

Higher Order Moments, Cumulants, Spectral and Bispectral Density Functions of the ZTPINAR(1) Process

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Abstract: In this paper, the higher order moments, cumulants, spectral, bispectral and normalized bispectral density functions of zero truncated Poisson first-order integer-valued autoregressive (ZTPINAR(1)) model are calculated. We estimated the spectrum, bispectrum and normalized bispectrum using the smoothed periodogram method with different lag windows. Finally, we used the bispectral density function and normalized bispectral density function in order for studying the linearity of integer valued time series models.

Keywords: INAR(1); ZTPINAR(1); Moments; Cumulants; Spectrum; Bispectrum; Normalized bispectrum; Daniell lag window; Parzen lag window; Tukey Hamming lag window; 2-dimensional Subba Rao and Gabr lag window.

1 Introduction

Integer-valued time series play an important role in statistical researches in the last few decades. This series are fairly common, such as the number of births at a hospital in successive months, count of chromosome interchanges in cells, count of accidents, number of transmitted messages, count of patients and so on. There have been many attempts in modeling such series through history. Particularly attractive is integer-valued autoregressive (INAR) model introduced by [1]. INAR model was based on a binomial thinning operator generated by Bernoulli distributed counting series. It considered present number of data as the sum of those that remained from the previous period, and those that entered in the observed period. This model was further developed by several authors, for example [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Models based on negative binomial thinning operator generated by geometric distributed counting series were also considered [15, 16, 17, 18]. Integer-valued time series generated by mixtures of binomial and negative binomial thinning operators are considered in [19] and [20]. Also, random coefficient integer-valued time series are introduced [21, 22, 23]. The common feature of all these models is the assumption of independence of count variables. Some generalizations

concerning relaxing the assumption of independence can be found in [24]. [25] introduce a new stationary first-order INAR process with geometric marginals on the basis of the generalized binomial thinning operator, which contains dependent Bernoulli counting series. He relax the assumption of independence underlying the basic INAR model and make the model more readily available for applications in practise. [26] study some higher order moments, spectral and bispectral density functions for some integer autoregressive of order one (INAR(1)) models. These models are the new skew INAR(1) (NSINAR(1)), the shifted geometric INAR(1) type II (SGINAR(1)-II) and the dependent counting geometric INAR(1) (DCGINAR(1)).

In this paper, we study some higher order moments, spectral and bispectral density functions for a new stationary first-order zero truncated Poisson integer valued autoregressive process which symbolized by ZTPINAR(1). So, motivation and our interest of this process is of such process arises from its ability in modelling and analysis of counting point processes and positive integer-valued time series. An example of a situation in which the zeros often are not observed (truncated) is the number of weekly bus trips where the analyst in this case do not observe the entire distribution of counts.

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As in literature, the integer-valued autoregressive time series of the first-order (INAR(1)) is given by

$$X_t = \alpha \circ X_{t-1} + \varepsilon_t, t \in Z, \quad (1)$$

where X_t is a non-negative integer-valued random variable and the compound random variable $\alpha \circ X_{t-1}$ is defined by

$$\alpha \circ X_{t-1} = \sum_{i=1}^{X_{t-1}} Y_i, \alpha \in (0, 1) \quad (2)$$

where $\{Y_i\}$ is a sequence of i.i.d. Bernoulli r.v.'s with parameter α independent of X_{t-1} and $\{\varepsilon_t\}$ is a sequence of i.i.d. non negative integer valued r.v.'s and independent of X_{t-i} for $i \geq 1$ with mean μ_ε and finite variance σ_ε^2 . $\{X_t\}$ is stationary if $0 < \alpha < 1$. The operation in (2) is known as thinning and hence \circ is called (binomial) thinning operator. For more properties of \circ see [9].

