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Some Hermite-Hadamard Weighted Integral Inequalities for (h,m)-Convex Modified Functions

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Abstract: In this paper, we establish some new type integral inequalities for differentiable (h,m)-convex modified of the second type functions, using generalized integrals.

Keywords: Hermite-Hadamard integral inequality, integral operators generalized, (h,m)-convex modified functions

1 Introduction

A function $\phi : I \to \mathbb{R}$, $I := [\xi_1, \xi_2]$ is said to be convex if $\phi\big(\lambda x + (1-\lambda)y\big) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$ holds for all $x, y \in I$ and $\lambda \in [0,1]$. If the above inequality is reversed, then the function ϕ will be the concave on $[\xi_1, \xi_2]$.

From a geometric point of view, this means that if we take three different points on the graph of ϕ, for example, A, B and C, with B between A and C, then B is located below the chord AC .

Convex functions have been the object of attention in recent decades and the original notion has been extended and generalized in various directions. Readers interested in the aforementioned development, can consult [1], where a panorama, practically complete, of these branches is presented.

One of the most important inequalities, for convex functions, is the famous Hermite–Hadamard inequality:

$$\phi\left(\frac{\xi_1+\xi_2}{2}\right) \leq \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\phi(x)dx \leq \frac{\phi(\xi_1)+\phi(\xi_2)}{2} \quad (1)$$

holds for any function ϕ convex on the interval $[\xi_1, \xi_2]$. This inequality was published by Hermite ([2]) in 1883 and, independently, by Hadamard in 1893 ([3]). It gives an estimation of the mean value of a convex function, and it is important to note that it also provides a refinement to the Jensen inequality. Several results can be consulted in [4,5,6,7,8,9,10,11,12,13,14,15,16,17] and references therein for more information and other extensions of the Hermite–Hadamard inequality.

Toader defined m-convexity in the following way:

Definition 1. *[18] The function $\phi : [0,\xi_2] \to \mathbb{R}$, $\xi_2 > 0$, is said to be m-convex, where $m \in [0,1]$, if*

$$\phi\,(tx + (1-t)y) \leq t\phi(x) + m(1-t)\phi(y)$$

holds for all $x, y \in [0,\xi_2]$ and $t \in [0,1]$.

If the above inequality holds in reverse, then we say that the function ϕ is m-concave.

The following definitions are successive extensions of the concept of convex function and, as we will see later, they are particular cases of our Definition.

In [19], Hudzik and Maligranda introduced the following definitions.

Definition 2. *Let $s \in (0,1]$. A function $\phi : [0,\xi_2] \to [0,\infty)$ with $\xi_2 > 0$, is said to be s-convex in the first sense if*

$$\phi(tx + (1-t)y) \leq t^s\phi(x) + (1-t^s)\phi(y),$$

for all $x, y \in [0,\xi_2]$ and $t \in [0,1]$.

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Definition 3. *Let $s \in (0,1]$. A function $\phi : [0,\xi_2] \to [0,\infty)$ with $\xi_2 > 0$, is said to be s-convex in the second sense if*

$$\phi(tx + (1-t)y) \leq t^s \phi(x) + (1-t)^s \phi(y)$$

for all $x, y \in [0,\xi_2]$ and $t \in [0,1]$.

An extended definition is given in [20].

Definition 4. *Let $s \in [-1,1]$. A function $\phi : [0,\xi_2] \to [0,\infty)$ with $\xi_2 > 0$ is said to be extended s-convex if*

$$\phi(tx + (1-t)y) \leq t^s \phi(x) + (1-t)^s \phi(y)$$

for all $x, y \in [0,\xi_2]$ and $t \in (0,1)$.

Mihesan [21] present the class of (a,m)-convex functions as follows.

Definition 5. *The function $\phi : [0,\xi_2] \to [0,\infty)$ with $\xi_2 > 0$, is said to be (a,m)-convex, where $a, m \in [0,1]$, if, for every $x, y \in [0,\xi_2]$ and $t \in [0,1]$*

$$\phi(tx + m(1-t)y) \leq t^a \phi(x) + m(1-t^a)\phi(y).$$

In [22] the following definition is introduced.

Definition 6. *Let $h : [0,1] \to \mathbb{R}$ be a nonnegative function and $h \not\equiv 0$. The nonnegative function $\phi : [0,\xi_2] \to [0,\infty)$ with $\xi_2 > 0$, is said to be (h,m)-convex on $[0,\xi_2]$ if*

$$\phi\,(tx + (1-t)y) \leq h(t)\phi(x) + mh(1-t)\phi(y)$$

is fulfilled for $m, t \in [0,1]$ and for all $x, y \in [0,\xi_2]$.

