Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/160106

On Solving Partial Differential Equations by a Coupling of the Homotopy Perturbation Method and a New Integral Transform

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Received: 11 Aug. 2021, Revised: 1 Nov. 2021, Accepted: 2 Dec. 2021
Published online: 1 Jan. 2022

Abstract: In this work, we use a combination between a new integral transform and the homotopy perturbation method. This combination presents an accurate methodology to obtain an exact and numerical solutions for linear and nonlinear partial differential equations. The aim of using this new integral transform is to overcome the deficiency. It is mainly caused by unsatisfied conditions in the other semi-analytical methods such as HPM, VIM, and ADM. More than, this new method appears more applicable, it needs fewer computations. As in the provided numerical example, this method can be used in engineering computations.

Keywords: Homotopy perturbation technique; The new integral transform; Biological population equation; Evaluation equation; The parabolic partial differential equation.

1 Introduction

The numerical solutions of differential equations of integer order has been a hot topic in numerical and computional mathematics for a long time. There are many different methods have been used to estimate he solution of partial differential equations, such as Adomain decomposition method [1, 2], variational iteration method [3, 4] and homotopy perturbation method, it is work mentioning that the HPM, proposed first by Ji-Huan He [5–7], for solving linear and nonlinear partial differential equations.

The HPM has been introduced as a means to solve singular nonlinear differential equations [4], nonlinear wave equations [8] and nonlinear oscillators [9]

On the other hand the integral transformations played an essential role in many fields of science [10, 11], especially, engineering mathematics [12], mathematical physics [13], optics [14], image processing [15]. Many of these transforms have been used and appplied on theory and applications, such as Sumudu [16], Laplace [17, 18], Elzaki [19] and new integral transform [20]. Among these the most widely used is Laplace transform. here, the new integral transform has many interesting properties which make it rival to the Laplace transform. Our method yields the solution in terms of a rapid convergent series with easily compactable components for linear and nonlinear partial differential equations.

This article is organized as follows: In section 2, we introduce some basic definitions, proprieties for the new integral transform and we give some of the advantage of the considered methods. In section 3, we discussed the method used in this work. Some applications are given in section 4 to show accuracy. Finally, numerical results are discussed in section 5.

2 Basic Definition of the New Integral Transform (NT)

In this section we mension the following basic definitions and theorems of the new transform which are used in the present paper [20].

Definition of the New Transform. The transform of a function $f(t)$ is defined by

$$F(s) = Tf(t) = \int_0^\infty e^{-t} f\left(\frac{t}{s}\right) dt,\ s \in R,$$

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from this definition, we get

$$T(t^n) = \frac{n!}{s^n}.$$

Theorem 1. (Sufficient Condition). If a function h is piecewise continuous on $[0,\infty)$ and exponential order s_0, then the transform of h exists for $s > s_0$.

Theorem 2. (n^{th} Derivatives). If the functions $Tu, Tu', \cdots, Tu^{(n)}$ are well defined, $n = 1,2,3,\cdots$, then

$$Tu^{(n)} = s^n Tu - \sum_{k=0}^{n-1} s^{n-k} u^{(k)}(0), \tag{1}$$

3 The Advantage of the New Transform

The new transform has many interesting properties which make it reival to the Laplace transform. Some of these properties are:

1-The domain of the new transform is wider than or equal to the domain of Laplace transform.
2-The new transform can solve all the problem which would be solved by Laplace transform.
3-The unit step function in the t-domain is transformed to unity in the u-domain.
4-The differentiation and integration in the t-domain are equivalent to multipliciation and division of the transformed function $F(u)$ by u in the u-domain.

For more details see [20].