This paper is organized as follows. In Section 2, the construction of the zero truncated Poisson first-order integer-valued autoregressive (ZTPINAR(1)) process and its properties are mentioned. In Section 3, the higher order moments and cumulants up to order three are calculated. In Section 4, the spectral, bispectral and normalized bispectral density functions are computed. In Section 5, the spectrum, bispectrum and normalized bispectrum are estimated using the smoothed periodogram based on different lag windows (Daniell, Tukey Hamming and Parzen lag window) and using a generated realizations of size $N=500$ from the ZTPINAR(1).

2 The ZTPINAR(1) Process

The ZTPINAR(1) model was presented by [14]. The sequence $\{X_t\}$ is called ZTPINAR(1) process, which is given by

$$X_t = \begin{cases} \varepsilon_t, & \text{with probability } e^{-\lambda} \\ \alpha \circ X_{t-1} + \varepsilon_t, & \text{with probability } 1 - e^{-\lambda} \end{cases} \quad (3)$$

where $\alpha \in (0, \frac{1}{2}]$, $\lambda > 0$, $\{\varepsilon_t\}$ is a sequence of i.i.d. random variables and the marginal distribution of $\{X_t\}$ is zero truncated Poisson(λ) distribution (denoted as ZTP(λ)), $P(X_t = k) = \lambda^k e^{-\lambda} / (k!(1 - e^{-\lambda}))$, $k = 1, 2, \dots$. The probability generating function of X_t and ε_t are respectively given by

$$\begin{aligned} \phi_X(s) &= \frac{e^{\lambda s} - 1}{e^\lambda - 1}, \\ \phi_\varepsilon(s) &= \frac{\phi_X(s)}{e^{-\lambda} + (1 - e^{-\lambda})\phi_X(1 - \alpha + \alpha s)} \\ &= \frac{e^{\lambda s} - 1}{(1 - e^{-\lambda})e^{\lambda(1 - \alpha + \alpha s)}}. \end{aligned}$$

Therefore, the probability mass function of the random variable ε_t is given by

$$P(\varepsilon_t = k) = \frac{[(1 - \alpha)^k - (-\alpha)^k] \lambda^k e^{\lambda \alpha}}{k!(e^\lambda - 1)}, k = 1, 2, \dots$$

The mean and variance of X_t and ε_t are then given by respectively

$$\begin{aligned} \mu_X &= \frac{\lambda}{1 - e^{-\lambda}}, & \sigma_X^2 &= \frac{\lambda e^\lambda (e^\lambda - \lambda - 1)}{(e^\lambda - 1)^2}, \\ \mu_\varepsilon &= \frac{\lambda(\alpha + e^\lambda - \alpha e^\lambda)}{e^\lambda - 1}, \\ \sigma_\varepsilon^2 &= \frac{\lambda(e^{2\lambda} - \alpha - e^\lambda + 2\alpha e^\lambda - \lambda e^\lambda - \alpha e^{2\lambda})}{(e^\lambda - 1)^2}. \end{aligned}$$

The ZTPINAR(1) that defined by (3) may be rewritten as

$$X_t = \alpha_t \circ X_{t-1} + \varepsilon_t, \quad (4)$$

where $\{\alpha_t\}$ is a sequence of i.i.d r.v.'s independent of $\{X_t\}$ and $\{\varepsilon_t\}$ with the following distribution

$$P(\alpha_t = 0) = 1 - P(\alpha_t = \alpha) = \frac{1}{1 + \mu}.$$

For more information about the model see [14] and [27].

3 Higher Order Joint Moments and Cumulants Up to Order Three

Theorem 1. Let $\{X_t\}$ be a stationary process satisfying (3), then:

The second-order joint moment are calculated as

$$\begin{aligned} \mu_{(0)} &= \frac{\lambda e^\lambda (1 + \lambda)}{e^\lambda - 1}, \\ \mu_{(s)} &= (\alpha(1 - e^{-\lambda}))^s [\mu_{(0)} - \mu_X^2] + \mu_X^2 \\ &= \frac{\lambda e^\lambda (1 + \lambda)(\alpha - \alpha e^{-\lambda})^s}{e^\lambda - 1} - \frac{\lambda^2 e^{2\lambda} [\alpha^s (1 - e^{-\lambda})^s - 1]}{(e^\lambda - 1)^2}. \end{aligned}$$