If the above inequality is reversed, then ϕ is said to be (h,m)-concave. Note that if $h(t) = t$ then the ϕ above definition reduces to the definition of m-convex function, if in addition, we put $m = 1$ then we obtain the definition of convex function.

In [23] the authors presented the class of s-(a,m)-convex functions as follows:

Definition 7. *A function $\phi : [0,\infty) \to [0,\infty)$ is said to be $s-(a,m)$-convex in the first sense, if for all $x, y \in [0,\infty)$ and $t \in [0,1]$, we obtain*

$$\phi\left(tx + m(1-t)y\right) \leq t^{as}\phi(x) + m\left(1 - t^{as}\right)\phi\left(\frac{y}{m}\right),$$

where $a, m \in [0,1]$ and for some fixed $s \in (0,1]$.

Definition 8. *A function $\phi : [0,\infty) \to [0,\infty)$ is said to be $s-(a,m)$-convex in the second sense, if for all $x, y \in [0,\infty)$ and $t \in [0,1]$, we obtain*

$$\phi\left(tx + m(1-t)y\right) \leq \left(t^a\right)^s \phi(x) + m\left(1 - t^a\right)^s \phi\left(\frac{y}{m}\right),$$

where $a, m \in [0,1]$ and for some fixed $s \in (0,1]$.

On the basis of these definitions, we will present the classes of functions that will be the basis of our work (see [24]).

Definition 9. *Let $h : [0,1] \to \mathbb{R}$ be a nonnegative function and $h \not\equiv 0$. The nonnegative function $\phi : [0,\infty) \to [0,\infty)$ is said to be $s-(h,m)$-convex modified of first type on $[0,\infty)$ if inequality*

$$\phi\,(tx + m(1-t)y) \leq h^s(t)\phi(x) + m(1 - h^s(t))\phi(y)$$

is fulfilled for $m, t \in [0,1], s \in [-1,1]$ and for all $x, y \in [0,\infty)$.

Definition 10. *Let $h : [0,1] \to \mathbb{R}$ be a positive function. The nonnegative function $\phi : [0,\infty) \to [0,\infty)$ is said to be $s-(h,m)$-convex modified of second type on $[0,\infty)$ if inequality*

$$\phi\,(tx + m(1-t)y) \leq h^s(t)\phi(x) + m(1 - h(t))^s \phi(y)$$

is fulfilled for $m, t \in [0,1]$, $s \in [-1,1]$ and for all $x, y \in [0,\infty)$.

Remark. From Definitions 9 and 10 we have

1. If $h(t) = t$, then ϕ is a m-convex function on $[0,\infty)$.
2. If $h(t) = t^a$ with $a \in (0,1]$, then ϕ is a $s-(a,m)$-convex function on $[0,\infty)$.
3. If $s = 1$, then ϕ is an (h,m)-convex function on $[0,\infty)$.
4. If $h(t) = t$, $s \in (0,1]$ and $m = 1$, then ϕ is a s-convex function on $[0,\infty)$.
5. If $h(t) = t$, $s \in [-1,1]$ and $m = 1$, then ϕ is an extended s-convex function on $[0,\infty)$.
6. If $h(t) = t$ and $s = m = 1$, then ϕ is a convex function on $[0,\infty)$.

All through the work we utilize the functions Γ (see [25,26]) and Γ_k (see [27]):

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt,$$

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{-t^k/k} dt.$$

where $z \in \mathbb{C}$, with $Re(z) > 0$ and $k > 0$.

Note that if $k \to 1$, then $\Gamma_k(z) \to \Gamma(z)$, $\Gamma_k(z) = (k)^{\frac{z-k}{k}} \Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$.

The following functions will also be required:

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt,$$

$$B_1(a,b) = B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt.$$

To encourage comprehension of the subject, we present the definition of Riemann-Liouville fractional integral (with $0 \leq \xi_1 < t < \xi_2 < \infty$). The first is the classic Riemann-Liouville fractional integrals.

Definition 11. *[28] Let $\phi \in L_1[\xi_1,\xi_2]$. The Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ are defined by (right and left respectively)*

$$^{\alpha}I_{\xi_1^+}(\phi(x)) = \frac{1}{\Gamma(\alpha)}\int_{\xi_1}^{x}(x-t)^{\alpha-1}\phi(t)\,dt, \quad x > \xi_1$$

$$^{\alpha}I_{\xi_2^-}(\phi(x)) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\xi_2}(t-x)^{\alpha-1}\phi(t)\,dt, \quad x < \xi_2.$$

Next we present the weighted integral operators, which will be the basis of our work.

