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Some Coincidence Point Theorems and an Application to Integral Equation in Partially Ordered Metric Spaces

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Abstract: In ordered metric space, the results on coincidence point of the mappings satisfying generalized rational contractions are investigated. Also discussed the integral contractions of the mappings in the same context to obtain the coincidence points. Two numerical examples are presented to justify the results obtained. Apart from in view of an application, the existence and the unique solution of an integral equation is discussed.

Keywords: Monotone $\mathscr{g}$-nondecreasing, rational contraction, coincidence point, compatable and weakly compatiable mappings, ordered metric spaces.

1 Introduction

First, Banach [1] introduced the contraction condition for a self-mapping in complete metric space for the existence of a fixed point. It has many applications in nonlinear analysis, applied mathematics and also in sciences. Later, it has been enhanced by many researcher in several directions by considering weaker conditions on either a space or on the mappings. Some generalizations and extensions of the Banach's contraction principle can be found from the works of [2,3,4,5,6,7,8,9,10,11].

Ran and Reurings [12] investigated a result on a fixed point of the mapping in partial order set and also provided some applications in matrix algebra. While Nieto et al. [13,14] generalized the results of [12] in partially ordered sets and also explored applications on ordinary differential equations. In the same context, several authors have developed important results in different spaces which have many applications in applied sciences and nonlinear analysis. On various ordered spaces, fixed point results have been obtained by considering different contraction conditions, some of such works can be found from [15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44], which creates natural interest to work more on it.

The work in this paper presents the results on coincidence point for the mappings satisfying rational contractions in partially ordered metric spaces. These results extended the results of [13,14,23,24,32]and other well known results in literature. Also discussed integral contractions of the mappings in the same context for the similar conclusions. Further, some numerical illustrations and the existence of a unique solution of an integral equation are discussed.

2 Main Results

This section starts with the following theorem in partially ordered metric spaces.

Theorem 21*The two continuous self-mappings $\mathscr{B}$ and $\mathscr{g}$ defined in a complete partially ordered metric space (c.p.o.m.s) $\mathscr{G}$ have a coincidence point if*

(i).$\mathscr{B}$ is a monotone $\mathscr{g}$ non-decreasing,

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(ii). $\mathscr{B}(\mathscr{G}) \subseteq g(\mathscr{G})$,
(iii).

$$\Omega(\mathscr{B}\vartheta,\mathscr{B}\zeta) \leq a\frac{\Omega(g\vartheta,\mathscr{B}\vartheta)\,\Omega(g\zeta,\mathscr{B}\zeta)}{\Omega(g\vartheta,g\zeta)} + b\,[\Omega(g\vartheta,\mathscr{B}\vartheta)+\Omega(g\zeta,\mathscr{B}\zeta)] + c\,[\Omega(g\vartheta,\mathscr{B}\zeta)+\Omega(g\zeta,\mathscr{B}\vartheta)] + d\,\Omega(g\vartheta,g\zeta), \quad (1)$$

for all $\vartheta,\zeta \in \mathscr{G}$ *with* $g(\vartheta) \neq g(\zeta)$ *are comparable and* $0 \leq a+2(b+c)+d < 1$ *for* $0 \leq a,b,c,d < 1$,
(iv). $g\vartheta_0 \preceq \mathscr{B}\vartheta_0$, *for certain* $\vartheta_0 \in \mathscr{G}$ *and* g, $\mathscr{B}$ *are compatible.*

Proof. If certain $\vartheta_0 \in \mathscr{G}$ with $g\vartheta_0 \preceq \mathscr{B}\vartheta_0$, then there is a point $\vartheta_1 \in \mathscr{G}$ such that $g\vartheta_1 = \mathscr{B}\vartheta_0$ by the hypotheses. Since $\mathscr{B}\vartheta_1 \in g(\mathscr{G})$, then there exists another point $\vartheta_2 \in \mathscr{G}$ such that $g\vartheta_2 = \mathscr{B}\vartheta_1$. Repeating the same process, we obtain a sequence $\{\vartheta_n\} \subset \mathscr{G}$ such that $g\vartheta_{n+1} = \mathscr{B}\vartheta_n, n \geq 0$.