The second order joint central moment (cumulant) is calculated as

$$\begin{aligned} C_2(s) &= (\alpha(1 - e^{-\lambda}))^s C_2(0) \\ &= (\alpha(1 - e^{-\lambda}))^s \frac{\lambda e^\lambda (e^\lambda - \lambda - 1)}{(e^\lambda - 1)^2}. \end{aligned}$$

The third-order joint moments are

$$\begin{aligned} \mu_{(0,0)} &= \frac{\lambda e^\lambda (\lambda^2 + 3\lambda + 1)}{e^\lambda - 1}, \\ \mu_{(0,s)} &= (\alpha(1 - e^{-\lambda}))^s [\mu_{(0,0)} - \mu_{(0)}\mu_X] + \mu_{(0)}\mu_X \\ &= \frac{\lambda^2 e^{2\lambda} (\lambda + 1)}{(e^\lambda - 1)^2} - \frac{\lambda e^\lambda (\alpha - \alpha e^{-\lambda})^s (3\lambda - e^\lambda - 2\lambda e^\lambda + \lambda^2 + 1)}{(e^\lambda - 1)^2}, \\ \mu_{(s,s)} &= \frac{\lambda e^\lambda (\lambda^2 + 3\lambda + 1)(-\alpha^2 (e^{-\lambda} - 1))^s}{e^\lambda - 1} \\ &\quad - \left[\frac{\lambda e^\lambda (\lambda - e^\lambda + 1)(\alpha + e^\lambda + 2\alpha\lambda - \alpha e^\lambda + 2\lambda e^\lambda - 2\alpha\lambda e^\lambda - 1)}{(e^\lambda - 1)^3 (\alpha - 1)} \times \right. \\ &\quad \left. ((-\alpha^2 (e^{-\lambda} - 1))^s - (\alpha - \alpha e^{-\lambda})^s) \right] \end{aligned}$$

$$\begin{aligned}
 & - \frac{\lambda^2 e^{2\lambda} (\lambda + 1) ((\alpha^2 e^{-\lambda} (e^\lambda - 1))^s - 1)}{(e^\lambda - 1)^2}, \\
 \mu_{(s, \tau)} &= (\alpha(1 - e^{-\lambda}))^{\tau-s} (\mu_{(s, s)} - \mu_{(s)} \mu_X) + \mu_{(s)} \mu_X. \\
 \text{The third-order joint central moments (cumulants) are} \\
 C_3(0, 0) &= \frac{\lambda e^\lambda (3\lambda + e^{2\lambda} - 2e^\lambda - 3\lambda e^\lambda + \lambda^2 + \lambda^2 e^\lambda + 1)}{(e^\lambda - 1)^3}, \\
 C_3(0, s) &= (\alpha(1 - e^{-\lambda}))^s C_3(0, 0) \\
 &= \frac{\lambda e^\lambda (\alpha(1 - e^{-\lambda}))^s (3\lambda + e^{2\lambda} - 2e^\lambda - 3\lambda e^\lambda + \lambda^2 + \lambda^2 e^\lambda + 1)}{(e^\lambda - 1)^3}, \\
 C_3(s, s) &= \\
 & (\alpha^2(1 - e^{-\lambda}))^s C_3(0, 0) + gC_2(0) \frac{(\alpha(1 - e^{-\lambda}))^s - (\alpha^2(1 - e^{-\lambda}))^s}{\alpha(1 - e^{-\lambda}) - \alpha^2(1 - e^{-\lambda})} \\
 &= \frac{\lambda e^\lambda (\alpha^2(1 - e^{-\lambda}))^s (3\lambda + e^{2\lambda} - 2e^\lambda - 3\lambda e^\lambda + \lambda^2 + \lambda^2 e^\lambda + 1)}{(e^\lambda - 1)^3} - \\
 & \frac{\lambda e^\lambda (e^\lambda + 2\alpha\lambda - 1)(\lambda - e^\lambda + 1)[(-\alpha^2(e^{-\lambda} - 1))^s - (\alpha - \alpha e^{-\lambda})^s]}{(e^\lambda - 1)^3(\alpha - 1)}, \\
 C_3(s, \tau) &= \alpha^{\tau-s} C_3(s, s).
 \end{aligned}$$