Definition 12. *Let $\phi \in L_1(\xi_1,\xi_2)$ and let $w:[0,\infty)\to[0,\infty)$ be a continuous function with first and second order derivatives piecewise continuous on $[0,\infty)$. The weighted fractional integrals are defined by (right and left, respectively)*

$$^{(n+1)}I^{w}_{\xi_1^+}(\phi(x)) = \int_{\xi_1}^{x} w'\left(\frac{n+1}{\xi_2-\xi_1}(x-t)\right)\phi(t)\,dt, \quad x > \xi_1,$$

$$^{(n+1)}I^{w}_{\xi_2^-}(\phi(x)) = \int_{x}^{\xi_2} w'\left(\frac{n+1}{\xi_2-\xi_1}(t-x)\right)\phi(t)\,dt, \quad x < \xi_2.$$

Remark. If $w'(t) = \frac{(\xi_2-\xi_1)t^{\alpha-1}}{\Gamma(\alpha)}$ and $n=0$, then we obtain the Riemann-Liouville fractional integral, right and left respectively.

Remark. Putting $w'(t) \equiv 1$ and $n=0$, we obtain the classical Riemann integral.

In this work, we present some variants of the inequality (1), for (h,m)-convex modified functions, within the framework of the generalized integral operators of the Definition 12.

2 Hermite-Hadamard type inequalities for (h,m)-convex modified functions

To establish our results, we need the following Lemma.

Lemma 1. *Let ϕ be a real function defined on some interval $[\xi_1,\xi_2]\subset\mathbb{R}$, and differentiable on (ξ_1,ξ_2). If $\phi' \in L_1(a,b)$, then we have the following equality*

$$\begin{aligned}
&\frac{\xi_2-\xi_1}{n+1}\times\\
&\left\{w(1)\left[\phi(\xi_2)-\phi(\xi_1)\right]-w(0)\left[\phi\left(\frac{\xi_1+n\xi_2}{n+1}\right)-\phi\left(\frac{n\xi_1+\xi_2}{n+1}\right)\right]\right\}\\
&+\,^{(n+1)}I^{w}_{\xi_1^+}\left(\phi\left(\frac{n\xi_1+\xi_2}{n+1}\right)\right)-\,^{(n+1)}I^{w}_{\xi_2^-}\left(\phi\left(\frac{\xi_1+n\xi_2}{n+1}\right)\right)\\
&=\left(\frac{\xi_2-\xi_1}{n+1}\right)^2\times\\
&\int_0^1 w(t)\left[\phi'\left(\frac{(1-t)\xi_1+(n+t)\xi_2}{n+1}\right)+\phi'\left(\frac{(n+t)\xi_1+(1-t)\xi_2}{n+1}\right)\right]dt. \qquad (2)
\end{aligned}$$

Proof. First note that

$$\begin{aligned}
&\int_0^1 w(t)\left[\phi'\left(\frac{(1-t)\xi_1+(n+t)\xi_2}{n+1}\right)+\phi'\left(\frac{(n+t)\xi_1+(1-t)\xi_2}{n+1}\right)\right]dt\\
&=\int_0^1 w(t)\phi'\left(\frac{(1-t)\xi_1+(n+t)\xi_2}{n+1}\right)dt+\int_0^1 w(t)\phi'\left(\frac{(n+t)\xi_1+(1-t)\xi_2}{n+1}\right)dt\\
&=I_1+I_2.
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
I_1&=\frac{n+1}{\xi_2-\xi_1}\left[w(1)\phi(\xi_2)-w(0)\phi\left(\frac{\xi_1+n\xi_2}{n+1}\right)\right]\\
&\quad-\frac{n+1}{\xi_2-\xi_1}\int_0^1 w'(t)\phi\left(\frac{1-t}{n+1}\xi_1+\frac{n+t}{n+1}\xi_2\right)dt\\
&=\frac{n+1}{\xi_2-\xi_1}\left[w(1)\phi(\xi_2)-w(0)\phi\left(\frac{\xi_1+n\xi_2}{n+1}\right)\right]\\
&\quad-\left(\frac{n+1}{\xi_2-\xi_1}\right)^2\int_{\frac{\xi_1+n\xi_2}{n+1}}^{\xi_2} w'\left(\frac{T-\frac{\xi_1+n\xi_2}{n+1}}{\frac{\xi_2-\xi_1}{n+1}}\right)\phi(T)dT\\
&=\frac{n+1}{\xi_2-\xi_1}\left[w(1)\phi(\xi_2)-w(0)\phi\left(\frac{\xi_1+n\xi_2}{n+1}\right)\right]\\
&\quad-\left(\frac{n+1}{\xi_2-\xi_1}\right)^2\,^{(n+1)}I^{w}_{\xi_2^-}\left(\phi\left(\frac{\xi_1+n\xi_2}{n+1}\right)\right).
\end{aligned}$$