As we know from the hypothesis that $g\vartheta_0 \preceq \mathscr{B}\vartheta_0 = g\vartheta_1$. Hence from the condition (1), we obtained that $\mathscr{B}\vartheta_0 \preceq \mathscr{B}\vartheta_1$. Consequently, we have

$$\mathscr{B}\vartheta_0 \preceq \mathscr{B}\vartheta_1 \preceq ... \preceq \mathscr{B}\vartheta_n \preceq \mathscr{B}\vartheta_{n+1} \preceq ... .$$

Now, the remaining proof will be discussed in the following two cases.
Case:(i): If for certain $n \in \mathbb{N}$, $\Omega(\mathscr{B}\vartheta_n,\mathscr{B}\vartheta_{n+1}) = 0$, then $\mathscr{B}\vartheta_{n+1} = \mathscr{B}\vartheta_n$. Therefore, $\mathscr{B}$ and g have a coincidence point ϑ_{n+1}.
Case:(ii): Suppose that $\Omega(\mathscr{B}\vartheta_n,\mathscr{B}\vartheta_{n+1}) \neq 0, \forall n \in \mathbb{N}$. Then equation (1) becomes

$$\Omega(\mathscr{B}\vartheta_{n+1},\mathscr{B}\vartheta_n) \leq a\frac{\Omega(g\vartheta_{n+1},\mathscr{B}\vartheta_{n+1})\,\Omega(g\vartheta_n,\mathscr{B}\vartheta_n)}{\Omega(g\vartheta_{n+1},g\vartheta_n)} + b\,[\Omega(g\vartheta_{n+1},\mathscr{B}\vartheta_{n+1})+\Omega(g\vartheta_n,\mathscr{B}\vartheta_n)] + c\,[\Omega(g\vartheta_{n+1},\mathscr{B}\vartheta_n)+\Omega(g\vartheta_n,\mathscr{B}\vartheta_{n+1})] + d\,\Omega(g\vartheta_{n+1},g\vartheta_n),$$

which implies that

$$\Omega(\mathscr{B}\vartheta_{n+1},\mathscr{B}\vartheta_n) \leq a\,\Omega(\mathscr{B}\vartheta_n,\mathscr{B}\vartheta_{n+1}) + b\,[\Omega(\mathscr{B}\vartheta_n,\mathscr{B}\vartheta_{n+1})+\Omega(\mathscr{B}\vartheta_{n-1},\mathscr{B}\vartheta_n)] + c\,[\Omega(\mathscr{B}\vartheta_n,\mathscr{B}\vartheta_n)+\Omega(\mathscr{B}\vartheta_{n-1},\mathscr{B}\vartheta_{n+1})] + d\,\Omega(\mathscr{B}\vartheta_n,\mathscr{B}\vartheta_{n-1}).$$

Thus we have

$$\Omega(\mathscr{B}\vartheta_{n+1},\mathscr{B}\vartheta_n) \leq \left(\frac{b+c+d}{1-a-b-c}\right)\Omega(\mathscr{B}\vartheta_n,\mathscr{B}\vartheta_{n-1}).$$

Finally, we arrive by induction that

$$\Omega(\mathscr{B}\vartheta_{n+1},\mathscr{B}\vartheta_n) \leq \Gamma^n\,\Omega(\mathscr{B}\vartheta_1,\mathscr{B}\vartheta_0), \quad (2)$$

where $\Gamma = \frac{b+c+d}{1-a-b-c} < 1$.

For $m \geq n$, and then by the triangular inequality of a metric, we have

$$\begin{aligned}\Omega(\mathscr{B}\vartheta_m,\mathscr{B}\vartheta_n) &\leq \Omega(\mathscr{B}\vartheta_m,\mathscr{B}\vartheta_{m-1})+\Omega(\mathscr{B}\vartheta_{m-1},\mathscr{B}\vartheta_{m-2}) + ...+\Omega(\mathscr{B}\vartheta_{n+1},\mathscr{B}\vartheta_n)\\ &\leq \left(\Gamma^{m-1}+\Gamma^{m-2}+...+\Gamma^n\right)\Omega(\mathscr{B}\vartheta_1,\mathscr{B}\vartheta_0)\\ &\leq \frac{\Gamma^n}{1-\Gamma}\,\Omega(\mathscr{B}\vartheta_1,\mathscr{B}\vartheta_0),\end{aligned}$$

as $m,n \to +\infty$, $\Omega(\mathscr{B}\vartheta_m, \mathscr{B}\vartheta_n) \to 0$, which implies that $\{\mathscr{B}\vartheta_n\}$ is a Cauchy sequence. Hence by the completeness of $\mathscr{G}$ there exists $\mu \in \mathscr{G}$ such that $\mathscr{B}\vartheta_n \to \mu$.

