, $\{\widetilde{\varpi}_\beta\}$, $\{\widetilde{\eta}_\beta\}$ in $\mho$ verifying

$$\pi\xi_\beta=\Pi(\xi_{\beta-1},\varkappa_{\beta-1},\varpi_{\beta-1},\eta_{\beta-1}),$$
$$\pi\varkappa_\beta=\Pi(\varkappa_{\beta-1},\varpi_{\beta-1},\eta_{\beta-1},\xi_{\beta-1}),$$
$$\pi\varpi_\beta=\Pi(\varpi_{\beta-1},\eta_{\beta-1},\xi_{\beta-1},\varkappa_{\beta-1}),$$
$$\pi\eta_\beta=\Pi(\eta_{\beta-1},\xi_{\beta-1},\varkappa_{\beta-1},\varpi_{\beta-1}),\text{ for }\beta\in\mathbb{N},$$

and

$$\pi\widetilde{\xi}_\beta=\Pi\left(\widetilde{\xi}_{\beta-1},\widetilde{\varkappa}_{\beta-1},\widetilde{\varpi}_{\beta-1},\widetilde{\eta}_{\beta-1}\right),$$
$$\pi\widetilde{\varkappa}_\beta=\Pi\left(\widetilde{\varkappa}_{\beta-1},\widetilde{\varpi}_{\beta-1},\widetilde{\eta}_{\beta-1},\widetilde{\xi}_{\beta-1}\right),$$
$$\pi\widetilde{\varpi}_\beta=\Pi\left(\widetilde{\varpi}_{\beta-1},\widetilde{\eta}_{\beta-1},\widetilde{\xi}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),$$
$$\pi\widetilde{\eta}_\beta=\Pi\left(\widetilde{\eta}_{\beta-1},\widetilde{\xi}_{\beta-1},\widetilde{\varkappa}_{\beta-1},\widetilde{\varpi}_{\beta-1}\right),\text{ for }\beta\in\mathbb{N}.$$

Taking into account the characteristics of coincidence points, easily we can obtain $\xi_\beta=\xi$, $\varkappa_\beta=\varkappa$, $\varpi_\beta=\varpi$, $\eta_\beta=\eta$ and $\widetilde{\xi}_\beta=\widetilde{\xi}$, $\widetilde{\varkappa}_\beta=\widetilde{\varkappa}$, $\widetilde{\varpi}_\beta=\widetilde{\varpi}$, $\widetilde{\eta}_\beta=\widetilde{\eta}$, hence,

$$\pi\xi_\beta=\Pi(\xi,\varkappa,\varpi,\eta),$$
$$\pi\varkappa_\beta=\Pi(\varkappa,\varpi,\eta,\xi),$$
$$\pi\varpi_\beta=\Pi(\varpi,\eta,\xi,\varkappa),$$
$$\text{and }\pi\eta_\beta=\Pi(\eta,\xi,\varkappa,\varpi),\text{ for }\beta\in\mathbb{N}.$$

Also,

$$\pi\widetilde{\xi}_\beta=\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right),$$
$$\pi\widetilde{\varkappa}_\beta=\Pi\left(\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi}\right),$$
$$\pi\widetilde{\varpi}_\beta=\Pi\left(\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa}\right),$$
$$\text{and }\pi\widetilde{\eta}_\beta=\Pi\left(\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi}\right),\text{ for }\beta\in\mathbb{N}.$$

Since $(\xi,\varkappa,\varpi,\eta)$ and $(\xi_0^*,\varkappa_0^*,\varpi_0^*,\eta_0^*)=(\xi^*,\varkappa^*,\varpi^*,\eta^*)\in\mho^4$; therefore

$$(\pi\xi,\pi\xi_0^*),\ (\pi\varkappa,\pi\varkappa_0^*),(\pi\varpi,\pi\varpi_0^*)\text{ and }(\pi\eta,\pi\eta_0^*)\in\Gamma(\eth).$$

Because Π and π are $\pi-$edge preserving, we get

$$(\Pi(\xi,\varkappa,\varpi,\eta),\Pi(\xi_0^*,\varkappa_0^*,\varpi_0^*,\eta_0^*))=(\pi\xi,\pi\xi_1^*),$$
$$(\Pi(\varkappa,\varpi,\eta,\xi),\Pi(\varkappa_0^*,\varpi_0^*,\eta_0^*,\xi_0^*))=(\pi\varkappa,\pi\varkappa_1^*),$$
$$(\Pi(\varpi,\eta,\xi,\varkappa),\Pi(\varpi_0^*,\eta_0^*,\xi_0^*,\varkappa_0^*))=(\pi\varpi,\pi\varpi_1^*),$$
$$(\Pi(\eta,\xi,\varkappa,\varpi),\Pi(\eta_0^*,\xi_0^*,\varkappa_0^*,\varpi_0^*))=(\pi\eta,\pi\eta_1^*)\in\Gamma(\eth),$$

and continuing with the same manner, we have

$$\left(\pi\xi,\pi\xi_\beta^*\right),\ \left(\pi\varkappa,\pi\varkappa_\beta^*\right),$$
$$\left(\pi\varpi,\pi\varpi_\beta^*\right)\text{ and }\left(\pi\eta,\pi\eta_\beta^*\right)\in\Gamma(\eth).$$

Applying (1), we get

$$\begin{aligned}&\varphi\left(\Im\left(\pi\xi,\pi\xi_{\beta+1}^*\right)\right)\\&=\varphi\left(\Im\left(\Pi(\xi,\varkappa,\varpi,\eta),\Pi\left(\xi_\beta^*,\varkappa_\beta^*,\varpi_\beta^*,\eta_\beta^*\right)\right)\right)\\&\leq\psi\begin{pmatrix}\Im\left(\pi\xi,\pi\xi_\beta^*\right),\Im\left(\pi\varkappa,\pi\varkappa_\beta^*\right),\\\Im\left(\pi\varpi,\pi\varpi_\beta^*\right),\Im\left(\pi\eta,\pi\eta_\beta^*\right)\end{pmatrix}\\&\quad\times\varphi\left(\aleph\left(\pi\xi,\pi\xi_\beta^*,\pi\varkappa,\pi\varkappa_\beta^*,\pi\varpi,\pi\varpi_\beta^*,\pi\eta,\pi\eta_\beta^*\right)\right),\end{aligned}$$