4 Analysis of the Method

To illustrate the modification algorithm of the NTHPM, we consider the following nonlinear partial differential equation with time derivatives of any order

$$L(u(x,t)) + R(u(x,t)) + N(u(x,t)) = g(x), \tag{2}$$

where L is linear differential operator $(L = d^n/dt^n)$ and $g(x,t)$ is the source term, subject to the initial conditions

$$\frac{\partial^m u(x,0)}{\partial t^m} = h_m(x), \quad m = 0,1,2,\cdots,n-1. \tag{3}$$

In view of the homotopy technique, we can constract the following homotopy

$$\begin{aligned} H(u(x,t),p)) &= (1-p)[L(u(x,t)) - L(u(x,0))] \\ &\quad + p[L(u(x,t)) + R(u(x,t)) - g(x,t)] \\ &= 0, \end{aligned} \tag{4}$$

where $p \in [0,1]$, the homotopy parameter t always changes from zero to unity. When $p = 0$, equation (4) becomes

$$L(u(x,t)) = L(u(x,0)), \tag{5}$$

and when $p = 1$, equation (4) turns out to the original Equation (2). Since $u(x,0)$ is a function of x only, equation (4) can be rewritten to be in the following form

$$\frac{\partial^n u(x,t)}{\partial t^n} + p[N(u(x,t)) + R(u(x,t)) - g(x,t)] = 0. \tag{6}$$

According to the homotopy technique, the basic assumption is that the solution of equation (6) can be written as a power series in p as

$$u(x,t) = \sum_{i=0}^{\infty} p^i u_i(x,t), \tag{7}$$

where $u_i(x,t)$ are unknown functions to be determined. Now taking in mind the initial conditions (3), the NT for Equation (6) gives

$$\begin{aligned} s^n T\{u(x,t)\} - \sum_{i=0}^{\infty} s^{n-k} u^k(x,0) + pT[R(u(x,t)) + N(u(x,t)) \\ - g(x,t)] = 0, \end{aligned} \tag{8}$$

R and N are represents the general linear and nonlinear differential operators respectively, again taking the inverse of the NT for Equation (8), we obtain

$$\begin{aligned} u(x,t) - T^{-1}\left\{\sum_{k=0}^{n-1} \frac{1}{s^k} u^k(x,0)\right\} + T^{-1}\left\{\frac{1}{s^n} pT\{[R(u(x,t)) \right. \\ \left. + N(u(x,t)) - g(x,t)]\}\right\} = 0. \end{aligned} \tag{9}$$

Substituting from Equation (7) into equation (9), yields

$$\begin{aligned} \sum_{i=0}^{\infty} p^i u_i(x,t) - T^{-1}\left\{\sum_{k=0}^{n-1} \frac{1}{s^k}\left(\sum_{i=0}^{\infty} p^i u_i^{(k)}(x,0)\right)\right\} \\ + T^{-1}\left\{\frac{1}{s^n} pT\left\{N\left(\sum_{i=0}^{\infty} p^i u_i(x,t)\right)\right.\right. \\ \left.\left. + R\left(\sum_{i=0}^{\infty} p^i u_i(x,t)\right) - g(x,t)\right\}\right\} = 0. \end{aligned} \tag{10}$$

Equating the identical powers of p, therefore, after doing some calculations for the NT and the inverse of NT we get the unknown functions $u_0, u_1, u_2, \cdots,$. Now substituting into Equation (7) with $p = 1$, we get the solution of the problems (2), (3).

5 Applications on NTHPM

we will discuss this method by giving some examples of linear and nonlinear partial differential equations.

Example 4.1. Consider the generalized biological population model of the form

$$u_t - u_{xx}^2 - u_{yy}^2 - u(1 - ru) = 0, \tag{11}$$

subject to the initial conditions

$$u(x,u,0)=e^{\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)}. \tag{12}$$

Assume that the solution of Equation 11 can be written as a power series of follows

$$u(x,y,t)=\sum_{i=0}^{\infty}p^{i}u_i(x,y,t), \tag{13}$$

substituting from (13) into (10) for $n=1$, $g=0$, $N=-(u_{xx}^2+u_{yy}^2-ru^2+u)$, $R=0$ and using the initial condition (12), yields

$$\sum_{i=0}^{\infty}p^{i}u_i(x,y,t)-T^{-1}\left(e^{\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)}\right)$$
$$-T^{-1}\left\{\frac{1}{s}pT\{\sum_{i=0}^{\infty}p^{i}(u_{ixx}^2+u_{ryy}^2-u_i(1-ru_i))\}\right\}=0. \tag{14}$$