Analogously

$$\begin{aligned}
I_2&=-\frac{n+1}{\xi_2-\xi_1}\left[w(1)\phi(\xi_1)-w(0)\phi\left(\frac{n\xi_1+\xi_2}{n+1}\right)\right]\\
&\quad+\left(\frac{n+1}{\xi_2-\xi_1}\right)^2\,^{(n+1)}I^{w}_{\xi_1^+}\left(\phi\left(\frac{n\xi_1+\xi_2}{n+1}\right)\right).
\end{aligned}$$

After adding I_1+I_2, and grouping appropriately, we obtain the inequality sought.

To realize the scope and generality of our previous result, we will present several particular cases, known from the literature.

Let's consider $n=0$. If we take $w(t)=w_1(t)+w_2(t)$ and write the right side of (2) as

$$I_1=\int_0^1 w_1(t)\phi'\left(\frac{(1-t)\xi_1+(n+t)\xi_2}{n+1}\right)dt \quad \text{and}$$
$$I_2=\int_0^1 w_2(t)\phi'\left(\frac{(n+t)\xi_1+(1-t)\xi_2}{n+1}\right)dt,$$

where $w_1=B_t(n+1,\alpha-n)-B(n+1,\alpha-n)$ and $w_2=B(n+1,\alpha-n)-B_t(n+1,\alpha-n)$, we get Lemma 1 of [39].

If we use only I_2 and consider $w(t)=\frac{1}{2}\left[B_t(n+1,\alpha-n)-B_t(n+1,\alpha-n)\right]$ the Lemma 3.1 of [52] is obtained.

If we only work with I_2 and consider $w(t)$ defined by

$$w(t)=\begin{cases} t, & [0,1/2) \\ t-1, & [1/2,1] \end{cases}$$

so we have Lemma 2.1 of [40].

Similarly, putting $w(t)=\dfrac{1-2t}{2}$ we will have Lemma 2.1 of [34] (also see Lemma 2.1 of [38]).

Working only with I_2, $n=0$ and using $w(t)=(1-t)^{\alpha/k}-t^{\alpha/k}$, the Lemma 1 of [37] is obtained from Lemma 1.

Lemma 2 of [50] (see also [43]) is obtained from the previous result taking $w(t)=\dfrac{(1-t)^{\alpha}-t^{\alpha}}{2}$.

In the same way, putting $w(t)=\dfrac{(1-t)^{\alpha/k}-t^{\alpha/k}}{2}$ and using I_1, we get Lemma 1 of [41], if we use I_2, then Lemma 2.3 of [36] is obtained.

Working with I_2 and defining

$$w(t)=\begin{cases} t, & t\in\left[0,\dfrac{\xi_2-x}{\xi_2-\xi_1}\right] \\ t-1, & t\in\left[\dfrac{\xi_2-x}{\xi_2-\xi_1},1\right], \end{cases}$$

a variant of Lemma 1 of [31] is obtained.

Lemma 1 of [44] is derived from our result, putting

$$w(t)=\begin{cases} t-\dfrac{\xi_2-x}{\xi_2-\xi_1}\lambda, & t\in\left[0,\dfrac{\xi_2-x}{\xi_2-\xi_1}\right] \\ t-1+\dfrac{\xi_2-x}{\xi_2-\xi_1}\lambda, & t\in\left[\dfrac{\xi_2-x}{\xi_2-\xi_1},1\right], \end{cases}$$

$\lambda\in[0,1]$ and considering I_2.

The reader will be able to verify, without much difficulty, that under different variants of the function $w(t)$ we can obtain Lemma 2 of [45], Lemma 1.1 of [53] (see also Lemma 2 of [42]), Lemma 2.1 from [49], Lemma 2.1 from [60], Lemma 2.1 from [57], Lemma 1.6 from [48], Lemma 2.1 from [29], Lemma 1 of [32], Lemma 2.1 of [51], and Lemma 2.1 of [47].