Proof. $\mu_{(0)}$ is computed by

$$\begin{aligned}
 \mu_{(0)} &= E(X_t^2) = E[(\alpha_t \circ X_{t-1} + \varepsilon_t)^2] \\
 &= E(\alpha_t \circ X_{t-1})^2 + 2E(\alpha_t \circ X_{t-1})\mu_\varepsilon + E(\varepsilon_t^2) \\
 &= \alpha^2(1 - e^{-\lambda})E(X_{t-1}^2) + \alpha(1 - e^{-\lambda})E(X_{t-1}) \\
 &\quad - \alpha^2(1 - e^{-\lambda})E(X_{t-1}) + 2\alpha(1 - e^{-\lambda})E(X_{t-1})\mu_\varepsilon \\
 &\quad + E(\varepsilon_t^2) \\
 &= \alpha^2(1 - e^{-\lambda})\mu_{(0)} + \alpha(1 - e^{-\lambda})\mu_X - \alpha^2(1 - e^{-\lambda})\mu_X \\
 &\quad + 2\alpha(1 - e^{-\lambda})\mu_X\mu_\varepsilon + E(\varepsilon_t^2) \\
 &= \frac{\alpha(1 - e^{-\lambda})\mu_X - \alpha^2(1 - e^{-\lambda})\mu_X}{1 - \alpha^2(1 - e^{-\lambda})} \\
 &\quad + \frac{2\alpha(1 - e^{-\lambda})\mu_X\mu_\varepsilon + E(\varepsilon_t^2)}{1 - \alpha^2(1 - e^{-\lambda})} \\
 &= \frac{\lambda e^\lambda (1 + \lambda)}{e^\lambda - 1},
 \end{aligned}$$

then,

$$C_2(0) = \mu_{(0)} - \mu_X^2 = \frac{\lambda e^\lambda (1 + \lambda)}{e^\lambda - 1} - \frac{\lambda^2 e^{2\lambda}}{(e^\lambda - 1)^2} = \frac{\lambda e^\lambda (e^\lambda - \lambda - 1)}{(e^\lambda - 1)^2}.$$

$\mu_{(s)}$ is obtained by

$$\begin{aligned}
 \mu_{(s)} &= E(X_t X_{t+s}) = E(X_t (\alpha_{t+s} \circ X_{t+s-1} + \varepsilon_{t+s})) \\
 &= \alpha(1 - e^{-\lambda})E(X_t X_{t+s-1}) + \mu_X \mu_\varepsilon \\
 &= \alpha(1 - e^{-\lambda})\mu_{(s-1)} + \mu_X \mu_\varepsilon \\
 &= (\alpha(1 - e^{-\lambda}))^s \mu_{(0)} + \mu_X \mu_\varepsilon \frac{1 - (\alpha(1 - e^{-\lambda}))^s}{1 - (\alpha(1 - e^{-\lambda}))} \\
 &= (\alpha(1 - e^{-\lambda}))^s \mu_{(0)} + \mu_X (1 - \alpha(1 - e^{-\lambda}))\mu_X \times \\
 &\quad \frac{1 - (\alpha(1 - e^{-\lambda}))^s}{1 - (\alpha(1 - e^{-\lambda}))} \\
 &= (\alpha(1 - e^{-\lambda}))^s \mu_{(0)} + \mu_X^2 (1 - (\alpha(1 - e^{-\lambda}))^s) \\
 &= (\alpha(1 - e^{-\lambda}))^s [\mu_{(0)} - \mu_X^2] + \mu_X^2 \\
 &= \frac{\lambda e^\lambda (1 + \lambda)(\alpha - \alpha e^{-\lambda})^s}{e^\lambda - 1} - \frac{\lambda^2 e^{2\lambda} [\alpha^s (1 - e^{-\lambda})^s - 1]}{(e^\lambda - 1)^2},
 \end{aligned}$$