On putting the coefficients to the power of p equal to zero in Equation (14), we obtain series of linear equations which are easily to solve by using Mathematica software to give

$$u_0=T^{-1}\{e^{\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)}\}=e^{\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)},$$

$$u_1=T^{-1}\left\{\frac{1}{s}T\{u_0+ru_0^2+2u_{0y}^2+2u_0+u_{0yy}+2u_{0x}^2+2u_0u_{0xx}\}\right\}=T^{-1}\left\{\frac{1}{s}e^{\frac{\sqrt{r}(x+y)}{2\sqrt{2}}}\right\}=e^{\frac{\sqrt{r}(x+y)}{2\sqrt{2}}}t,$$

$$u_2=T^{-1}\left\{\frac{1}{s}T\{-2ru_0u_1+u_1+4u_{0y}u_{1x}+2u_1u_{0yy}+2u_0u_{1yy}+4u_{0x}u_{1x}+2u_1u_{0xx}+2u_0u_{1xx}\}\right\}$$
$$=T^{-1}\left\{\frac{1}{s}\{e^{\frac{\sqrt{r}(x+y)}{2\sqrt{2}}}\cdot t\}\right\}=T^{-1}\left\{\frac{e^{\frac{\sqrt{2}(x+y)}{2\sqrt{2}}}}{s^2}\right\}$$
$$=e^{\frac{\sqrt{2}(x+y)}{2\sqrt{2}}}\frac{t^2}{2!},$$

$$u_3=T^{-1}\left\{\frac{1}{s}T\{u_2-ru_1^2-2ru_0u_2+2u_{1y}^2+4u_{0y}u_{2y}+2u_2u_{0yy}+2u_1u_{1yy}+2u_0u_{2yy}+2u_{1x}^2+4u_{0x}u_{2x}+2u_2u_{0xx}+2u_1u_{1xx}+2u_0u_{2xx}\}\right\}$$
$$=T^{-1}\left\{\frac{1}{s}T\left\{\frac{e^{\frac{\sqrt{r}(x+y)}{2\sqrt{2}}}}{2s}\cdot t^2\right\}\right\}$$
$$=T^{-1}\left\{\frac{1}{s}T\left\{\frac{e^{\frac{\sqrt{r}(x+y)}{2\sqrt{2}}}}{s^3}\right\}\right\}=e^{\frac{\sqrt{r}(x+y)}{2\sqrt{2}}}\frac{t^3}{3!}$$
$$\vdots$$

and so on. Proceeding as before the rest of compnents were obtained, and the 4-term approximate solution of the initial value problem (10)-(12) is given by

$$u(x,y,t)=e^{\frac{\sqrt{r}(x+y)}{2\sqrt{2}}}\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}\right].$$

In the closed form the solution $u(x,y,t)$ is readily to be

$$u(x,y,t)=e^{\frac{\sqrt{r}(x+y)}{2\sqrt{2}}}t. \tag{15}$$

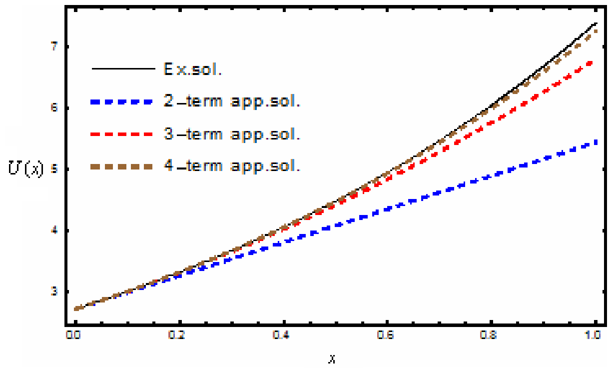

Fig. 1: The curves of the exact and approximate solutions of Example (4.1)