With $w(t)=(1-t)^{\alpha}$ and $n=0$, we obtain a new result, for Riemann-Liouville integrals.

If $n=1$, Lemma 1 of [30] and Lemma 1 of [56] can be obtained from our result, under the appropriate definition of $w(t)=w_1+w_2$ (see also [55]).

Our first main result is the following.

Theorem 1. *Let $\phi: I\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable function on I°(interior of I) such that $\phi'\in L_1[\xi_1,\xi_2/m]$. Under the assumptions of Lemma 1 if $|\phi'|$ is s-(h,m)-convex modified of second type on $[\xi_1,\xi_2/m]$, we have the following inequality*

$$\left|\mathscr{A}+{}^{(n+1)}I^{w}_{\xi_1^+}\phi\left(\frac{n\xi_1+\xi_2}{n+1}\right)-{}^{(n+1)}I^{w}_{\xi_2^-}\phi\left(\frac{\xi_1+n\xi_2}{n+1}\right)\right|$$
$$\leq\left(\frac{\xi_2-\xi_1}{n+1}\right)^2\left(|\phi'(\xi_1)|\mathscr{B}+m\left|\phi'\left(\frac{\xi_2}{m}\right)\right|\mathscr{C}\right),$$

where

$$\mathscr{A}=\frac{\xi_2-\xi_1}{n+1}\Big\{w(1)(\phi(\xi_1)+\phi(\xi_2))$$
$$-w(0)\left(\phi\left(\frac{\xi_1+n\xi_2}{n+1}\right)-\phi\left(\frac{n\xi_1+\xi_2}{n+1}\right)\right)\Big\},$$

$$\mathscr{B}=\int_0^1 w(t)\left[h^s\left(\frac{n+t}{n+1}\right)+h^s\left(\frac{1-t}{n+1}\right)\right]dt,$$

$$\mathscr{C}=\int_0^1 w(t)\left[\left(1-h\left(\frac{1-t}{n+1}\right)\right)^s+\left(1-h\left(\frac{n+t}{n+1}\right)\right)^s\right]dt.$$

Proof. From (2) and the s-(h,m)-convex modified of second type of $|\phi'|$, we obtain

$$\left|\mathscr{A} + {}^{(n+1)}I^{w}_{\xi_1^+}\phi\left(\frac{n\xi_1+\xi_2}{n+1}\right) - {}^{(n+1)}I^{w}_{\xi_2^-}\phi\left(\frac{\xi_1+n\xi_2}{n+1}\right)\right|$$
$$\leq \left(\frac{\xi_2-\xi_1}{n+1}\right)^2 \times \left[\int_0^1 w(t)\left|\phi'\left(\frac{(1-t)\xi_1+(n+t)\xi_2}{n+1}\right)\right|dt + \int_0^1 w(t)\left|\phi'\left(\frac{(n+t)\xi_1+(1-t)\xi_2}{n+1}\right)\right|dt\right]$$
$$\leq \left(\frac{\xi_2-\xi_1}{n+1}\right)^2 \times \left[\int_0^1 w(t)\left(h^s\left(\frac{1-t}{n+1}\right)|\phi'(\xi_1)| + m\left(1-h\left(\frac{n+t}{n+1}\right)\right)^s\left|\phi'\left(\frac{\xi_2}{m}\right)\right|\right)dt + \int_0^1 w(t)\left(h^s\left(\frac{n+t}{n+1}\right)|\phi'(\xi_1)| + m\left(1-h\left(\frac{1-t}{n+1}\right)\right)^s\left|\phi'\left(\frac{\xi_2}{m}\right)\right|\right)dt\right]$$
$$= \left(\frac{\xi_2-\xi_1}{n+1}\right)^2 \times \left[|\phi'(\xi_1)|\int_0^1 w(t)\left(h^s\left(\frac{1-t}{n+1}\right)+h^s\left(\frac{n+t}{n+1}\right)\right)dt + m\left|\phi'\left(\frac{\xi_2}{m}\right)\right|\int_0^1 w(t)\left[\left(1-h\left(\frac{n+t}{n+1}\right)\right)^s + \left(1-h\left(\frac{1-t}{n+1}\right)\right)^s\right]dt\right].$$

*Remark.*Considering $n=0$, some known results, which can be obtained as particular cases of the previous Theorem, are the following:

(a)Theorem 1 of [31], only for I_2, $w(t) = \begin{cases} t, & t\in\left[0,\frac{\xi_2-x}{\xi_2-\xi_1}\right); \\ t-1, & t\in\left[\frac{\xi_2-x}{\xi_2-\xi_1},1\right], \end{cases}$ and convex functions.