then,

$$\begin{aligned}
 C_2(s) &= \mu_{(s)} - \mu_X^2 \\
 &= \frac{\lambda e^\lambda (1 + \lambda)(\alpha - \alpha e^{-\lambda})^s}{e^\lambda - 1} - \frac{\lambda^2 e^{2\lambda} [\alpha^s (1 - e^{-\lambda})^s - 1]}{(e^\lambda - 1)^2} \\
 &\quad - \frac{\lambda^2}{(1 - e^{-\lambda})^2} \\
 &= \frac{\lambda e^\lambda (\alpha - \alpha e^{-\lambda})^s (e^\lambda - \lambda - 1)}{(e^\lambda - 1)^2} \\
 &= (\alpha - \alpha e^{-\lambda})^s \frac{\lambda e^\lambda (e^\lambda - \lambda - 1)}{(e^\lambda - 1)^2} \\
 &= \alpha^s (1 - e^{-\lambda})^s C_2(0).
 \end{aligned}$$

$\mu_{(0,0)}$ is calculated by

$$\begin{aligned}
 \mu_{(0,0)} &= E(X_t^3) = E[(\alpha_t \circ X_{t-1} + \varepsilon_t)^3] \\
 &= E(\alpha_t \circ X_{t-1})^3 + E(\varepsilon_t^3) + 3E((\alpha_t \circ X_{t-1})^2 \varepsilon_t) \\
 &\quad + 3E[(\alpha_t \circ X_{t-1}) \varepsilon_t^2] \\
 &= \alpha^3(1 - e^{-\lambda})E(X_{t-1}^3) + 3\alpha^2(1 - e^{-\lambda})E(X_{t-1}^2) \\
 &\quad - 3\alpha^3(1 - e^{-\lambda})E(X_{t-1}) \\
 &\quad + [\alpha(1 - e^{-\lambda}) - 3\alpha^2(1 - e^{-\lambda}) + 2\alpha^3(1 - e^{-\lambda})] \times \\
 &\quad E(X_{t-1}) + E(\varepsilon_t^3) + 3\mu_\varepsilon [\alpha^2(1 - e^{-\lambda})E(X_{t-1}^2) \\
 &\quad + \alpha(1 - e^{-\lambda})E(X_{t-1}) - \alpha^2(1 - e^{-\lambda})E(X_{t-1})] \\
 &\quad + 3\alpha(1 - e^{-\lambda})E(X_{t-1})E(\varepsilon_t^2) \\
 &= \alpha^3(1 - e^{-\lambda})\mu_{(0,0)} + 3\alpha^2(1 - e^{-\lambda})\mu_{(0)} \\
 &\quad - 3\alpha^3(1 - e^{-\lambda})\mu_{(0)} \\
 &\quad + [\alpha(1 - e^{-\lambda}) - 3\alpha^2(1 - e^{-\lambda}) \\
 &\quad + 2\alpha^3(1 - e^{-\lambda})]E(X_{t-1}) + E(\varepsilon_t^3) \\
 &\quad + 3\mu_\varepsilon [\alpha^2(1 - e^{-\lambda})\mu_{(0)} \\
 &\quad + \alpha(1 - e^{-\lambda})\mu_X - \alpha^2(1 - e^{-\lambda})\mu_X] \\
 &\quad + 3\alpha(1 - e^{-\lambda})E(X_{t-1})E(\varepsilon_t^2) \\
 &= \frac{\lambda e^\lambda (\lambda^2 + 3\lambda + 1)}{e^\lambda - 1},
 \end{aligned}$$

then,

$$\begin{aligned}
 C_3(0, 0) &= \mu_{(0,0)} - 3\mu_X \mu_{(0)} + 2\mu_X^3 \\
 &= \frac{\lambda e^\lambda (\lambda^2 + 3\lambda + 1)}{e^\lambda - 1} - \frac{3\lambda^2 e^{2\lambda} (1 + \lambda)}{(e^\lambda - 1)^2} + 2\left(\frac{\lambda e^\lambda}{e^\lambda - 1}\right)^3 \\
 &= \frac{\lambda e^\lambda (3\lambda + e^{2\lambda} - 2e^\lambda - 3\lambda e^\lambda + \lambda^2 + \lambda^2 e^\lambda + 1)}{(e^\lambda - 1)^3},
 \end{aligned}$$