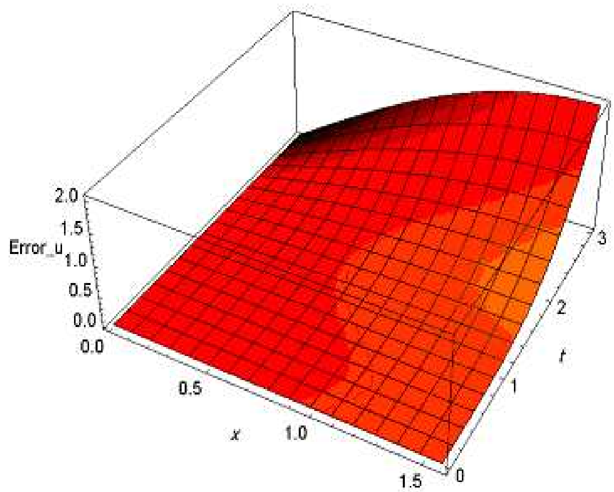

Fig. 2: The figure explain the surface error for Example (4.1)

Example 4.2. Consider the following linear evaluation equation

$$u_t+u_{xxxx}=0, \tag{16}$$

subject to

$$u(x,0)=\sin x. \tag{17}$$

By applying the same steps used in example (4.1) we can easily get

$$u_0(x,t)=\sin x,$$

$$u_1(x,t)=T^{-1}\left\{\frac{1}{s}T\{-u_{0xxxx}\}\right\}=T^{-1}\left\{\frac{1}{s}T\{-\sin x\}\right\}$$
$$=T^{-1}\left\{-\frac{\sin x}{s}\right\}=-\sin x\cdot t,$$

$$u_2=T^{-1}\left\{\frac{1}{s}T\{-t\cdot u_{1xxxx}\}\right\}=T^{-1}\left\{\frac{1}{s}T\{-t\sin x\}\right\}$$
$$=T^{-1}\left\{\frac{1}{s^2}\sin x\right\}=\sin x\cdot\frac{t^2}{2},$$

$$u_3=T^{-1}\left\{\frac{1}{s}T\left\{-u_{2xxxx}\cdot\frac{t^2}{2}\right\}\right\}$$
$$=T^{-1}\left\{\frac{1}{s}T\left\{\sin x\cdot\frac{t^2}{2}\right\}\right\}=T^{-1}\left\{\frac{1}{s}T\left\{\sin x\cdot\frac{t^2}{2}\right\}\right\}$$
$$=T^{-1}\left\{\frac{1}{s^3}\sin x\right\}=\sin x\cdot\frac{t^3}{3!}$$

$$\vdots$$

and so on. The 4-term approximate solution of the initial value problems (16)-(17) is given by

$$u(x,t)=\sum_{i=0}^{\infty}=\sin x-t\sin x+\frac{t^2}{!2}\sin x-\frac{t^3}{!3}\sin x$$
$$=\sin x\left(-t+\frac{t^2}{2!}-\frac{t^3}{3!}\right),$$

which in the closed form gives

$$u(x,t)=e^{-t}\sin x. \tag{18}$$

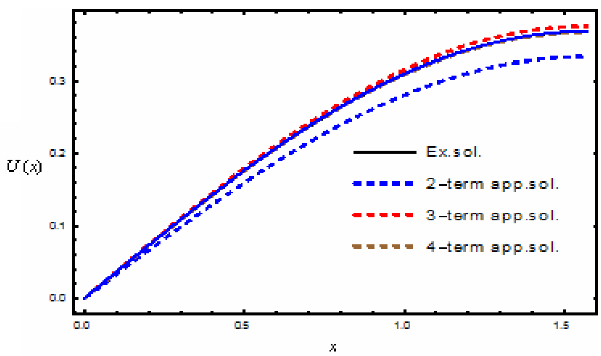

Fig. 3: The curves of the exact and approximate solutions of Example (4.2)

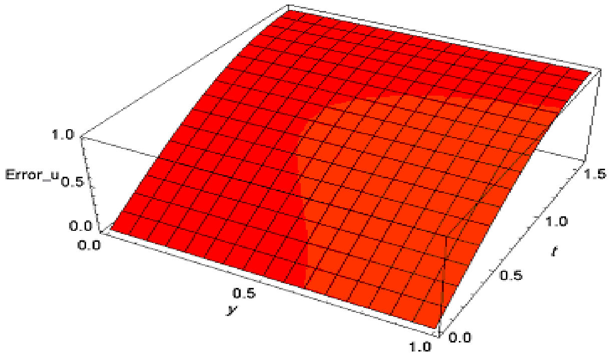

Fig. 4: The figure explain the surface error for Example (4.2)

