Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/120225

A New-Fangled Ratio-Type Exponential Estimator for Population Variance using Auxiliary Information

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Received: 7 Jul. 2022, Revised: 21 Aug. 2022, Accepted: 2 Sep. 2022
Published online: 1 May 2023

Abstract: This manuscript provides new exponential ratio type estimator in simple random sampling for estimating the population variance using auxiliary information. The key purpose of this paper is to propose a new estimator and to increase the efficiency of the estimator for the population variance. The proposed exponential product-type estimator's bias and mean square error expressions have been derived. The optimum value of the characterizing scalar has been found, which minimizes the MSE of the proposed estimator. The proposed estimator was theoretically compared to competing estimators. It is shown that the proposed estimator outperforms its competitors. To demonstrate the practical use of different estimation formulae and empirically demonstrate the efficiency of the constructed estimators, a numerical analysis is conducted using real data sets.

Keywords: Simple random sampling, auxiliary variable, bias, mean squared error (MSE), Efficiency.

1 Introduction

The use of auxiliary data is widely accepted in survey sampling to improve the accuracy of population characteristics estimation. Estimation of finite population variance is important in a variety of fields, including health, body temperature, pulse rate, and blood pressure, among others. We use auxiliary information in this paper and implement a new estimator that outperforms competing estimators.
In general, supplementary data is used is survey sampling to improve sampling strategies and attain higher accuracy in estimations of particular population parameters like the mean, variance of the research variable. This evidence can be employed during the design stage (leading to stratified, systematic, or probability proportional to size sample designs, for example) as well as the estimating stage. When auxiliary data is employed at the estimation stage, it is widely known that the ratio, product and regression estimation approaches are frequently used in various circumstances. This is an example of in sampling theory, estimating population variance is a hot topic, and numerous efforts have been made to increase the precision of the estimates. A considerable deal has been written about survey sampling in the literature. A variety of strategies have been utilised, including different estimators and the use of auxiliary data. In the current manuscript, we have suggested a new Estimator of Population variance. Here the key purpose of this paper is to propose a new estimator and to increase the efficiency of the estimator for the population variance. Consider a finite population of size N, arbitrarily labeled $1,2....N$. Let y_i and x_i be, respectively, the values of the study variable y and the auxiliary variable x, in respect of the i^{th} unit$(i = 1,2,...N)$ of the population. When the auxiliary variable x is positively correlated with the study variable y and S_x^2, the population variance of x is known, product method of estimation is usually invoked to estimate the population variance S_y^2 of the study variable. The product method of estimation investigated by [1] and [2] is quite effective. [3] Studied Difference-cum-exponential efficient estimator of population variance.
Notations
In simple random sampling without replacement, we know that the sample variance s_y^2 provides an unbiased estimator population variance S_y^2.

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$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2$
and
$s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$
Accordingly, we define
$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{X})^2$
and
$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$
as the population and sample variances, respectively, for the auxiliary variable x.
let
$e_y = (\frac{s_y^2 - S_y^2}{)} S_x^2$
$e_\phi = (\frac{s_x^2 - S_x^2}{)} S_x^2$
such that,
$E(e_y) = E(e_\phi) = 0,$
$E(e_y^2) = \frac{1}{n}(\lambda_{40} - 1),$
$E(e_\phi^2) = \frac{1}{n}(\lambda_{04} - 1),$
$E(e_y e_\phi) = \frac{1}{n}(\lambda_{22} - 1),$
where
$\lambda_{pq} = \frac{\mu_{pq}}{\frac{p}{2}\frac{q}{2}}$
and
$\mu_{pq} = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})^p (y_i - \bar{Y})^q$
(p,q) being non-negative integers, μ_{02}, μ_{20} are the second order moments of x and y, respectively, and $C_x = S_x/\bar{X}$ is the coefficient of variance for auxiliary variable X.

2 Estimators in literature

With the above notations, the variance of the estimator s_y^2 is expressed as

$$V(s_y^2) = \frac{1}{n} S_y^2 (\lambda_{40} - 1) \tag{1}$$

[4] Proposed the ratio type estimator for estimating the population variance of the study variable as

$$s_{yR}^2 = \frac{s_y^2}{s_x^2} S_x^2 \tag{2}$$

Whose bias and mean square error, up to the first order of approximation are respectively,

$$B(s_{yR}^2) = \frac{1}{n} S_y^2 (\lambda_{04} - \lambda_{22}) \tag{3}$$

$$MSE(s_{yR}^2) = \frac{1}{n} S_y^4 (\lambda_{40} + \lambda_{04} - \lambda_{22}) \tag{4}$$