(b)Theorem 2.1 from [33] (case $q=1$), for m-convex functions, $h(t)=t$ and $s=1$.

(c)Theorem 2.2 from [34], obtained for convex functions, using $w(t)=1-2t$ and using only I_2.

(d)Theorem 2.4 of [36] for convex functions, $h(t)=t$ and $s=m=1$, a known result for k-fractional integrals.

(e)Theorem 2.2 of [40], obtained for convex functions and
$$w(t) = \begin{cases} t, & [0,1/2) \\ t-1, & [1/2,1] \end{cases}$$

(f)Theorem 2.3 of [23], with I_2, $w(t)=1-2t$ and s-(a,m)-convex functions are considered.

(g)Theorem 7 from [42], for s-convex functions.

(h)Theorem 3.1 from [47], where I_2 is used and the interval $[0,1]$ is divided, using $w_1=\lambda-t$ for $[0,1/2]$ and $w_2=\mu-t$ for $[1/2,1]$, where λ and μ real numbers such that $0\leq\lambda\leq 1/2\leq\mu\leq 1$.

(i)Theorem 3 of [50], for convex functions and taking $w(t)=(1-t)^\alpha - t^\alpha$ and I_2, a result for Riemann-Liouville fractional integrals.

(j)The first part of Theorem 5 of [54], statement for fractional integrals of the Riemann-Liouville type, using only I_2 and $w(t)=(1-t)^\alpha-t^\alpha$, in function class h-convex.

(k)Theorem 5 of [59] working with $w(t)=(1-t)^\alpha-t^\alpha$ and using I_2, a valid inequality for fractional integrals.

Remark. If in the Theorem 1 we make $n=1$, $m=1$ and $h(s)=s$ (that is, we consider s-convex functions in the second sense), we will obtain Theorem 10 of [56], putting $w(t)=w_1(t)+w_2(t)$ where

$$w_1 = \frac{t\xi_2+(1-t)\xi_1}{4}; \qquad w_2 = \frac{\xi_1+(1-t)\xi_2}{4}.$$

Refinements of the previous results, can be obtained by imposing new additional conditions on $|\phi'|^q$, $1/p+1/q=1$.

Theorem 2.*Let $\phi : I\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable function on I° such that $\phi'\in L_1\left[\xi_1,\frac{\xi_2}{m}\right]$. Under the assumptions of Lemma 1 if $|\phi'|^q$, $(q\geq 1)$ is s-(h,m)-convex modified of second type on $\left[\xi_1,\frac{\xi_2}{m}\right]$, we have*

$$\left|\mathscr{A} + {}^{(n+1)}I^{w}_{\xi_1^+}\left(\phi\left(\frac{n\xi_1+\xi_2}{n+1}\right)\right) - {}^{(n+1)}I^{w}_{\xi_2^-}\left(\phi\left(\frac{\xi_1+n\xi_2}{n+1}\right)\right)\right|$$
$$\leq \left(\frac{\xi_2-\xi_1}{n+1}\right)^2 \|w\|_p \times \left[\left(|\phi'(\xi_1)|^q\int_0^1 h^s\left(\frac{n+t}{n+1}\right)dt + m\left|\phi'\left(\frac{\xi_2}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{1-t}{n+1}\right)\right)^s dt\right)^{1/q} + \left(|\phi'(\xi_1)|^q\int_0^1 h^s\left(\frac{1-t}{n+1}\right)dt + m\left|\phi'\left(\frac{\xi_2}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{n+t}{n+1}\right)\right)^s dt\right)^{1/q}\right]$$

where

$$\mathscr{A} = \frac{\xi_2-\xi_1}{n+1}\left\{w(1)\big(\phi(\xi_1)+\phi(\xi_2)\big) - w(0)\left(\phi\left(\frac{\xi_1+n\xi_2}{n+1}\right) - \phi\left(\frac{n\xi_1+\xi_2}{n+1}\right)\right)\right\},$$

and $1/p+1/q=1$.