Moreover from the continuity property of $\mathscr{B}$, we have

$$\lim_{n\to+\infty} \mathscr{B}(\mathscr{B}\vartheta_n) = \mathscr{B}\left(\lim_{n\to+\infty} \mathscr{B}\vartheta_n\right) = \mathscr{B}\mu.$$

Therefore, $\lim_{n\to+\infty} g\vartheta_{n+1} = \mu$ as $g\vartheta_{n+1} = \mathscr{B}\vartheta_n$.

Also from the condition (iv), we have

$$\lim_{n\to+\infty} \Omega(\mathscr{B}h\vartheta_n, g\mathscr{B}\vartheta_n) = 0.$$

Therefore, the metric triangular inequality suggest that

$$\begin{aligned}\Omega(\mathscr{B}\mu, g\mu) = \Omega(\mathscr{B}\mu, \mathscr{B}g\vartheta_n) + \Omega(\mathscr{B}g\vartheta_n, g\mathscr{B}\vartheta_n)\\ + \Omega(g\mathscr{B}\vartheta_n, g\mu).\end{aligned} \tag{3}$$

By letting $n \to +\infty$ in (3) and the continuity of $\mathscr{B}$ and g, we obtained that $\Omega(\mathscr{B}\mu, g\mu) = 0$. Therefore, $\mathscr{B}\mu = g\mu$. Hence the result.

From Theorem 21, we have the following corollary by setting $c = 0$ and $b = 0$ in equation (1)

Corollary 22*A coincidence point exists for the continuous self-mappings $\mathscr{B}$ and g defined on $\mathscr{G}$, where $\mathscr{G}$ is c.p.o.m.s with the following assumptions:*

(i).$\mathscr{B}(\mathscr{G}) \subseteq g(\mathscr{G})$,
(ii).$\mathscr{B}$ is a monotone g non-decreasing,
(iii).(a).

$$\begin{aligned}\Omega(\mathscr{B}\vartheta, \mathscr{B}\zeta) \le a\frac{\Omega(g\vartheta, \mathscr{B}\vartheta)\ \Omega(g\zeta, \mathscr{B}\zeta)}{\Omega(g\vartheta, g\zeta)}\\ + b\,[\Omega(g\vartheta, \mathscr{B}\vartheta) + \Omega(g\zeta, \mathscr{B}\zeta)]\\ + d\,\Omega(g\vartheta, g\zeta),\end{aligned}$$

for $0 \le a, b, d < 1$ with $0 \le a + 2b + d < 1$,
(b).

$$\begin{aligned}\Omega(\mathscr{B}\vartheta, \mathscr{B}\zeta) \le a\frac{\Omega(g\vartheta, \mathscr{B}\vartheta)\ \Omega(g\zeta, \mathscr{B}\zeta)}{\Omega(g\vartheta, g\zeta)}\\ + c\,[\Omega(g\vartheta, \mathscr{B}\zeta) + \Omega(g\zeta, \mathscr{B}\vartheta)]\\ + d\,\Omega(g\vartheta, g\zeta),\end{aligned}$$

for $0 \le a, c, d < 1$ such that $0 \le a + 2c + d < 1$,
for all $\vartheta, \zeta \in \mathscr{G}$ with $g(\vartheta) \ne g(\zeta)$ are comparable and
(iv).$g\vartheta_0 \preceq \mathscr{B}\vartheta_0$, for certain $\vartheta_0 \in \mathscr{G}$ and g and $\mathscr{B}$ are compatible.

Corollary 23*A continuous self-mapping $\mathscr{B}$ defined on a comparable set $\mathscr{G}$ has a fixed point in Theorem 21 and Corollary 22, if $\mathscr{B}(\mathscr{B}\vartheta) \preceq \mathscr{B}\vartheta, \vartheta \in \mathscr{G}$ and $\vartheta_0 \preceq \mathscr{B}\vartheta_0$ for certain $\vartheta_0 \in \mathscr{G}$.*

*Proof.*The proof can be obtained by letting $g = I_{\mathscr{G}}$ in Theorem 21.