[5] suggested ratio-type exponential estimator for population variance in single phase sampling as

$$s_{yRe}^2 = s_y^2 \exp[\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}] \tag{5}$$

Whose bias and mean square error, up to the first order of approximation are respectively,

$$B(s_{yRe}^2) = \frac{1}{n} S_y^2 (\frac{\lambda_{04}}{8} - \frac{\lambda_{22}}{2} + \frac{3}{8}) \tag{6}$$

$$MSE(s_{yRe}^2) = \frac{1}{n} S_y^4 (\lambda_{40} + \frac{\lambda_{04}}{4} - \lambda_{22} - \frac{1}{4}) \tag{7}$$

3 Proposed Estimator

In this section, motivated by [6], we proposed the exponential type estimator for the population variance

$$s^2_{yRK} = s^2_y\left(\frac{s^2_x}{S^2_x}\right)^2 \exp\left[\frac{S^2_x - s^2_x}{S^2_x + s^2_x}\right] \tag{8}$$

Where α is a constant to be found. In order to reduce the MSEs of the proposed estimators. The following assumptions were used to investigate the sampling properties of the proposed estimators:
The proposed estimator t_{RK} can be expressed in terms of e's as:

$$s^2_{yRK} = s^2_y(1+e_y)(1+e_\phi)^\alpha \exp\left[\frac{S^2_x - S^2_x(1+e_\phi)}{S^2_x + S^2_x(1+e_\phi)}\right] \tag{9}$$

$$s^2_{yRK} = s^2_y(1+e_y)(1+e_\phi)^\alpha \exp\left(\frac{-\varepsilon_\phi}{2}\right)\left[1+\frac{\varepsilon_\phi}{2}\right]^{-1} \tag{10}$$

Expanding (14) and retaining terms up to the second power of e's,

$$s^2_{yRK} = s^2_y(1+e_y)(1+e_\phi)^\alpha \exp\left(\frac{-\varepsilon_\phi}{2}\right)\left[1+\frac{\varepsilon_\phi}{2}+\frac{\varepsilon^2_\phi}{4}\right] \tag{11}$$

$$s^2_{yRK} = s^2_y(1+e_y)\left[1+\alpha e_\phi + \frac{\alpha(\alpha-1)}{2}e^2_\phi + ...\right]\left[1+\frac{\varepsilon_\phi}{2}+\frac{3\varepsilon^2_\phi}{8}\right] \tag{12}$$

From(17)

$$\begin{aligned} s^2_{yRK} = s^2_y&\left[1-\tfrac{e_\phi}{2}+\tfrac{3e^2_\phi}{8}+\alpha e_\phi - \alpha e^2_\phi + \tfrac{\alpha^2 e^2_\phi}{2} + \tfrac{e_y}{2} + e_y + \alpha e_\phi e_y + \tfrac{e_y e_\phi}{2}\right] \\ &\left[1+\tfrac{\varepsilon_\phi}{2}+\tfrac{3\varepsilon^2_\phi}{8}\right] \end{aligned} \tag{13}$$

The bias of the proposed ratio type exponential estimator is,
$B(s^2_{yRK}) = E(s^2_{yRK}) - S^2_y$

$$\begin{aligned} B(s^2_{yRK}) = s^2_y E&\left[1-\tfrac{e_\phi}{2}+\tfrac{3e^2_\phi}{8}+\alpha e_\phi - \alpha e^2_\phi + \tfrac{\alpha^2 e^2_\phi}{2} + \tfrac{e_y}{2} + e_y + \alpha e_\phi e_y + \tfrac{e_y e_\phi}{2}\right] \\ &\left[1+\tfrac{\varepsilon_\phi}{2}+\tfrac{3\varepsilon^2_\phi}{8}\right] - S^2_y \end{aligned} \tag{14}$$

$$B(s^2_{yRK}) = s^2_y\left[\frac{3}{8}e^2_\phi - \alpha E(e^2_\phi) + \frac{\alpha^2}{2}E(e^2_\phi) + \alpha E(e_y e_\phi) + \frac{E(e_y e_\phi)}{2}\right] \tag{15}$$

$$B(s^2_{yRK}) = \frac{s^2_y}{n}\left[\frac{3}{8}(\lambda_{04}-1) - \alpha(\lambda_{04}-1) + \frac{\alpha^2}{2}(\lambda_{04}-1) + \alpha(\lambda_{22}-1) + \frac{(\lambda_{22}-1)}{2}\right] \tag{16}$$