$\mu_{(0,s)}$ is proved by

$$\begin{aligned} \mu_{(0,s)} &= E(X_t X_t X_{t+s}) \\ &= E(X_t X_t [\alpha_{t+s} \circ X_{t+s-1} + \varepsilon_{t+s}]) \\ &= \alpha(1 - e^{-\lambda})E(X_t X_t X_{t+s-1}) + E(X_t^2)E(\varepsilon_{t+s}) \\ &= \alpha(1 - e^{-\lambda})\mu_{(0,s-1)} + \mu_{(0)}\mu_\varepsilon \\ &= (\alpha(1 - e^{-\lambda}))^s \mu_{(0,0)} + \mu_{(0)}\mu_\varepsilon \frac{1 - (\alpha(1 - e^{-\lambda}))^s}{1 - \alpha(1 - e^{-\lambda})} \\ &= (\alpha(1 - e^{-\lambda}))^s \mu_{(0,0)} + \mu_{(0)}\mu_X [1 - (\alpha(1 - e^{-\lambda}))^s] \\ &= (\alpha(1 - e^{-\lambda}))^s [\mu_{(0,0)} - \mu_{(0)}\mu_X] + \mu_{(0)}\mu_X. \end{aligned}$$

After substituting about $\mu_{(0,0)}$, $\mu_{(0)}$ and μ_X and simplifying, we get

$$\mu_{(0,s)} = \frac{\lambda^2 e^{2\lambda} (\lambda + 1)}{(e^\lambda - 1)^2} - \frac{\lambda e^\lambda (\alpha - \alpha e^{-\lambda})^s (3\lambda - e^\lambda - 2\lambda e^\lambda + \lambda^2 + 1)}{(e^\lambda - 1)^2},$$

hence,

$$\begin{aligned} C_3(0,s) &= E[(X_t - \mu)^2 (X_{t+s} - \mu)] \\ &= \mu_{(0,s)} - 2\mu_X \mu_{(s)} - \mu_X \mu_{(0)} + 2\mu_X^3 \\ &= (1 - e^{-\lambda})\mu_{(0,s-1)} + \mu_{(0)}\mu_\varepsilon \\ &\quad - 2\mu_X [\alpha(1 - e^{-\lambda})\mu_{(s-1)} + \mu_X \mu_\varepsilon] - \mu_X \mu_{(0)} + 2\mu_X^3 \\ &= \alpha(1 - e^{-\lambda})\mu_{(0,s-1)} + \mu_{(0)}(\mu_X - \alpha\lambda) \\ &\quad - 2\alpha(1 - e^{-\lambda})\mu_X \mu_{(s-1)} - 2\mu_X^2 \mu_\varepsilon - \mu_X \mu_{(0)} + 2\mu_X^3 \\ &= \alpha(1 - e^{-\lambda})\mu_{(0,s-1)} - 2\alpha(1 - e^{-\lambda})\mu_X \mu_{(s-1)} \\ &\quad - \alpha\lambda \mu_{(0)} - 2\mu_X^2 \mu_\varepsilon + 2\mu_X^3 \\ &= \alpha(1 - e^{-\lambda})\mu_{(0,s-1)} - 2\alpha(1 - e^{-\lambda})\mu_X \mu_{(s-1)} \\ &\quad - \alpha\lambda \mu_{(0)} - 2\mu_X^2 (\mu_X - \alpha\lambda) + 2\mu_X^3 \\ &= \alpha(1 - e^{-\lambda})\mu_{(0,s-1)} - 2\alpha(1 - e^{-\lambda})\mu_X \mu_{(s-1)} \\ &\quad - \alpha\lambda \mu_{(0)} - 2\mu_X^3 + 2\alpha\lambda \mu_X^2 + 2\mu_X^3 \\ &= \alpha(1 - e^{-\lambda})[\mu_{(0,s-1)} - 2\mu_X \mu_{(s-1)} - \frac{\lambda \mu_{(0)}}{1 - e^{-\lambda}} \\ &\quad + \frac{2\lambda \mu_X^2}{1 - e^{-\lambda}}] \\ &= \alpha(1 - e^{-\lambda})[\mu_{(0,s-1)} - 2\mu_X \mu_{(s-1)} - \frac{\lambda e^\lambda}{e^\lambda - 1} \mu_{(0)} \\ &\quad + \frac{2\lambda e^\lambda}{e^\lambda - 1} \mu_X^2] \\ &= \alpha(1 - e^{-\lambda})[\mu_{(0,s-1)} - 2\mu_X \mu_{(s-1)} - \mu_X \mu_{(0)} + 2\mu_X^3] \\ &= \alpha(1 - e^{-\lambda})C_3(0, s - 1). \end{aligned}$$