Relaxing the continuity property of $\mathscr{B}$ and $\mathscr{G}$ and satisfy the following condition still have the same conclusion of the mappings in Theorem 21:

$$\begin{aligned}&\text{A sequence } \{\vartheta_n\} \text{ in } \mathscr{G} \text{ is non decrasing with } \vartheta_n \to \vartheta \text{ then}\\ &\vartheta_n \preceq \vartheta,\ (n \ge 0).\end{aligned} \tag{4}$$

Theorem 24*If $\mathscr{G}$ has the property of (4) in Theorem 21, then*

(a).A coincidence point for $\mathscr{B}$ and g exists, if $g(\mathscr{G}) \subset \mathscr{G}$ is complete,

(b).A common fixed point for $\mathscr{B}$ and $\mathscr{g}$ exists, if $\mathscr{B}$ and $\mathscr{g}$ are weakly compatible,
(c).$\mathscr{B}$ and $\mathscr{g}$ have only one common fixed point if $\mathscr{B}$ and $\mathscr{g}$ have well ordered common fixed points set.

*Proof.*If $\mathscr{g}(\mathscr{G})$ is complete then from Theorem 21, there exists a Cauchy sequence $\{\mathscr{g}\vartheta_n\}$ such that

$$\lim_{n\to+\infty} \mathscr{B}\vartheta_n = \lim_{n\to+\infty} \mathscr{g}\vartheta_n = \mathscr{g}u, \text{ for } \mathscr{g}u \in \mathscr{g}(\mathscr{G}). \tag{5}$$

Since $\{\mathscr{B}\vartheta_n\}$ and $\{\mathscr{g}\vartheta_n\}$ are non-decreasing sequences then as a result we obtained that $\mathscr{B}\vartheta_n \preceq \mathscr{g}u$ and $\mathscr{g}\vartheta_n \preceq \mathscr{g}u$. Therefore, $\mathscr{B}\vartheta_n \preceq \mathscr{B}u, (n \geq 0)$ by the monotone property of $\mathscr{B}$. As by limiting case, we arrive at $\mathscr{g}u \preceq \mathscr{B}u$.

Assume that $\mathscr{g}u \prec \mathscr{B}u$. Let $u_0 = u$ and define a sequence $\{u_n\}$ in $\mathscr{G}$ by $\mathscr{g}u_{n+1} = \mathscr{B}u_n, (n \geq 0)$. Hence, by Theorem 21 there exists a convergent non-decreasing Cauchy sequence $\{\mathscr{g}u_n\}$ such that $\lim_{n\to+\infty} \mathscr{g}(u_n) = \lim_{n\to+\infty} \mathscr{B}u_n = \mathscr{g}v, v \in \mathscr{G}$. Hence, we have $\sup \mathscr{g}u_n \preceq \mathscr{g}v$ and $\sup \mathscr{B}u_n \preceq \mathscr{g}v$, $n \geq 0$ from the hypotheses.

Thus,

$$\mathscr{g}\vartheta_n \preceq \mathscr{g}u \preceq \mathscr{g}u_1 \preceq \ldots \preceq \mathscr{g}u_n \preceq \ldots \leq hv. \tag{6}$$

The conclusions will see from the following cases:
Case:(a) Suppose $\mathscr{g}\vartheta_{n_0} = \mathscr{g}u_{n_0}$ for certain $n_0 \geq 1$. Then

$$\mathscr{g}\vartheta_{n_0} = \mathscr{g}u = \mathscr{g}u_{n_0} = \mathscr{g}u_1 = \mathscr{B}u. \tag{7}$$

From (7), $\mathscr{B}$ and $\mathscr{g}$ have a coincidence point u.
Case:(b) Suppose $\mathscr{g}\vartheta_{n_0} \neq \mathscr{g}u_{n_0}, \forall n \in \mathbb{N}$ then from (1), we have

$$\begin{aligned} \Omega(\mathscr{g}\vartheta_{n+1}, \mathscr{g}u_{n+1}) &= \Omega(\mathscr{B}\vartheta_n, \mathscr{B}u_n) \\ &\leq \mathscr{a}\frac{\Omega(\mathscr{g}\vartheta_n, \mathscr{B}\vartheta_n)\ \Omega(\mathscr{g}u_n, \mathscr{B}u_n)}{\Omega(\mathscr{g}\vartheta_n, \mathscr{g}u_n)} \\ &\quad + \mathscr{b}\left[\Omega(\mathscr{g}\vartheta_n, \mathscr{B}\vartheta_n) + \Omega(\mathscr{g}u_n, \mathscr{B}u_n)\right] \\ &\quad + \mathscr{c}\left[\Omega(\mathscr{g}\vartheta_n, \mathscr{B}u_n) + \Omega(\mathscr{g}u_n, \mathscr{B}\vartheta_n)\right] \\ &\quad + \mathscr{d}\,\Omega(\mathscr{g}\vartheta_n, \mathscr{g}u_n). \end{aligned} \tag{8}$$