$$\begin{aligned}&\varphi\left(\Im\left(\pi\varkappa,\pi\varkappa_{\beta+1}^*\right)\right)\\&=\varphi\left(\Im\left(\Pi(\varkappa,\varpi,\eta,\xi),\Pi\left(\varkappa_\beta^*,\varpi_\beta^*,\eta_\beta^*,\xi_\beta^*\right)\right)\right)\\&\leq\psi\begin{pmatrix}\Im\left(\pi\varkappa,\pi\varkappa_\beta^*\right),\Im\left(\pi\varpi,\pi\varpi_\beta^*\right),\\\Im\left(\pi\eta,\pi\eta_\beta^*\right),\Im\left(\pi\xi,\pi\xi_\beta^*\right)\end{pmatrix}\\&\quad\times\varphi\left(\aleph\left(\pi\xi,\pi\xi_\beta^*,\pi\varkappa,\pi\varkappa_\beta^*,\pi\varpi,\pi\varpi_\beta^*,\pi\eta,\pi\eta_\beta^*\right)\right)\\&=\psi\begin{pmatrix}\Im\left(\pi\xi,\pi\xi_\beta^*\right),\Im\left(\pi\varkappa,\pi\varkappa_\beta^*\right),\\\Im\left(\pi\varpi,\pi\varpi_\beta^*\right),\Im\left(\pi\eta,\pi\eta_\beta^*\right)\end{pmatrix}\\&\quad\times\varphi\left(\aleph\left(\pi\xi,\pi\xi_\beta^*,\pi\varkappa,\pi\varkappa_\beta^*,\pi\varpi,\pi\varpi_\beta^*,\pi\eta,\pi\eta_\beta^*\right)\right),\end{aligned}$$

$$\begin{aligned}
&\varphi\left(\Im\left(\pi\varpi,\pi\varpi^*_{\beta+1}\right)\right)\\
&=\varphi\left(\Im\left(\Pi(\varpi,\eta,\xi,\varkappa),\Pi\left(\varpi^*_\beta,\eta^*_\beta,\xi^*_\beta,\varkappa^*_\beta\right)\right)\right)\\
&\le\psi\begin{pmatrix}\Im\left(\pi\varpi,\pi\varpi^*_\beta\right),\Im\left(\pi\eta,\pi\eta^*_\beta\right),\\ \Im\left(\pi\xi,\pi\xi^*_\beta\right),\Im\left(\pi\varkappa,\pi\varkappa^*_\beta\right)\end{pmatrix}\\
&\quad\times\varphi\left(\aleph\left(\pi\xi,\pi\xi^*_\beta,\pi\varkappa,\pi\varkappa^*_\beta,\pi\varpi,\pi\varpi^*_\beta,\pi\eta,\pi\eta^*_\beta\right)\right)\\
&=\psi\begin{pmatrix}\Im\left(\pi\xi,\pi\xi^*_\beta\right),\Im\left(\pi\varkappa,\pi\varkappa^*_\beta\right),\\ \Im\left(\pi\varpi,\pi\varpi^*_\beta\right),\Im\left(\pi\eta,\pi\eta^*_\beta\right)\end{pmatrix}\\
&\quad\times\varphi\left(\aleph\left(\pi\xi,\pi\xi^*_\beta,\pi\varkappa,\pi\varkappa^*_\beta,\pi\varpi,\pi\varpi^*_\beta,\pi\eta,\pi\eta^*_\beta\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
&\varphi\left(\Im\left(\pi\eta,\pi\eta^*_{\beta+1}\right)\right)\\
&=\varphi\left(\Im\left(\Pi(\eta,\xi,\varkappa,\varpi),\Pi\left(\eta^*_\beta,\xi^*_\beta,\varkappa^*_\beta,\varpi^*_\beta\right)\right)\right)\\
&\le\psi\begin{pmatrix}\Im\left(\pi\eta,\pi\eta^*_\beta\right),\Im\left(\pi\xi,\pi\xi^*_\beta\right),\\ \Im\left(\pi\varkappa,\pi\varkappa^*_\beta\right),\Im\left(\pi\varpi,\pi\varpi^*_\beta\right)\end{pmatrix}\\
&\quad\times\varphi\left(\aleph\left(\pi\xi,\pi\xi^*_\beta,\pi\varkappa,\pi\varkappa^*_\beta,\pi\varpi,\pi\varpi^*_\beta,\pi\eta,\pi\eta^*_\beta\right)\right)\\
&=\psi\begin{pmatrix}\Im\left(\pi\xi,\pi\xi^*_\beta\right),\Im\left(\pi\varkappa,\pi\varkappa^*_\beta\right),\\ \Im\left(\pi\varpi,\pi\varpi^*_\beta\right),\Im\left(\pi\eta,\pi\eta^*_\beta\right)\end{pmatrix}\\
&\quad\times\varphi\left(\aleph\left(\pi\xi,\pi\xi^*_\beta,\pi\varkappa,\pi\varkappa^*_\beta,\pi\varpi,\pi\varpi^*_\beta,\pi\eta,\pi\eta^*_\beta\right)\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\varphi\left(\aleph\begin{pmatrix}\pi\xi,\pi\xi^*_{\beta+1},\pi\varkappa,\pi\varkappa^*_{\beta+1},\\ \pi\varpi,\pi\varpi^*_{\beta+1},\pi\eta,\pi\eta^*_{\beta+1}\end{pmatrix}\right)\\
&\le\psi\begin{pmatrix}\Im\left(\pi\eta,\pi\eta^*_\beta\right),\Im\left(\pi\xi,\pi\xi^*_\beta\right),\\ \Im\left(\pi\varkappa,\pi\varkappa^*_\beta\right),\Im\left(\pi\varpi,\pi\varpi^*_\beta\right)\end{pmatrix}\\
&\quad\times\varphi\left(\aleph\left(\pi\xi,\pi\xi^*_\beta,\pi\varkappa,\pi\varkappa^*_\beta,\pi\varpi,\pi\varpi^*_\beta,\pi\eta,\pi\eta^*_\beta\right)\right)\\
&<\varphi\left(\aleph\left(\pi\eta,\pi\eta^*_\beta,\pi\xi,\pi\xi^*_\beta,\pi\varkappa,\pi\varkappa^*_\beta,\pi\varpi,\pi\varpi^*_\beta\right)\right). \quad (12)
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\varphi\left(\aleph\begin{pmatrix}\pi\xi,\pi\xi^*_{\beta+1},\pi\varkappa,\pi\varkappa^*_{\beta+1},\\ \pi\varpi,\pi\varpi^*_{\beta+1},\pi\eta,\pi\eta^*_{\beta+1}\end{pmatrix}\right)\\
&<\varphi\left(\aleph\left(\pi\eta,\pi\eta^*_\beta,\pi\xi,\pi\xi^*_\beta,\pi\varkappa,\pi\varkappa^*_\beta,\pi\varpi,\pi\varpi^*_\beta\right)\right).
\end{aligned}$$