Example 4.3. Consider the following singular fourth-order parabolic partial differential equation in the two space variable

$$u_{tt}+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)u_{xxxx}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)u_{yyyy}=0, \tag{19}$$

subject to the initial condition

$$u(x,y,0)=0,\ \ u_t(x,y,0)=2+\frac{x^6}{6!}+\frac{y^6}{6!}, \tag{20}$$

substituting from (13) into (10), taking $n=2$, $R=0$

$$N=2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)u_{xxxx}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)u_{yyyy},$$

and using the initial condition (20), we get

$$\sum_{i=0}^{\infty}p^i u_i(x,y,t)-T^{-1}\left\{2+\frac{x^6}{6!}+\frac{y^6}{6!}\right\}$$
$$+T^{-1}\left\{\frac{1}{s}pT\left\{2\cdot\left(\frac{1}{x}+\frac{x^4}{6!}\right)\sum_{i=0}^{\infty}u_{ixxxx}\right.\right.$$
$$\left.\left.+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\sum_{i=0}^{\infty}u_{iyyyy}\right\}\right\}=0, \tag{21}$$

by butting the coefficients to the powers of p in (21) equal zero, we get

$$u_0(x,t)=T^{-1}\left\{u(x,y,0)+\frac{1}{s}u_t'(x,y,0)\right\}$$
$$=T^{-1}\left\{\frac{1}{s}\left(2+\frac{x^6}{6!}\right)\right\}=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\cdot t$$
$$=a(x,y)\cdot t,$$

where $a(x,y)=2+\frac{x^6}{6!}+\frac{y^6}{6!}$

$$u_1=T^{-1}\left\{\frac{-1}{s^2}T\left\{\frac{2u_{0yyyy}}{y^2}+\frac{y^4u_{0yyyy}}{360}+\frac{2u_{0xxxx}}{x^2}\right.\right.$$

$$+\frac{x^4 x_{0xxxx}}{360}\Big\}\Big\}$$
$$=T^{-1}\Big\{-\frac{1}{s^2}T\{a(x,y)\cdot t\}=T^{-1}\Big\{-a(x,y)\frac{1}{s^3}\Big\}$$
$$=-a(x,y)\cdot\frac{t^3}{3!},$$

$$u_2=T^{-1}\Big\{-\frac{1}{s^2}T\Big\{\frac{2u_{1yyyy}}{y^2}+\frac{y^4u_{1yyyy}}{360}+\frac{2u_{1xxxx}}{x^2}$$
$$\frac{x^4u_{1xxxx}}{360}\Big\}\Big\}$$
$$=T^{-1}\Big\{\frac{1}{s^2}T\Big\{\frac{a(x,y}{3!}\cdot t^3\Big\}\Big\}=T^{-1}\Big\{\frac{a(x,y)}{s^5}\Big\}$$
$$=a(x,y)\cdot\frac{t^5}{5!},$$

$$u_3=T^{-1}\Big\{\frac{1}{s^2}T\Big\{\frac{2y_{2yyyy}}{y^2}+\frac{y^4u_{2yyyy}}{360}+\frac{2u_{2xxxx}}{x^2}$$
$$\frac{x^4u_{2xxxx}}{360}\Big\}\Big\}$$
$$=T^{-1}\Big\{\frac{1}{s^2}T\Big\{-a(x,y)\frac{t^5}{5!}\Big\}=T^{-1}\frac{a(x,y)}{s^7}\Big\}$$
$$=a(x,y)\cdot\frac{t^7}{7!},$$

The 4-term approximate solution of the initial value problems (19)-(20) takes the form

$$u(x,y,t)=a(x,y)\left[t-\frac{t^3}{3!}+\frac{t^5}{5!}\frac{t^7}{7!}\right],$$

in closed form

$$u(x,y,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{7!}\right)\sin t. \tag{22}$$