Letting $n \to +\infty$ in (8), we get

$$\begin{aligned} \Omega(\mathscr{g}u, \mathscr{g}v) &\leq (2\mathscr{c} + \mathscr{d})\Omega(\mathscr{g}u, \mathscr{g}v) \\ &< \Omega(\mathscr{g}u, \mathscr{g}v), \text{ since } 2\mathscr{c} + \mathscr{d} < 1. \end{aligned} \tag{9}$$

Therefore,

$$\mathscr{g}u = \mathscr{g}v = \mathscr{g}u_1 = \mathscr{B}u,$$

Hence, the mappings $\mathscr{B}$ and $\mathscr{g}$ have a coincidence point.

Suppose that $\mathscr{q}$ is a coincidence point and, $\mathscr{B}$ and $\mathscr{g}$ are weakly compatible mappings, then

$$\mathscr{B}\mathscr{q} = \mathscr{B}\mathscr{g}z = \mathscr{g}\mathscr{B}z = \mathscr{g}\mathscr{q}, \text{ since } \mathscr{q} = \mathscr{B}z = \mathscr{g}z, \text{ for some } z \in \mathscr{G}.$$

Therefore (1) becomes,

$$\begin{aligned} \Omega(\mathscr{B}z, \mathscr{B}\mathscr{q}) &\leq \mathscr{a}\frac{\Omega(\mathscr{g}z, \mathscr{B}z)\ \Omega(\mathscr{g}\mathscr{q}, \mathscr{B}\mathscr{q})}{\Omega(\mathscr{g}z, \mathscr{g}\mathscr{q})} \\ &\quad + \mathscr{b}\left[\Omega(\mathscr{g}z, \mathscr{B}z) + \Omega(\mathscr{g}\mathscr{q}, \mathscr{B}\mathscr{q})\right] \\ &\quad + \mathscr{c}\left[\Omega(\mathscr{g}z, \mathscr{B}\mathscr{q}) + \Omega(\mathscr{g}\mathscr{q}, \mathscr{B}z)\right] \\ &\quad + \mathscr{d}\,\Omega(\mathscr{g}z, \mathscr{g}\mathscr{q}) \\ &\leq (2\mathscr{c} + \mathscr{d})\Omega(\mathscr{B}z, \mathscr{B}\mathscr{q}). \end{aligned} \tag{10}$$

Finally we arrive at $\Omega(\mathscr{B}z, \mathscr{B}\mathscr{q}) = 0$ as $2\mathscr{c} + \mathscr{d} < 1$ from (10). Hence, $\mathscr{B}z = \mathscr{B}\mathscr{q} = \mathscr{g}\mathscr{q} = \mathscr{q}$ and suggest that $\mathscr{q}$ is a common fixed point of $\mathscr{B}$ and $\mathscr{g}$.

Next, suppose that $\mathscr{B}$ and g have well ordered common fixed point set. For uniqueness, let u and v be any two distinct common fixed points. Then from (1),

$$\begin{aligned}\Omega(u,v) \leq\ & a\frac{\Omega(gu,\mathscr{B}u)\ \Omega(gv,\mathscr{B}v)}{\Omega(gu,gv)}\\ & + b\left[\Omega(gu,\mathscr{B}u)+\Omega(gv,\mathscr{B}v)\right]\\ & + c\left[\Omega(gu,\mathscr{B}v)+\Omega(gv,\mathscr{B}u)\right]+d\,\Omega(gu,gv)\\ \leq\ & (2c+d)\ \Omega(u,v)\\ <\ & \Omega(u,v),\ \text{since } 2c+d<1.\end{aligned} \tag{11}$$

This is a contradiction in (11). Conversely, suppose that $\mathscr{B}$ and g have only one common fixed point. Therefore, the set of common fixed points of $\mathscr{B}$ and g being a singleton. Hence it is well ordered set.

One can have the same conclusions as of Theorem 21 and Corollary 22 by omitting the continuity property of a mapping $\mathscr{B}$ and implementing the condition (4) on $\mathscr{G}$.

Corollary 25*A self-mapping $\mathscr{B}$ defined on c.p.o.m.s $\mathscr{G}$ has a fixed point, if it satisfies the contraction condition* (1), *$\mathscr{B}(\mathscr{B}\vartheta) \preceq \mathscr{B}\vartheta, \forall \vartheta \in \mathscr{G}$, for any non-decreasing sequence $\{\vartheta_n\}$ with $\vartheta_n \to \vartheta \in \mathscr{G}$ such that $\vartheta_n \preceq \vartheta, (n \geq 0)$ and $\vartheta_0 \preceq \mathscr{B}\vartheta_0$, for certain $\vartheta_0 \in \mathscr{G}$.*

*Proof.*Setting $g = I_{\mathscr{G}}$ in Theorem 24, the required proof can be obtained.