The properties of φ leads to

$$\begin{aligned}
&\aleph\begin{pmatrix}\pi\xi,\pi\xi^*_{\beta+1},\pi\varkappa,\pi\varkappa^*_{\beta+1},\\ \pi\varpi,\pi\varpi^*_{\beta+1},\pi\eta,\pi\eta^*_{\beta+1}\end{pmatrix}\\
&<\aleph\begin{pmatrix}\pi\eta,\pi\eta^*_\beta,\pi\xi,\pi\xi^*_\beta,\\ \pi\varkappa,\pi\varkappa^*_\beta,\pi\varpi,\pi\varpi^*_\beta\end{pmatrix}.
\end{aligned}$$

Therefore, the sequence

$$\Im_\beta=\aleph\begin{pmatrix}\pi\xi,\pi\xi_{\beta+1},\pi\varkappa,\pi\varkappa_{\beta+1},\\ \pi\varpi,\pi\varpi_{\beta+1},\pi\eta,\pi\eta_{\beta+1}\end{pmatrix}$$

is decreasing, then $\Im_\beta\to\Im$ as $\beta\to\infty$ for some $\Im\ge0$.

Now, we show that $\Im=0$. Suppose to the contrary that $\Im>0$; then from *(12)*, one can get

$$\begin{aligned}
&\frac{\varphi\left(\aleph\begin{pmatrix}\pi\xi,\pi\xi^*_{\beta+1},\pi\varkappa,\pi\varkappa^*_{\beta+1},\\ \pi\varpi,\pi\varpi^*_{\beta+1},\pi\eta,\pi\eta^*_{\beta+1}\end{pmatrix}\right)}{\varphi\left(\aleph\begin{pmatrix}\pi\xi,\pi\xi^*_\beta,\pi\varkappa,\pi\varkappa^*_\beta,\\ \pi\varpi,\pi\varpi^*_\beta,\pi\eta,\pi\eta^*_\beta\end{pmatrix}\right)}\\
&\le\psi\begin{pmatrix}\Im\left(\pi\xi,\pi\xi^*_\beta\right),\Im\left(\pi\varkappa,\pi\varkappa^*_\beta\right),\\ \Im\left(\pi\varpi,\pi\varpi^*_\beta\right),\Im\left(\pi\eta,\pi\eta^*_\beta\right)\end{pmatrix}<1.
\end{aligned}$$

Passing $\beta\to\infty$, we have

$$\psi\begin{pmatrix}\Im\left(\pi\xi,\pi\xi^*_\beta\right),\Im\left(\pi\varkappa,\pi\varkappa^*_\beta\right),\\ \Im\left(\pi\varpi,\pi\varpi^*_\beta\right),\Im\left(\pi\eta,\pi\eta^*_\beta\right)\end{pmatrix}\to1.$$

Since $\varphi\in\Phi$, we obtain

$$\begin{aligned}
&\Im\left(\pi\xi,\pi\xi^*_\beta\right)\to0,\ \Im\left(\pi\varkappa,\pi\varkappa^*_\beta\right)\to0,\\
&\Im\left(\pi\varpi,\pi\varpi^*_\beta\right)\to0,\text{and }\Im\left(\pi\eta,\pi\eta^*_\beta\right)\to0,
\end{aligned}$$

as $\beta\to\infty$. Therefore

$$\lim_{\beta\to\infty}\Im_\beta=\lim_{\beta\to\infty}\aleph\begin{pmatrix}\pi\xi,\pi\xi^*_\beta,\\ \pi\varkappa,\pi\varkappa^*_\beta,\\ \pi\varpi,\pi\varpi^*_\beta,\\ \pi\eta,\pi\eta^*_\beta\end{pmatrix}=0,$$

which contradicts with the assumption $\Im>0$. Hence, we have

$$\lim_{\beta\to\infty}\Im_\beta=\lim_{\beta\to\infty}\aleph\begin{pmatrix}\pi\xi,\pi\xi^*_\beta,\\ \pi\varkappa,\pi\varkappa^*_\beta,\\ \pi\varpi,\pi\varpi^*_\beta,\\ \pi\eta,\pi\eta^*_\beta\end{pmatrix}=0,$$

we conclude that

$$\begin{aligned}
&\lim_{\beta\to\infty}\left(\pi\xi,\pi\xi^*_\beta\right)=0,\ \lim_{\beta\to\infty}\left(\pi\varkappa,\pi\varkappa^*_\beta\right)=0,\\
&\lim_{\beta\to\infty}\left(\pi\varpi,\pi\varpi^*_\beta\right)=0\text{ and }\lim_{\beta\to\infty}\left(\pi\eta,\pi\eta^*_\beta\right)=0.
\end{aligned}$$

In similar scenario, we get

$$\begin{aligned}
&\lim_{\beta\to\infty}\left(\pi\tilde{\xi},\pi\xi^*_\beta\right)=0,\ \lim_{\beta\to\infty}\left(\pi\tilde{\varkappa},\pi\varkappa^*_\beta\right)=0,\\
&\lim_{\beta\to\infty}\left(\pi\tilde{\varpi},\pi\varpi^*_\beta\right)=0\text{ and }\lim_{\beta\to\infty}\left(\pi\tilde{\eta},\pi\eta^*_\beta\right)=0.
\end{aligned}$$