Proof. From (2), Hölder's inequality and the $s-(h,m)$-convex modified of second type of $\left|\phi'\right|^q$, we obtain

$$\begin{aligned}
&\left|\mathscr{A}+{}^{(n+1)}I^w_{\xi_1^+}\phi\left(\frac{n\xi_1+\xi_2}{n+1}\right)-{}^{(n+1)}I^w_{\xi_2^-}\phi\left(\frac{\xi_1+n\xi_2}{n+1}\right)\right|\\
&\le\left(\frac{\xi_2-\xi_1}{n+1}\right)^2\times\\
&\quad\left[\int_0^1 w(t)\left|\phi'\left(\frac{(1-t)\xi_1+(n+t)\xi_2}{n+1}\right)\right|dt\right.\\
&\quad\left.+\int_0^1 w(t)\left|\phi'\left(\frac{(n+t)\xi_1+(1-t)\xi_2}{n+1}\right)\right|dt\right]\\
&\le\left(\frac{\xi_2-\xi_1}{n+1}\right)^2\left(\int_0^1 w^p(t)dt\right)^{1/p}\times\\
&\quad\left[\left(\int_0^1\left|\phi'\left(\frac{(1-t)\xi_1+(n+t)\xi_2}{n+1}\right)\right|^q dt\right)^{1/q}\right.\\
&\quad\left.+\left(\int_0^1\left|\phi'\left(\frac{(n+t)\xi_1+(1-t)\xi_2}{n+1}\right)\right|^q dt\right)^{1/q}\right]\\
&\le\left(\frac{\xi_2-\xi_1}{n+1}\right)^2\|w\|_p\times\\
&\quad\left[\left(\int_0^1\left(h^s\left(\frac{1-t}{n+1}\right)|\phi'(\xi_1)|^q\right.\right.\right.\\
&\quad\left.\left.+m\left(1-h\left(\frac{n+t}{n+1}\right)\right)^s\left|\phi'\left(\frac{\xi_2}{m}\right)\right|^q\right)dt\right)^{1/q}\\
&\quad+\left(\int_0^1\left(h^s\left(\frac{n+t}{n+1}\right)|\phi'(\xi_1)|^q\right.\right.\\
&\quad\left.\left.\left.+m\left(1-h\left(\frac{1-t}{n+1}\right)\right)^s\left|\phi'\left(\frac{\xi_2}{m}\right)\right|^q\right)dt\right)^{1/q}\right]\\
&\le\left(\frac{\xi_2-\xi_1}{n+1}\right)^2\|w\|_p\times\\
&\quad\left[\left(|\phi'(\xi_1)|^q\int_0^1\left(h^s\left(\frac{1-t}{n+1}\right)dt\right.\right.\right.\\
&\quad\left.+m\left|\phi'\left(\frac{\xi_2}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{n+t}{n+1}\right)\right)^s\right)dt\right)^{1/q}\\
&\quad+\left(|\phi'(\xi_1)|^q\int_0^1\left(h^s\left(\frac{n+t}{n+1}\right)dt\right.\right.\\
&\quad\left.\left.\left.+m\left|\phi'\left(\frac{\xi_2}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{1-t}{n+1}\right)\right)^s\right)dt\right)^{1/q}\right].
\end{aligned}$$

Remark. If we take $n=1$, we obtain the Theorem 11 of [56] as particular case. Other known results from the literature that can be obtained as particular cases of the previous Theorem are the following: Theorem 3.2 of [35], Theorem 6 of [59], Theorem 2.1 (second part), Theorem 2.3 [34], Theorem 6 of [55], Theorem 8 of [42], the second part of Theorem 1 of [44], Theorem 1 of [46], Theorem 2.11 of [23], Theorem 5 of [54] and Theorem 2 of [31].

Theorem 3.*Let $\phi : I\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable function on I° such that $\phi'\in L_1\left[\xi_1,\frac{\xi_2}{m}\right]$. Under the assumptions of Lemma 1 if $\left|\phi'\right|^q$, $(q\ge 1)$ is s-(h,m)-convex modified of second type on $\left[\xi_1,\frac{\xi_2}{m}\right]$, we obtain*

$$\begin{aligned}
&\left|\mathscr{A}+{}^{(n+1)}I^w_{\xi_1^+}\left(\phi\left(\frac{n\xi_1+\xi_2}{n+1}\right)\right)-{}^{(n+1)}I^w_{\xi_2^-}\left(\phi\left(\frac{\xi_1+n\xi_2}{n+1}\right)\right)\right|\\
&\le\left(\frac{\xi_2-\xi_1}{n+1}\right)^2\Delta\left[\left(|\phi'(\xi_1)|^q\mathscr{D}_{11}+m\left|\phi'\left(\frac{\xi_2}{m}\right)\right|^q\mathscr{D}_{12}\right)^{1/q}\right.\\
&\quad\left.+\left(|\phi'(\xi_1)|^q\mathscr{D}_{21}+m\left|\phi'\left(\frac{\xi_2}{m}\right)\right|^q\mathscr{D}_{22}\right)^{1/q}\right]
\end{aligned}$$