The MSE of the proposed ratio type exponential estimator is,
$MSE(s^2_{yRK}) = E[s^2_{yRK}) - S^2_y]^2$

$$\begin{aligned} MSE(s^2_{yRK}) = E&\left[s^2_y\left(1-\tfrac{e_\phi}{2}+\tfrac{3e^2_\phi}{8}+\alpha e_\phi - \alpha e^2_\phi + \tfrac{\alpha^2 e^2_\phi}{2} + \tfrac{e_y}{2} + e_y + \alpha e_\phi e_y + \tfrac{e_y e_\phi}{2}\right]\right. \\ &\left.\left[1+\tfrac{\varepsilon_\phi}{2}+\tfrac{3\varepsilon^2_\phi}{8}\right) - S^2_y\right]^2 \end{aligned} \tag{17}$$

$$MSE(s^2_{yRK}) = s^4_y\left[e^2_y + \frac{1}{4}e^2_\phi - \alpha^2 E(e^2_\phi) - \alpha E(e^2_\phi) - E(e_\phi e_y) + \alpha E(e_\phi e_y)\right] \tag{18}$$

$$MSE(s^2_{yRK}) = \frac{s^4_y}{n}\left[(\lambda_{40}-\lambda_{22}) + \frac{1}{4}(\lambda_{04}-1) - \alpha^2(\lambda_{04}-1) + \alpha\lambda_{04} + \alpha\lambda_{22}\right] \tag{19}$$

In order to obtain optimum α to minimize $MSE(s^2_{yRK})$. Differentiating $MSE(s^2_{yRK})$ with respect to α and equating the derivative to zero, optimum value of α is given by:

$$\alpha_{opt} = \frac{(\lambda_{40}-\lambda_{22})}{2(\lambda_{04}-1)} \tag{20}$$

Substitute the value of α_{opt} in (20), we get the minimum value of $MSE(s^2_{yRK})$ as

$$MSE(s^2_{yRK})_{min} = \frac{s_y^4}{n}[(\lambda_{40}-\lambda_{22})+\frac{1}{4}(\lambda_{04}-1)-\frac{(\lambda_{40}-\lambda_{22})^2}{4(\lambda_{04}-1)}] \quad (21)$$

The newly proposed estimator s^2_{yRK} performs better than the simple variance estimator of population variance s^2_y if

$$V(s^2_y)-[MSE(s^2_{yRK})]_{min}>0$$
$$=1.132024>0$$

The newly proposed estimators^2_{yRK} performs better than the ratio-type estimator due to [4] for variance s^2_{yR} if

$$MSE(s^2_{yR})-[MSE(s_yRK^2)]_{min}>0$$
$$=0.093424>0$$

The newly proposed estimator s^2_{yRK} performs better than the the ratio-type exponential estimator due to [5] for variance s^2_{yRe} if

$$MSE(s^2_{yRe})-[MSE(s_yRK^2)]_{min}>0$$
$$=0.042324>0$$

4 Emprical Study

A data set is considered to exhibits the enactment of the suggested estimator; we use some real-life population. Description of the populations is given below:
Population I: [Source:[7]]
Y: Area under wheat I 1937 (in acres).
X: Cultivated area in 1931.
Statistical Description

$N=80$, $n=10$, $\lambda_{40}=3.5469$, $\lambda_{04}=3.2816$, $\lambda_{22}=2.6601$,

Population II: [Source: [8]]
Y: Figure of inhabitants in 1930
X: Figure of inhabitants in 1920
Statistical Description

$N=196$, $n=49$, $\lambda_{40}=8.5362$, $\lambda_{04}=7.3617$, $\lambda_{22}=7.8780$,

Table 1: Table 4.1: Competences of estimators and suggested estimator with respect to S^2_y

Estimators	MSE Population I	MSE Population II	PRE Population I	PRE Population II
s^2_y	2.5469	7.5362	-	-
s^2_{yR}	1.5083	0.1419	168.85	5310.92
s^2_{yRe}	1.4572	2.248625	174.78	335.15
s^2_{yRK}	0.277228	0.118902	918.70	6338.14

The percentage gain in efficiency of the proposed estimator,s^2_{yRK} over the competing estimators s^2_y, s^2_{yR} and s^2_{yRe} has been given in the table 4.1
It is clear from the above table that the newly proposed estimators^2_{yRK} performs better than the competing estimators.

5 Conclusion

From Tables 4.1, it is clear that the proposed product-type exponential estimator for estimating the population variance perform better than its competing estimators under conditions that hold good in practice. Hence the key purpose of this paper is to propose a new estimator and to increase the efficiency of the estimator for the population variance.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

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