By using the triangle inequality, one can get

$$\Im\left(\pi\xi,\pi\widetilde{\xi}\right)\leq\Im\left(\pi\xi,\pi\xi_{\beta}^{*}\right)+\Im\left(\pi\xi_{\beta}^{*},\pi\widetilde{\xi}\right),$$
$$\Im\left(\pi\varkappa,\pi\widetilde{\varkappa}\right)\leq\Im\left(\pi\varkappa,\pi\varkappa_{\beta}^{*}\right)+\Im\left(\pi\varkappa_{\beta}^{*},\pi\widetilde{\varkappa}\right),$$
$$\Im\left(\pi\varpi,\pi\widetilde{\varpi}\right)\leq\Im\left(\pi\varpi,\pi\varpi_{\beta}^{*}\right)+\Im\left(\pi\varpi_{\beta}^{*},\pi\widetilde{\varpi}\right),$$
$$\text{and }\Im\left(\pi\eta,\pi\widetilde{\eta}\right)\leq\Im\left(\pi\xi,\pi\eta_{\beta}^{*}\right)+\Im\left(\pi\eta_{\beta}^{*},\pi\widetilde{\eta}\right),\ \forall\,\beta\in\mathbb{N}.$$

Passing $\beta\to\infty$, we have

$$\Im\left(\pi\xi,\pi\widetilde{\xi}\right)=0,\ \Im\left(\pi\varkappa,\pi\widetilde{\varkappa}\right)=0,$$
$$\Im\left(\pi\varpi,\pi\widetilde{\varpi}\right)=0\text{ and }\Im\left(\pi\eta,\pi\widetilde{\eta}\right)=0.$$

Hence we obtain

$$\pi\xi=\pi\widetilde{\xi},\ \pi\varkappa=\pi\widetilde{\varkappa},\ \pi\varpi=\pi\widetilde{\varpi}\text{ and }\pi\eta=\pi\widetilde{\eta}.$$

Now, letting

$$\xi^{c}=\pi\xi,\ \varkappa^{c}=\pi\varkappa,\ \varpi^{c}=\pi\varpi\text{ and }\eta^{c}=\pi\eta.$$

Therefore, we get

$$\begin{aligned}\pi\xi^{c}&=\pi(\pi\xi)=\pi\Pi(\xi,\varkappa,\varpi,\eta),\\ \pi\varkappa^{c}&=\pi(\pi\varkappa)=\pi\Pi(\varkappa,\varpi,\eta,\xi),\\ \pi\varpi^{c}&=\pi(\pi\varpi)=\pi\Pi(\varpi,\eta,\xi,\varkappa),\\ \text{and }\pi\eta^{c}&=\pi(\pi\eta)=\pi\Pi(\eta,\xi,\varkappa,\varpi).\end{aligned}$$

The definition of sequences (ξ_β), $(\varkappa_\beta)$, (ϖ_β) and (η_β) implies that

$$\begin{aligned}\pi\xi_{\beta}&=\Pi(\xi,\varkappa,\varpi,\eta)=\Pi\left(\xi_{\beta-1},\varkappa_{\beta-1},\varpi_{\beta-1},\eta_{\beta-1}\right),\\ \pi\varkappa_{\beta}&=\Pi(\varkappa,\varpi,\eta,\xi)=\Pi\left(\varkappa_{\beta-1},\varpi_{\beta-1},\eta_{\beta-1},\xi_{\beta-1}\right),\\ \pi\varpi_{\beta}&=\Pi(\varpi,\eta,\xi,\varkappa)=\Pi\left(\varpi_{\beta-1},\eta_{\beta-1},\xi_{\beta-1},\varkappa_{\beta-1}\right),\\ \text{and }\pi\eta_{\beta}&=\Pi(\eta,\xi,\varkappa,\varpi)=\Pi\left(\eta_{\beta-1},\xi_{\beta-1},\varkappa_{\beta-1},\varpi_{\beta-1}\right),\end{aligned}$$

for $\beta\in\mathbb{N}$. So, one can write

$$\begin{aligned}\lim_{\beta\to\infty}\Pi\left(\xi_\beta,\varkappa_\beta,\varpi_\beta,\eta_\beta\right)&=\lim_{\beta\to\infty}\pi\xi_\beta=\Pi(\xi,\varkappa,\varpi,\eta),\\ \lim_{\beta\to\infty}\Pi\left(\varkappa_\beta,\varpi_\beta,\eta_\beta,\xi_\beta\right)&=\lim_{\beta\to\infty}\pi\varkappa_\beta=\Pi(\varkappa,\varpi,\eta,\xi),\\ \lim_{\beta\to\infty}\Pi\left(\varpi_\beta,\eta_\beta,\xi_\beta,\varkappa_\beta\right)&=\lim_{\beta\to\infty}\pi\varpi_\beta=\Pi(\varpi,\eta,\xi,\varkappa),\\ \text{and }\lim_{\beta\to\infty}\Pi\left(\eta_\beta,\xi_\beta,\varkappa_\beta,\varpi_\beta\right)&=\lim_{\beta\to\infty}\pi\eta_\beta=\Pi(\eta,\xi,\varkappa,\varpi),\end{aligned}$$

for $\beta\in\mathbb{N}$. Since π and Π are compatible, then, we obtain

$$\lim_{\beta\to\infty}\Im\left(\begin{array}{c}\pi\Pi\left(\xi_\beta,\varkappa_\beta,\varpi_\beta,\eta_\beta\right),\\ \Pi\left(\pi\xi_\beta,\pi\varkappa_\beta,\pi\varpi_\beta,\pi\eta_\beta\right)\end{array}\right)=0,$$

This implies that

$$\pi\Pi(\xi,\varkappa,\varpi,\eta)=\Pi(\pi\xi,\pi\varkappa,\pi\varpi,\pi\eta),$$

Hence, we have

$$\begin{aligned}\pi\xi^{c}&=\pi\Pi(\xi,\varkappa,\varpi,\eta)\\ &=\Pi(\pi\xi,\pi\varkappa,\pi\varpi,\pi\eta)=\Pi\left(\xi^{c},\varkappa^{c},\varpi^{c},\eta^{c}\right),\end{aligned}$$