Remark 26 *(i).Theorems 2.1 & 2.3 of Chandok [23] can be obtained by replacing $b = c = 0$ in Theorems 21 & 24.*
(ii).Theorems 2.1 & 2.3 of Harjani et al. [24] will be getting by letting $b = c = 0$ and $g = I$ in Theorems 21& 24

The following is a consequence of Theorem 21, which comprise an integral contraction.

A self-mapping $\nu(t)$ defined on $[0,+\infty)$ be such that

(a).$\int_0^{\varepsilon} \nu(t)dt > 0$, for $\varepsilon > 0, t \in [0,+\infty)$ and
(b).ν is Lebesgue integrable on any compact subset of $[0,+\infty)$.

Denote all such functions defined above by Θ.

Corollary 27*A coincidence point exists for the continuous self-mappings $\mathscr{B}$ and g on c.p.o.m.s. $\mathscr{G}$ with the following assumptions:*

(i).$\mathscr{B}(\mathscr{G}) \subseteq g(\mathscr{G})$,
(ii).$\mathscr{B}$ is a monotone g non-decreasing,
(iii).

$$\begin{aligned}\int_0^{\Omega(\mathscr{B}\vartheta,\mathscr{B}\zeta)} \nu(t)dt \leq\ & a\int_0^{\frac{\Omega(g\vartheta,\mathscr{B}\vartheta)\ \Omega(g\zeta,\mathscr{B}\zeta)}{\Omega(g\vartheta,g\zeta)}} \nu(t)dt\\ & + b\int_0^{\Omega(g\vartheta,\mathscr{B}\vartheta)+\Omega(g\zeta,\mathscr{B}\zeta)} \nu(t)dt\\ & + c\int_0^{\Omega(g\vartheta,\mathscr{B}\zeta)+\Omega(g\zeta,\mathscr{B}\vartheta)} \nu(t)dt\\ & + d\int_0^{\Omega(g\vartheta,g\zeta)} \nu(t)dt,\end{aligned} \tag{12}$$

for all $\vartheta,\zeta \in \mathscr{G}$ with $g(\vartheta) \neq g(\zeta)$ are comparable and $0 \leq a + 2(b+c) + d < 1$ for $0 \leq a,b,c,d < 1$, $\nu \in \Theta$,
(iv).$g\vartheta_0 \preceq \mathscr{B}\vartheta_0$, for certain $\vartheta_0 \in \mathscr{G}$ and, g and $\mathscr{B}$ are compatible.

Remark 28*(i). One can acquire the same conclusions as in Corollary 27 by setting $c = 0$ and $b = 0$ in* (12).
(ii). By putting $b = c = 0$ in Corollary 27, we get Corollary 2.5 of [23].

We illustrate few examples for Theorem 21.

Example 29*A coincidence point for the self-mappings $\mathscr{B}$ and g exists on $\mathscr{G} = [0,1]$ with $\mathscr{B}\vartheta = \frac{\vartheta^2}{2}$, $g\vartheta = \frac{2\vartheta^2}{1+\vartheta}$ by a metric $\Omega(\vartheta,\zeta) = |\vartheta - \zeta|$.*

Proof. By definition of the mappings and a metric, the assumptions (i), (ii) and (iv) of Theorem 21 are fulfilled with $\vartheta_0 = 0 \in \mathscr{G}$. For condition (iii), let $\vartheta < \zeta$, for $\vartheta, \zeta \in \mathscr{G}$. Then