with $\mathscr{A}$ as before, $\Delta=\left(\int_0^1 w(t)dt\right)^{1/p}$, $1/p+1/q=1$,

$$\begin{aligned}
\mathscr{D}_{11}&=\int_0^1 w(t)h^s\left(\frac{n+t}{n+1}\right)dt,\\
\mathscr{D}_{12}&=\int_0^1 w(t)\left(1-h\left(\frac{1-t}{n+1}\right)\right)^s dt,\\
\mathscr{D}_{21}&=\int_0^1 w(t)h^s\left(\frac{1-t}{n+1}\right)dt,\quad\text{and}\\
\mathscr{D}_{22}&=\int_0^1 w(t)\left(1-h\left(\frac{n+t}{n+1}\right)\right)^s dt.
\end{aligned}$$

Proof. From (2), power mean inequality and the s-(h,m)-convex modified of second type of $\left|\phi'\right|^q$, we obtain

$$\left|\mathscr{A}+{}^{(n+1)}I^w_{\xi_1^+}\phi\left(\frac{n\xi_1+\xi_2}{n+1}\right)-{}^{(n+1)}I^w_{\xi_2^-}\phi\left(\frac{\xi_1+n\xi_2}{n+1}\right)\right|$$

$$\begin{aligned}
&\le\left(\frac{\xi_2-\xi_1}{n+1}\right)^2\times\\
&\quad\left[\int_0^1 w(t)\left|\phi'\left(\frac{(1-t)\xi_1+(n+t)\xi_2}{n+1}\right)\right|dt\right.\\
&\quad\left.+\int_0^1 w(t)\left|\phi'\left(\frac{(n+t)\xi_1+(1-t)\xi_2}{n+1}\right)\right|dt\right]
\end{aligned}$$

$$\begin{aligned}
&\le\left(\frac{\xi_2-\xi_1}{n+1}\right)^2\left(\int_0^1 w(t)dt\right)^{1/p}\times\\
&\left[\left(\int_0^1 w(t)\left|\phi'\left(\frac{(1-t)\xi_1+(n+t)\xi_2}{n+1}\right)\right|^q dt\right)^{1/q}\right.\\
&\left.+\left(\int_0^1 w(t)\left|\phi'\left(\frac{(n+t)\xi_1+(1-t)\xi_2}{n+1}\right)\right|^q dt\right)^{1/q}\right]
\end{aligned}$$

$$\leq \left(\frac{\xi_2-\xi_1}{n+1}\right)^2 \Delta \times$$
$$\left[\left(\int_0^1 w(t)\left[h^s\left(\frac{1-t}{n+1}\right)|\phi'(\xi_1)|^q + m\left(1-\left(\frac{n+t}{n+1}\right)\right)^s \left|\phi'\left(\frac{\xi_2}{m}\right)\right|^q\right]dt\right)^{1/q} + \left(\int_0^1 w(t)\left[h^s\left(\frac{n+t}{n+1}\right)|\phi'(\xi_1)|^q + m\left(1-h\left(\frac{1-t}{n+1}\right)\right)^s \left|\phi'\left(\frac{\xi_2}{m}\right)\right|^q\right]dt\right)^{1/q}\right].$$

Remark. The Theorem 12 of [56] can be obtained from Theorem 2 putting $n = 1$ and considering s-convex functions. Additionally, the following results: Theorem 3.2 of [47], Theorem 2.13 of [23], Theorem 5 of [31], Theorem 9 of [42], Theorem 7 [55], Theorem 2 of [44], Theorem 2.3 of [33], Theorem 7 of [59] and Theorem 3.6 of [35], can be obtained as particular cases of the theorem previous.

3 Conclusions

In this work we have presented some integral inequalities, which generalize several of those known from the literature, whether for fractional operators or not. We have pointed out the breadth and strength of our results throughout the work.

The generality of the results obtained can also be understood as being valid for convex functions, h-convex functions, m-convex functions and s-convex functions in the second sense, defined in a closed interval of negative non-real numbers. It is clear that the problem of extending these results to the case of (h,m)-convex functions of the first type remains open.

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Conflict of interest

The authors declare that they have no conflict of interest.

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