Analogoulsy,

$$\begin{aligned}\pi\varkappa^{c}&=\pi\Pi(\varkappa,\varpi,\eta,\xi)\\ &=\Pi(\pi\varkappa,\pi\varpi,\pi\eta,\pi\xi)=\Pi\left(\varkappa^{c},\varpi^{c},\eta^{c},\xi^{c}\right),\end{aligned}$$

$$\begin{aligned}\pi\varpi^{c}&=\pi\Pi(\varpi,\eta,\xi,\varkappa)\\ &=\Pi(\pi\varpi,\pi\eta,\pi\xi,\pi\varkappa)=\Pi\left(\varpi^{c},\eta^{c},\xi^{c},\varkappa^{c}\right),\end{aligned}$$

and

$$\begin{aligned}\pi\eta^{c}&=\pi\Pi(\eta,\xi,\varkappa,\varpi)\\ &=\Pi(\pi\eta,\pi\xi,\pi\varkappa,\pi\varpi)=\Pi\left(\eta^{c},\xi^{c},\varkappa^{c},\varpi^{c}\right).\end{aligned}$$

This fulfills that $(\xi^c,\varkappa^c,\varpi^c,\eta^c)$ is also a QCP. This means that

$$\begin{aligned}\pi\xi^{c}&=\pi\xi=\xi^{c},\ \pi\varkappa^{c}=\pi\varkappa=\varkappa^{c},\\ \pi\varpi^{c}&=\pi\varpi=\varpi^{c}\text{ and }\pi\eta^{c}=\pi\eta=\eta^{c}.\end{aligned}$$

So,

$$\begin{aligned}\xi^{c}&=\pi\xi^{c}=\Pi\left(\xi^{c},\varkappa^{c},\varpi^{c},\eta^{c}\right),\\ \varkappa^{c}&=\pi\varkappa^{c}=\Pi\left(\varkappa^{c},\varpi^{c},\eta^{c},\xi^{c}\right),\\ \varpi^{c}&=\pi\varpi^{c}=\Pi\left(\varpi^{c},\eta^{c},\xi^{c},\varkappa^{c}\right),\\ \text{and }\eta^{c}&=\pi\eta^{c}=\Pi\left(\eta^{c},\xi^{c},\varkappa^{c},\varpi^{c}\right).\end{aligned}$$

Therefore $(\xi^c,\varkappa^c,\varpi^c,\eta^c)$ is a CQFP of π and Π. The uniqueness is easy to prove, and thus the proof ends.

4 Solve a system of nonlinear integral equations

In fact, this section is the pillar of our manuscript because it represents applications of the obtained theoretical results where the existence of the solution to a quadrilateral system of nonlinear integral equations is studied.

Consider the following system:

$$\begin{cases}\xi(\sigma)=\int_0^{\urcorner}\beth(\sigma,\varsigma,\xi(\varsigma),\varkappa(\varsigma),\varpi(\varsigma),\eta(\varsigma))\,d\varsigma+g(\sigma),\\ \varkappa(\sigma)=\int_0^{\urcorner}\beth(\sigma,\varsigma,\varkappa(\varsigma),\varpi(\varsigma),\eta(\varsigma),\xi(\varsigma))\,d\varsigma+g(\sigma),\\ \varpi(\sigma)=\int_0^{\urcorner}\beth(\sigma,\varsigma,\varpi(\varsigma),\eta(\varsigma),\xi(\varsigma),\varkappa(\varsigma))\,d\varsigma+g(\sigma),\\ \eta(\sigma)=\int_0^{\urcorner}\beth(\sigma,\varsigma,\eta(\varsigma),\xi(\varsigma),\varkappa(\varsigma),\varpi(\varsigma))\,d\varsigma+g(\sigma),\end{cases}\tag{13}$$

where $\sigma\in[0,\urcorner]$ and $\urcorner>0$.

Assume that $\mho=C\left([0,\urcorner],\mathbb{R}^{\beta}\right)$ endowed with

$$\|\xi\|=\max_{\sigma\in[0,\urcorner]}|\xi(\sigma)|,\text{ for }\xi\in C\left([0,\urcorner],\mathbb{R}^{\beta}\right).$$

Define a partial order relation $\preceq$ as follows:

$$\xi,\varkappa\in\mho,\ \xi\preceq\varkappa\Leftrightarrow\xi(\sigma)\preceq\varkappa(\sigma),\text{ for }\sigma\in[0,\urcorner].$$

Clearly if $\Im$ is the metric induced by the norm, then $(\mho,\Im)$ is a complete MS equipped with a directed graph $\eth$, where a graph $\eth$ is defined as

$$\Gamma(\eth)=\left\{(\xi,\varkappa)\in\mho^{2}:\xi\preceq\varkappa\right\},$$

then $\Gamma(\eth)$ fulfills the transitive property and the diagonal ∇ of $\mho^2$ is induced in $\Gamma(\eth)$. In addition to, $(\mho,\Im,\eth)$ has the property A. In this situation, we put

$$(\mho^{4})_{\pi}^{\Pi}=\left\{\begin{array}{c}(\xi,\varkappa,\varpi,\eta)\in\mho^{4}:\\(\pi\xi,\Pi(\xi,\varkappa,\varpi,\eta)),(\pi\varkappa,\Pi(\varkappa,\varpi,\eta,\xi)),\\(\pi\varpi,\Pi(\varpi,\eta,\xi,\varkappa)),(\pi\eta,\Pi(\eta,\xi,\varkappa,\varpi))\\\in\Gamma(\eth)\end{array}\right\},$$

for $\xi=(\xi_1,\xi_2,\xi_3,...,\xi_\beta)$ and $\varkappa=(\varkappa_1,\varkappa_2,\varkappa_3,...,\varkappa_\beta)\in\mathbb{R}^\beta$,

$$\xi\preceq\varkappa\Leftrightarrow\xi_i\preceq\varkappa_i,\forall i=1,2,3,...,\beta.$$

Now, our first basic results here are ready for presentation.