$\mu_{(s,s)}$ is proved by

$$\begin{aligned} \mu_{(s,s)} &= E(X_t X_{t+s}^2) = E(X_t [\alpha_{t+s} \circ X_{t+s-1} + \varepsilon_{t+s}]^2) \\ &= \alpha^2(1 - e^{-\lambda})E(X_t X_{t+s-1}^2) \end{aligned}$$

$$\begin{aligned} &+ \alpha(1 - \alpha)(1 - e^{-\lambda})E(X_t X_{t+s-1}) \\ &+ E(X_t)E(\varepsilon_{t+s}^2) \\ &+ 2\alpha(1 - e^{-\lambda})E(X_t X_{t+s-1})E(\varepsilon_{t+s}) \\ &= \alpha^2(1 - e^{-\lambda})\mu_{(s-1,s-1)} + \alpha(1 - \alpha)(1 - e^{-\lambda})\mu_{(s-1)} \\ &\quad + \mu_X E(\varepsilon^2) + 2\alpha(1 - e^{-\lambda})\mu_{(s-1)}\mu_\varepsilon \\ &= \alpha^2(1 - e^{-\lambda})\mu_{(s-1,s-1)} + [\alpha(1 - \alpha)(1 - e^{-\lambda}) \\ &\quad + 2\alpha(1 - e^{-\lambda})\mu_\varepsilon]\mu_{(s-1)} + \mu_X E(\varepsilon^2) \\ &= \alpha^2(1 - e^{-\lambda})\mu_{(s-1,s-1)} + [\alpha(1 - \alpha)(1 - e^{-\lambda}) \\ &\quad + 2\alpha(1 - e^{-\lambda})\mu_\varepsilon][(\alpha(1 - e^{-\lambda}))^{s-1} [\mu_{(0)} - \mu_X^2] \\ &\quad + \mu_X^2] + \mu_X E(\varepsilon^2) \\ &= \alpha^2(1 - e^{-\lambda})\mu_{(s-1,s-1)} + [\alpha(1 - \alpha)(1 - e^{-\lambda}) \\ &\quad + 2\alpha(1 - e^{-\lambda})\mu_\varepsilon](\alpha(1 - e^{-\lambda}))^{s-1} [\mu_{(0)} - \mu_X^2] \\ &\quad + [\alpha(1 - \alpha)(1 - e^{-\lambda}) + 2\alpha(1 - e^{-\lambda})\mu_\varepsilon]\mu_X^2 \\ &\quad + \mu_X E(\varepsilon^2) \end{aligned}$$