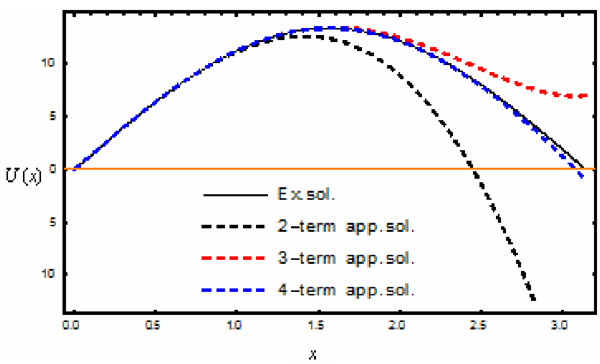

Fig. 5: The curves of the exact and approximate solutions of Example (4.3)

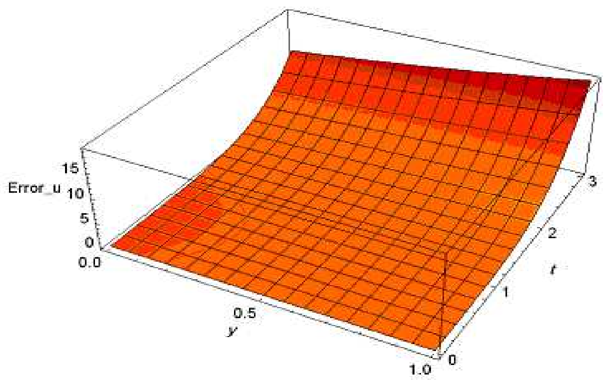

Fig. 6: The figure explain the surface error for Example (4.3)

Example 4.4. Consider the one dimentional linear system

$$u_t-v_x-u-v=0, \tag{23}$$

$$v_t-u_x+u+v=0, \tag{24}$$

subject to the initial conditions

$$u(x,0)=\sin x, \tag{25}$$

$$v(x,0)=\cos x. \tag{26}$$

Assume that the solutions of the equations (23) and (24) can be written as a power series of follows

$$u(x,t)=\sum_{i=0}^{\infty}p^iu_i(x,t), \tag{27}$$

$$v(x,t)=\sum_{i=0}^{\infty}p^iu_i(x,t), \tag{28}$$

substituting from equations (27) and (28) into Equation (10) and using the initial conditions (25) and (26) respectively, we get the following after equating the coefficients of the powers of p

$$u_1=T^{-1}\Big\{\frac{1}{s}T\{u_0+v_0+v_{0x}\}\Big\}=T^{-1}\Big\{\frac{1}{s}T\{\cos x\}\Big\}$$
$$=T^{-1}\Big\{\frac{\cos x}{s}\Big\}=t\cdot\cos x,$$

$$v_1=T^{-1}\Big\{\frac{1}{s}T\{-v_0-u_0-u_{0x}\}\Big\}=T^{-1}\Big\{-\frac{1}{s}T\{\sin x\}\Big\}$$
$$=-t\cdot\sin x,$$

$$u_2=T^{-1}\Big\{\frac{1}{s}T\{u_1+v_1+v_{1x}\}\Big\}=T^{-1}\Big\{-\frac{1}{s}T\{t\cdot\sin x\}\Big\}$$

$$= T^{-1}\left\{-\frac{1}{s^2}\sin x\right\} = -\frac{t^2}{2!}\cdot\sin x,$$

$$v_2 = T^{-1}\left\{-\frac{1}{s}T\{v_1+u_1+u_{1x}\}\right\}$$
$$= T^{-1}\left\{-\frac{1}{s}T\{t\cdot\cos x\}\right\} = T^{-1}\left\{-\frac{\cos x}{s^2}\right\}$$
$$= -\frac{t^2}{2!}\cdot\cos x,$$

$$u_3 = T^{-1}\left\{\frac{1}{s}T\{u_2+v_2+v_{2x}\}\right\}$$
$$= T^{-1}\left\{\frac{1}{s}T\{-\frac{t^2}{2!}\cdot\cos x\}\right\}T^{-1}\left\{-\frac{1}{s^3}\cos x\right\}$$
$$= -\frac{t^3}{3!}\cdot\sin x$$
$$\vdots$$