Theorem 3. *Consider the problem (13) under the postulates below:*

(i) the functions $\beth:[0,\daleth]^2\times(\mathbb{R}^\beta)^4\to\mathbb{R}^\beta$ and $g:[0,\daleth]\to\mathbb{R}^\beta$ are contiuous;

(ii) for $\xi,\varkappa,\varpi,\eta,\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\in\mathbb{R}^\beta$ with $\xi\le\widetilde{\xi},\varkappa\le\widetilde{\varkappa},\varpi\le\widetilde{\varpi},\eta\le\widetilde{\eta}$, we have

$$\beth(\sigma,\varsigma,\xi,\varkappa,\varpi,\eta)\le\beth\left(\sigma,\varsigma,\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right),\ \forall\,\sigma,\varsigma\in[0,\daleth];$$

(iii) there are $k\in[0,1)$ and $\daleth>0$ so that

$$\left|\beth(\sigma,\varsigma,\xi,\varkappa,\varpi,\eta)-\beth\left(\sigma,\varsigma,\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\right|\le\frac{k}{4\daleth}\left(\left|\xi-\widetilde{\xi}\right|+|\varkappa-\widetilde{\varkappa}|+|\varpi-\widetilde{\varpi}|+|\eta-\widetilde{\eta}|\right),$$

for each $\sigma,\varsigma\in[0,\daleth]$, $\xi,\varkappa,\varpi,\eta,\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\in\mathbb{R}^\beta$ and $\xi\le\widetilde{\xi},\varkappa\le\widetilde{\varkappa},\varpi\le\widetilde{\varpi},\eta\le\widetilde{\eta}$;

(iv) there is $(\xi_0,\varkappa_0,\varpi_0,\eta_0)\in\mho^4$ so that

$$\begin{cases}\xi_0(\sigma)\le\int_0^{\daleth}\beth(\sigma,\varsigma,\xi_0(\varsigma),\varkappa_0(\varsigma),\varpi_0(\varsigma),\eta_0(\varsigma))\,d\varsigma+g(\sigma),\\ \varkappa_0(\sigma)\le\int_0^{\daleth}\beth(\sigma,\varsigma,\varkappa_0(\varsigma),\varpi_0(\varsigma),\eta_0(\varsigma),\xi_0(\varsigma))\,d\varsigma+g(\sigma),\\ \varpi_0(\sigma)\le\int_0^{\daleth}\beth(\sigma,\varsigma,\varpi_0(\varsigma),\eta_0(\varsigma),\xi_0(\varsigma),\varkappa_0(\varsigma))\,d\varsigma+g(\sigma),\\ \eta_0(\sigma)\le\int_0^{\daleth}\beth(\sigma,\varsigma,\eta_0(\varsigma),\xi_0(\varsigma),\varkappa_0(\varsigma),\varpi_0(\varsigma))\,d\varsigma+g(\sigma),\end{cases}$$

where $\sigma\in[0,\daleth]$.

Then the system *(13)* has at least one solution in $\mho$.

Proof. Define an operator $\Pi:\mho^4\to\mho$ by

$$\Pi(\xi,\varkappa,\varpi,\eta)(\sigma)=\int_0^{\daleth}\beth(\sigma,\varsigma,\xi(\varsigma),\varkappa(\varsigma),\varpi(\varsigma),\eta(\varsigma))\,d\varsigma+g(\sigma),$$

as $\sigma\in[0,\daleth]$. And $\pi:\mho\to\mho$ is the identity mapping. Therefore, the problem *(13)* can be written as

$$\xi=\Pi(\xi,\varkappa,\varpi,\eta),\ \varkappa=\Pi(\varkappa,\varpi,\eta,\xi),$$
$$\varpi=\Pi(\varpi,\eta,\xi,\varkappa),\ \eta=\Pi(\eta,\xi,\varkappa,\varpi).$$

Suppose that $\xi,\varkappa,\varpi,\eta,\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\in\mho$ so that $\pi\xi\le\pi\widetilde{\xi}$, $\pi\varkappa\le\pi\widetilde{\varkappa}$, $\pi\varpi\le\pi\widetilde{\varpi}$ and $\pi\eta\le\pi\widetilde{\eta}$. For $\xi\le\widetilde{\xi}$, $\varkappa\le\widetilde{\varkappa}$, $\varpi\le\widetilde{\varpi}$ and $\eta\le\widetilde{\eta}$, we have for each $\sigma\in[0,\daleth]$,

$$\begin{aligned}\Pi(\xi,\varkappa,\varpi,\eta)(\sigma)&=\int_0^{\daleth}\beth(\sigma,\varsigma,\xi(\varsigma),\varkappa(\varsigma),\varpi(\varsigma),\eta(\varsigma))\,d\varsigma+g(\sigma)\\&\le\int_0^{\daleth}\beth\left(\sigma,\varsigma,\widetilde{\xi}(\varsigma),\widetilde{\varkappa}(\varsigma),\widetilde{\varpi}(\varsigma),\widetilde{\eta}(\varsigma)\right)d\varsigma+g(\sigma)\\&=\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)(\sigma),\end{aligned}$$

$$\begin{aligned}\Pi(\varkappa,\varpi,\eta,\xi)(\sigma)&=\int_0^{\daleth}\beth(\sigma,\varsigma,\varkappa(\varsigma),\varpi(\varsigma),\eta(\varsigma),\xi(\varsigma))\,d\varsigma+g(\sigma)\\&\le\int_0^{\daleth}\beth\left(\sigma,\varsigma,\widetilde{\varkappa}(\varsigma),\widetilde{\varpi}(\varsigma),\widetilde{\eta}(\varsigma),\widetilde{\xi}(\varsigma)\right)d\varsigma+g(\sigma)\\&=\Pi\left(\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi}\right)(\sigma),\end{aligned}$$

$$\begin{aligned}\Pi(\varpi,\eta,\xi,\varkappa)(\sigma)&=\int_0^{\daleth}\beth(\sigma,\varsigma,\varpi(\varsigma),\eta(\varsigma),\xi(\varsigma),\varkappa(\varsigma))\,d\varsigma+g(\sigma)\\&\le\int_0^{\daleth}\beth\left(\sigma,\varsigma,\widetilde{\varpi}(\varsigma),\widetilde{\eta}(\varsigma),\widetilde{\xi}(\varsigma),\widetilde{\varkappa}(\varsigma)\right)d\varsigma+g(\sigma)\\&=\Pi\left(\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa}\right)(\sigma),\end{aligned}$$

and

$$\Pi(\eta,\xi,\varkappa,\varpi)(\sigma)$$
$$=\int_0^{\daleth}\beth(\sigma,\varsigma,\eta(\varsigma),\xi(\varsigma),\varkappa(\varsigma),\varpi(\varsigma))d\varsigma+g(\sigma)$$
$$\leq\int_0^{\daleth}\beth\left(\sigma,\varsigma,\widetilde{\eta}(\varsigma),\widetilde{\xi}(\varsigma),\widetilde{\varkappa}(\varsigma),\widetilde{\varpi}(\varsigma)\right)d\varsigma+g(\sigma)$$
$$=\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)(\sigma).$$