$$\begin{aligned}
\Omega(\mathscr{B}\vartheta, \mathscr{B}\zeta) &= \frac{1}{2}|\vartheta^2 - \zeta^2| = \frac{1}{2}(\vartheta+\zeta)|\vartheta-\zeta| \\
&\leq \frac{2(\vartheta+\zeta+\vartheta\zeta)}{(1+\vartheta)(1+\zeta)}|\vartheta-\zeta| \\
&\leq \frac{a}{4}\frac{\vartheta^2\zeta^2}{(\vartheta+\zeta+\vartheta\zeta)}\frac{|\vartheta-3||\zeta-3|}{|\vartheta-\zeta|} \\
&\quad + \frac{b}{2}\frac{\vartheta^2(1+\zeta)|\vartheta-3| + \zeta^2(1+\vartheta)|\zeta-3|}{(1+\vartheta)(1+\zeta)} \\
&\quad + c\frac{(1+\zeta)|4\vartheta^2 - \zeta^2(1+\vartheta)| + (1+\vartheta)|4\zeta^2 - \vartheta^2(1+\zeta)|}{2(1+\vartheta)(1+\zeta)} \\
&\quad + d\frac{2(\vartheta+\zeta+\vartheta\zeta)}{(1+\vartheta)(1+\zeta)}|\vartheta-\zeta|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\Omega(\mathscr{B}\vartheta, \mathscr{B}\zeta) &\leq a\frac{\frac{\vartheta^2|\vartheta-3|}{2(1+\vartheta)}\cdot\frac{\zeta^2|\zeta-3|}{2(1+\zeta)}}{2|\vartheta-\zeta|\frac{\vartheta+\zeta+\vartheta\zeta}{(1+\vartheta)(1+\zeta)}} \\
&\quad + b\left[\frac{\vartheta^2|\vartheta-3|}{2(1+\vartheta)} + \frac{\zeta^2|\zeta-3|}{2(1+\zeta)}\right] \\
&\quad + c\left[\left|\frac{\vartheta^2}{(1+\vartheta)} - \frac{\zeta^2}{2}\right| + \left|\frac{2\zeta^2}{(1+\zeta)} - \frac{\vartheta^2}{2}\right|\right] \\
&\quad + d\frac{2(\vartheta+\zeta+\vartheta\zeta)}{(1+\vartheta)(1+\zeta)}|\vartheta-\zeta| \\
&\leq a\frac{\Omega(g\vartheta, \mathscr{B}\vartheta)\,\Omega(g\zeta, \mathscr{B}\zeta)}{\Omega(g\vartheta, g\zeta)} \\
&\quad + b\,[\Omega(g\vartheta, \mathscr{B}\vartheta) + \Omega(g\zeta, \mathscr{B}\zeta)] \\
&\quad + c\,[\Omega(g\vartheta, \mathscr{B}\zeta) + \Omega(g\zeta, \mathscr{B}\vartheta)] \\
&\quad + d\,\Omega(g\vartheta, g\zeta).
\end{aligned}$$

Hence the condition (iii) holds in Theorem 21 for $0 \leq a, b, c, d < 1$. Therefore, 0 is a coincidence point of the mappings $\mathscr{B}$ and g in $\mathscr{G}$.

Example 210 *The self-mappings $\mathscr{B}$ and g defined on $\mathscr{G} = [0,1]$ are such that $\mathscr{B}\vartheta = \vartheta^3$ and $g\vartheta = \vartheta^4$ have two coincidence points $0, 1$ with $\vartheta_0 = \frac{1}{4}$ from the metric $d(\vartheta, \zeta) = |\vartheta - \zeta|$ on $\mathscr{G}$.*

3 Applications

Consider the integral equation below:

$$\hat{h}(\vartheta) = \int_0^{\mathscr{V}} \mu(\vartheta, \zeta, \hat{h}(\zeta))d\zeta + g(\vartheta), \quad \vartheta \in [0, \mathscr{V}], \mathscr{V} > 0. \tag{13}$$

Let $\mathscr{G} = C[0, \mathscr{V}]$ be the set of all continuous functions defined on $[0, \mathscr{V}]$. Define a function $\Omega : \mathscr{G} \times \mathscr{G} \to \mathbb{R}^+$ by

$$\Omega(u, v) = \sup_{\vartheta \in [0, \mathscr{V}]} \{|u(\vartheta) - v(\vartheta)|\}$$

and $\mathscr{G} = C[0, \mathscr{V}]$ denote the set of all continuous functions on $[0, \mathscr{V}]$. Thus with usual order $\leq$, $(\mathscr{G}, \leq)$ is a partially ordered set.

Now, we discuss the solution of (13) in the following theorem.