$$v_3 = T^{-1}\left\{-\frac{1}{s}T\{v_2+u_2-u_{2x}\}\right\}$$
$$= T^{-1}\left\{\frac{1}{s}T\{\frac{t^2}{2!}\cdot\sin x\}\right\}T^{-1}\left\{-\frac{\sin x}{s^3}\right\}$$
$$= \frac{t^3}{3!}\cdot\sin x$$
$$\vdots$$

and so on. Then the 4-term approximate solution of the Equations (23)-(24) is subject to the initial condition are

$$u(x,t) = \sin x + t\cos x - \frac{t^2}{2!}\sin x - \frac{t^3}{3!}\cos x$$
$$= \sin x\left[1-\frac{t^2}{2!}+\frac{t^4}{4!}\right] + \cos x\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}\right),$$

$$v(x,t) = \cos x + t\sin x - \frac{t^2}{2!}\cos x + \frac{t^3}{3!}\sin x$$
$$= \cos x\left[1-\frac{t^2}{2!}+\frac{t^4}{4!}\right] - \sin x\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}\right),$$

in closed form

$$u(x,t) = \sin(x+t),$$

$$v(x,t) = \cos(x+t).$$

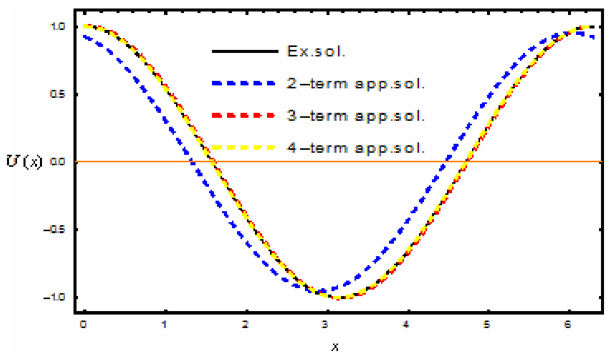

Fig. 7a

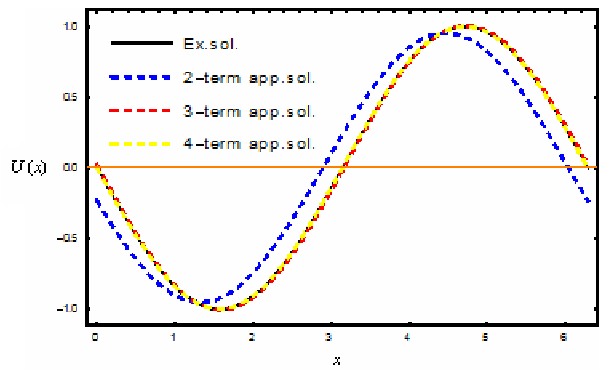

Fig. 7b

Fig. 7: The curves of the exact and approximate solutions of Example (4.4)

6 Discussion

In Fig. (1), Fig. (3), Fig. (5) and Fig. (7) we have plotted the exact, second, thaired and four's term-approximate solutions for the initial value problems (11-12), (16-17), (19-20) and (23-26). It is clear that, the approximate solutions $\to$ exact solutions and the rate of convergence can be increased by increasing the numbers of approximations.
Furthermore, in Fig. (2), Fig. (4) and Fig. (6) explain the surface errors for the initial value problems (11-12), (16-17) and (19-20), respectively where, error_u = $|$ex. solu.$u - 4^{\text{th}}$app. solu.$|$.
From these figures we achieved a very good approximation for the solution of the initial value problems and also for the system of partial differential equations.

7 Conclusion

In this paper, we present a numerical technique for solving the linear and nonlinear partial differential equations taking the advantage of the definition of our new integral transform and the simplicity of the homotopy perturbation method, we transform the initial value problems to a system of linear equations. By solving this system, the numerical solution is acquired. Numerical examples show that the numerical solutions is in very good accordance with the exact solutions.

Acknowledgement

The authors gratefully thank to Editor-in-Chief and the anonymous referees for their helpful and suggestive comments.

Conflict of interest

The authors declare that they have no conflict of interest.

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