Hence, if $\pi\xi\leq\pi\widetilde{\xi}$, $\pi\varkappa\leq\pi\widetilde{\varkappa}$, $\pi\varpi\leq\pi\widetilde{\varpi}$ and $\pi\eta\leq\pi\widetilde{\eta}$, then

$$\Pi(\xi,\varkappa,\varpi,\eta)\leq\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right),$$
$$\Pi(\varkappa,\varpi,\eta,\xi)\leq\Pi\left(\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi}\right),$$
$$\Pi(\varpi,\eta,\xi,\varkappa)\leq\Pi\left(\widetilde{\varpi},\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa}\right),$$

and $\Pi(\eta,\xi,\varkappa,\varpi)\leq\Pi\left(\widetilde{\eta},\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi}\right)$.

Based on the definition of $\Gamma(\eth)$, we obtain Π and π are $\pi-$preserving.

On the other hand,

$$\left|\Pi(\xi,\varkappa,\varpi,\eta)(\sigma)-\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)(\sigma)\right|$$
$$\leq\int_0^{\daleth}\left|\begin{array}{c}\beth(\sigma,\varsigma,\xi(\varsigma),\varkappa(\varsigma),\varpi(\varsigma),\eta(\varsigma))\\-\beth\left(\sigma,\varsigma,\widetilde{\xi}(\varsigma),\widetilde{\varkappa}(\varsigma),\widetilde{\varpi}(\varsigma),\widetilde{\eta}(\varsigma)\right)\end{array}\right|d\varsigma$$
$$\leq\frac{k}{4\daleth}\int_0^{\daleth}\left(\begin{array}{c}\left|\xi(\varsigma)-\widetilde{\xi}(\varsigma)\right|+|\varkappa(\varsigma)-\widetilde{\varkappa}(\varsigma)|\\+|\varpi(\varsigma)-\widetilde{\varpi}(\varsigma)|+|\eta(\varsigma)-\widetilde{\eta}(\varsigma)|\end{array}\right)d\varsigma$$
$$\leq k\left(\frac{\left\|\pi\xi-\pi\widetilde{\xi}\right\|+\|\pi\varkappa-\pi\widetilde{\varkappa}\|+\|\pi\varpi-\pi\widetilde{\varpi}\|+\|\pi\eta-\pi\widetilde{\eta}\|}{4}\right)$$
$$\leq k\aleph\left(\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\right),\text{ for }\sigma\in[0,\daleth].$$

Hence, there is $\varphi(\sigma)=\sigma$ and $\psi\in\Psi$ with $\psi(\sigma,\varsigma,\tau,\rho)=k$, for $\sigma,\varsigma,\tau,\rho\in[0,\infty)$ and $k\in[0,1)$ so that

$$\varphi\left(\left\|\Pi(\xi,\varkappa,\varpi,\eta)-\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\right\|\right)$$
$$\leq\psi\left(\left\|\pi\xi-\pi\widetilde{\xi}\right\|,\|\pi\varkappa-\pi\widetilde{\varkappa}\|,\|\pi\varpi-\pi\widetilde{\varpi}\|,\|\pi\eta-\pi\widetilde{\eta}\|\right)$$
$$\times\varphi\left(\aleph\left(\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\right)\right),$$

where

$$\aleph\left(\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\right)$$
$$=\max\left\{\begin{array}{c}\left\|\pi\xi-\pi\widetilde{\xi}\right\|,\|\pi\varkappa-\pi\widetilde{\varkappa}\|,\\\|\pi\varpi-\pi\widetilde{\varpi}\|,\|\pi\eta-\pi\widetilde{\eta}\|\end{array}\right\}.$$

This implies that Π and π are an $\psi-\varphi-$contraction.

Ultimately, hypothesis (iv) leads to

$$(\mho^4)^{\Pi}_{\pi}=\left\{\begin{array}{c}(\xi,\varkappa,\varpi,\eta)\in\mho^4:\\(\pi\xi,\Pi(\xi,\varkappa,\varpi,\eta)),\\(\pi\varkappa,\Pi(\varkappa,\varpi,\eta,\xi)),\\(\pi\varpi,\Pi(\varpi,\eta,\xi,\varkappa)),\\(\pi\eta,\Pi(\eta,\xi,\varkappa,\varpi))\end{array}\right\}\neq\emptyset.$$

Therefore $(\xi^*,\varkappa^*,\varpi^*,\eta^*)\in\mho^4$ is a CQFP of Π and π, which is the solution to the problem *(13)*.

If a slight change is made in one of the conditions of Theorem 3, we get the following theorem:

Theorem 4.*If we replaced the postulate (iii) of Theorem 3 with the following hypothesis with remain rest of the assumptions:*

$(h)$for each $\sigma,\varsigma\in[0,\daleth]$, $\xi,\varkappa,\varpi,\eta,\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\in\mathbb{R}^{\beta}$ and $\xi\leq\widetilde{\xi}$, $\varkappa\leq\widetilde{\varkappa}$, $\varpi\leq\widetilde{\varpi}$, $\eta\leq\widetilde{\eta}$, we have

$$\left|\beth(\sigma,\varsigma,\xi,\varkappa,\varpi,\eta)-\beth\left(\sigma,\varsigma,\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\right|$$
$$\leq\frac{1}{\daleth}\ln\left(1+\max\left\{\begin{array}{c}\left|\xi-\widetilde{\xi}\right|,|\varkappa-\widetilde{\varkappa}|,\\|\varpi-\widetilde{\varpi}|,|\eta-\widetilde{\eta}|\end{array}\right\}\right).$$

Then the system *(13)* has at least one solution in $\mho$.