Theorem 31 *Assume the following:*
(a). $\mu : [0,\mathscr{V}] \times [0,\mathscr{V}] \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are continuous,*
(b).for $\vartheta, \zeta \in [0,\mathscr{V}]$,

$$\mu(\vartheta,\zeta,\int_0^{\mathscr{V}} \mu(\vartheta,z,\hat{h}(z))dz + g(\vartheta)) \leq \mu(\vartheta,\zeta,\hat{h}(\zeta)),$$

(c).there is a continuous function $N : [0,\mathscr{V}] \times [0,\mathscr{V}] \to [0,+\infty]$ *with*

$$|\mu(\vartheta,\zeta,a) - \mu(\vartheta,\zeta,b)| \leq N(\vartheta,\zeta)|a-b| \text{ and}$$

(iv).

$$\sup_{\vartheta\in[0,\mathscr{V}]} \int_0^{\mathscr{V}} N(\vartheta,\zeta)d\zeta \leq c$$

where $c < 1$. *Then, for* $a \in C[0,\mathscr{V}]$, (13) *has a solution.*

Proof. Define $\mathscr{B} : C[0,\mathscr{V}] \to C[0,\mathscr{V}]$ by

$$\mathscr{B}w(\vartheta) = \int_0^{\mathscr{V}} \mu(\vartheta,\zeta,w(\vartheta))d\vartheta + g(\vartheta), \ \vartheta \in [0,\mathscr{V}].$$

Now, we have

$$\begin{aligned}
\mathscr{B}(\mathscr{B}w(\vartheta)) &= \int_0^{\mathscr{V}} \mu(\vartheta,\zeta,\mathscr{B}w(\vartheta))d\vartheta + g(\vartheta) \\
&= \int_0^{\mathscr{V}} \mu(\vartheta,\zeta,\int_0^{\mathscr{V}} \mu(\vartheta,z,w(z))dz + g(\vartheta))d\vartheta + g(\vartheta) \\
&\leq \int_0^{\mathscr{V}} \mu(\vartheta,\zeta,w(z))dz + g(\vartheta) \\
&= \mathscr{B}w(\vartheta).
\end{aligned}$$

Therefore, we have $\mathscr{B}(\mathscr{B}\vartheta) \leq \mathscr{B}\vartheta$ for any $\vartheta \in C[0,\mathscr{V}]$. Let $\vartheta^* \leq \zeta^*$ for $\vartheta^*, \zeta^* \in C[0,\mathscr{V}]$ then,

$$\begin{aligned}
\Omega(\mathscr{B}\vartheta^*,\mathscr{B}\zeta^*) &= \sup_{\vartheta\in[0,\mathscr{V}]} |\mathscr{B}\vartheta^*(\vartheta) - \mathscr{B}\zeta^*(\zeta)| \\
&= \sup_{\vartheta\in[0,\mathscr{V}]} |\int_0^{\mathscr{V}} \mu(\vartheta,\zeta,\vartheta^*(\vartheta)) - \mu(\vartheta,\zeta,\zeta^*(\vartheta))d\vartheta| \\
&\leq \sup_{\vartheta\in[0,\mathscr{V}]} |\int_0^{\mathscr{V}} \mu(\vartheta,\zeta,\vartheta^*(\vartheta)) - \mu(\vartheta,\zeta,\zeta^*(\vartheta))|d\vartheta \\
&\leq \sup_{\vartheta\in[0,\mathscr{V}]} |\int_0^{\mathscr{V}} N(\vartheta,\zeta)|\vartheta^*(\vartheta) - \zeta^*(\vartheta)|d\vartheta \\
&\leq \sup_{\vartheta\in[0,\mathscr{V}]} |\vartheta^*(\vartheta) - \zeta^*(\vartheta)| \sup_{\vartheta\in[0,\mathscr{V}]} \int_0^{\mathscr{V}} N(\vartheta,\zeta)d\vartheta \\
&= \Omega(\vartheta^*,\zeta^*) \sup_{\vartheta\in[0,\mathscr{V}]} \int_0^{\mathscr{V}} N(\vartheta,\zeta)d\vartheta \\
&\leq c\,d(\vartheta^*,\zeta^*).
\end{aligned}$$

Also the sequence $\{\vartheta_n^*\}$ is a non-decreasing in $C[0,\mathscr{V}]$ with $\vartheta_n^* \to \vartheta^*$ and suggest that $\vartheta_n^* \leq \vartheta^*, (n \geq 0)$. Hence from Corollary 25, the equation (13) has a solution for some $a \in [0,\mathscr{V}]$.

4 Conclusion

In this work, some coincidence point results of the self mappings satisfying generalized rational contractions with/without continuity property of the mappings are discussed. In obtaining the coincidence point of these results some topological properties are assumed on the space as well as on the self mappings. Few suitable numerical examples are given to support the findings. These results generalize and extend the well known results in the literature. Furthermore, the existence and the uniqueness of a solution of an integral equation is discussed at the end in view of an application of these obtained results.

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