*Proof.*Assume that $\Pi:\mho^4\to\mho$, $(\xi,\varkappa,\varpi,\eta)\mapsto\Pi(\xi,\varkappa,\varpi,\eta)$, where

$$\Pi(\xi,\varkappa,\varpi,\eta)(\sigma)$$
$$=\int_0^{\daleth}\beth(\sigma,\varsigma,\xi(\varsigma),\varkappa(\varsigma),\varpi(\varsigma),\eta(\varsigma))d\varsigma+g(\sigma),$$

for $\sigma\in[0,\daleth]$ and $\pi:\mho\to\mho$ by $\pi\xi(\sigma)=\xi(\sigma)$. According to Theorem 3, we obtain that Π and π are $\pi-$edge preserving.

On the other hand,

$$\left|\Pi(\xi,\varkappa,\varpi,\eta)(\sigma)-\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)(\sigma)\right|$$
$$\leq\int_0^{\daleth}\left|\begin{array}{c}\beth(\sigma,\varsigma,\xi(\varsigma),\varkappa(\varsigma),\varpi(\varsigma),\eta(\varsigma))\\-\beth\left(\sigma,\varsigma,\widetilde{\xi}(\varsigma),\widetilde{\varkappa}(\varsigma),\widetilde{\varpi}(\varsigma),\widetilde{\eta}(\varsigma)\right)\end{array}\right|d\varsigma$$
$$\leq\frac{1}{\daleth}\int_0^{\daleth}\ln\left(1+\max\left\{\begin{array}{c}\left|\xi(\varsigma)-\widetilde{\xi}(\varsigma)\right|,\\|\varkappa(\varsigma)-\widetilde{\varkappa}(\varsigma)|,\\|\varpi(\varsigma)-\widetilde{\varpi}(\varsigma)|,\\|\eta(\varsigma)-\widetilde{\eta}(\varsigma)|\end{array}\right\}\right)d\varsigma$$
$$\leq\ln\left(1+\max\left\{\begin{array}{c}\left\|\xi-\widetilde{\xi}\right\|,\|\varkappa-\widetilde{\varkappa}\|,\\\|\varpi-\widetilde{\varpi}\|,\|\eta-\widetilde{\eta}\|\end{array}\right\}\right)$$
$$=\ln\left(1+\aleph\left(\begin{array}{c}\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\\\pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\end{array}\right)\right),\ \forall\sigma\in[0,\daleth].$$

where

$$\aleph\left(\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\right) = \max\left\{\begin{array}{l}\left\|\pi\xi-\pi\widetilde{\xi}\right\|,\|\pi\varkappa-\pi\widetilde{\varkappa}\|,\\ \|\pi\varpi-\pi\widetilde{\varpi}\|,\|\pi\eta-\pi\widetilde{\eta}\|\end{array}\right\}.$$

So,

$$\begin{aligned}&\ln\left(\left|\Pi(\xi,\varkappa,\varpi,\eta)(\sigma)-\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)(\sigma)\right|+1\right)\\ &\leq \ln\left(\ln\left(1+\aleph\left(\begin{array}{c}\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\\ \pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\end{array}\right)\right)+1\right)\\ &= \frac{\ln\left(\ln\left(1+\aleph\left(\begin{array}{c}\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\\ \pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\end{array}\right)\right)+1\right)}{\ln\left(1+\aleph\left(\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\right)\right)}\\ &\quad\times\ln\left(1+\aleph\left(\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\right)\right).\end{aligned}$$

Thus, there is $\varphi(\xi)=\ln(\xi+1)$ and $\psi\in\Psi$ where

$$\psi(\sigma,\varsigma,\tau,\rho) = \begin{cases}\frac{\ln(\ln(1+\max\{\sigma,\varsigma,\tau,\rho\}))}{\ln(1+\max\{\sigma,\varsigma,\tau,\rho\})}, & \sigma>0 \text{ or } \varsigma>0 \text{ or } \tau>0,\\ \rho\in[0,1), & \sigma=0,\ \varsigma=0,\ \tau=0,\ \rho=0,\end{cases}$$

so that

$$\begin{aligned}&\varphi\left(\Im\left(\Pi(\xi,\varkappa,\varpi,\eta),\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\right)\right)\\ &= \varphi\left(\left\|\Pi(\xi,\varkappa,\varpi,\eta)-\Pi\left(\widetilde{\xi},\widetilde{\varkappa},\widetilde{\varpi},\widetilde{\eta}\right)\right\|\right)\\ &\leq \psi\left(\Im\left(\pi\xi,\pi\widetilde{\xi}\right),\Im(\pi\varkappa,\pi\widetilde{\varkappa}),\Im(\pi\varpi,\pi\widetilde{\varpi}),\Im(\pi\eta,\pi\widetilde{\eta})\right)\\ &\quad\times\varphi\left(\aleph\left(\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\right)\right),\end{aligned}$$

where

$$\aleph\left(\pi\xi,\pi\widetilde{\xi},\pi\varkappa,\pi\widetilde{\varkappa},\pi\varpi,\pi\widetilde{\varpi},\pi\eta,\pi\widetilde{\eta}\right) = \max\left\{\begin{array}{l}\left\|\pi\xi-\pi\widetilde{\xi}\right\|,\|\pi\varkappa-\pi\widetilde{\varkappa}\|,\\ \|\pi\varpi-\pi\widetilde{\varpi}\|,\|\pi\eta-\pi\widetilde{\eta}\|\end{array}\right\}.$$

This leads to Π and π are an $\psi-\varphi-$contraction.

Finally, assumption (iv) implies that

$$(\mho^4)^{\Pi}_{\pi} = \left\{\begin{array}{c}(\xi,\varkappa,\varpi,\eta)\in\mho^4:\\ (\pi\xi,\Pi(\xi,\varkappa,\varpi,\eta)),\\ (\pi\varkappa,\Pi(\varkappa,\varpi,\eta,\xi)),\\ (\pi\varpi,\Pi(\varpi,\eta,\xi,\varkappa)),\\ (\pi\eta,\Pi(\eta,\xi,\varkappa,\varpi))\end{array}\right\}\neq\emptyset.$$

Therefore $(\xi^*,\varkappa^*,\varpi^*,\eta^*)\in\mho^4$ is a CQFP of Π and π, which is the solution to the problem *(13)*.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors Contributions

All authors contributed equally and significantly in writing this article.

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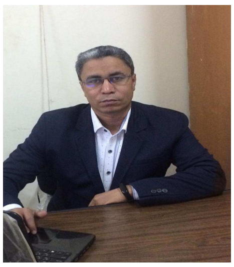

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