By using the iterations, we get

$$\begin{aligned} \mu_{(s,s)} &= (\alpha^2(1 - e^{-\lambda}))^s \mu_{(0,0)} + [\alpha(1 - \alpha)(1 - e^{-\lambda}) \\ &\quad + 2\alpha(1 - e^{-\lambda})\mu_\varepsilon][\mu_{(0)} - \mu_X^2] \sum_{i=0}^{s-1} (\alpha^2(1 - e^{-\lambda}))^i \\ &\quad \times (\alpha(1 - e^{-\lambda}))^{s-(i+1)} + [[\alpha(1 - \alpha)(1 - e^{-\lambda}) \\ &\quad + 2\alpha(1 - e^{-\lambda})\mu_\varepsilon]\mu_X^2 + \mu_X E(\varepsilon^2)] \sum_{j=0}^{s-1} (\alpha^2(1 - e^{-\lambda}))^j \\ &= (\alpha^2(1 - e^{-\lambda}))^s \mu_{(0,0)} + [\alpha(1 - \alpha)(1 - e^{-\lambda}) \\ &\quad + 2\alpha(1 - e^{-\lambda})\mu_\varepsilon][\mu_{(0)} - \mu_X^2] \times \\ &\quad \frac{(\alpha(1 - e^{-\lambda}))^s - (\alpha^2(1 - e^{-\lambda}))^s}{\alpha(1 - e^{-\lambda}) - \alpha^2(1 - e^{-\lambda})} \\ &\quad + ([\alpha(1 - \alpha)(1 - e^{-\lambda}) + 2\alpha(1 - e^{-\lambda})\mu_\varepsilon]\mu_X^2 \\ &\quad + \mu_X E(\varepsilon^2)) \frac{1 - (\alpha^2(1 - e^{-\lambda}))^s}{1 - \alpha^2(1 - e^{-\lambda})}, \end{aligned}$$

hence,

$$\begin{aligned} C_3(s,s) &= E[(X_t - \mu)(X_{t+s} - \mu)^2] \\ &= \mu_{(s,s)} - 2\mu_X \mu_{(s)} - \mu_X E(X_{t+s}^2) + 2\mu_X^3 \\ &= \alpha^2(1 - e^{-\lambda})\mu_{(s-1,s-1)} + [\alpha(1 - e^{-\lambda}) \\ &\quad - \alpha^2(1 - e^{-\lambda}) + 2\alpha(1 - e^{-\lambda})\mu_\varepsilon]\mu_{(s-1)} + \mu_X E(\varepsilon^2) \\ &\quad - 2\mu_X [\alpha(1 - e^{-\lambda})\mu_{(s-1)} + \mu_X \mu_\varepsilon] \\ &\quad - \mu_X E[(\alpha \circ X_{t+s-1} + \varepsilon_{t+s})^2] + 2\mu_X^3 \\ &= \alpha^2(1 - e^{-\lambda})\mu_{(s-1,s-1)} + [\alpha(1 - e^{-\lambda}) - \alpha^2(1 - e^{-\lambda}) \\ &\quad + 2\alpha(1 - e^{-\lambda})\mu_\varepsilon - 2\mu_X \alpha(1 - e^{-\lambda})]\mu_{(s-1)} \\ &\quad + \mu_X E(\varepsilon^2) + 2\mu_X^3 - 2\mu_X^2 \mu_\varepsilon \end{aligned}$$

$$\begin{aligned}
 & -\mu_X[\alpha^2(1-e^{-\lambda})E(X_{t+s-1}^2) + \alpha(1-e^{-\lambda})E(X_{t+s-1}) \\
 & - \alpha^2(1-e^{-\lambda})E(X_{t+s-1}) + E(\varepsilon^2) \\
 & + 2\alpha(1-e^{-\lambda})E(X_{t+s-1})E(\varepsilon_{t+s})] \\
 = & \alpha^2(1-e^{-\lambda})\mu_{(s-1,s-1)} + [\alpha(1-e^{-\lambda}) - \alpha^2(1-e^{-\lambda}) \\
 & + 2\alpha(1-e^{-\lambda})(1-\alpha(1-e^{-\lambda}))]\mu_X \\
 & - 2\mu_X\alpha(1-e^{-\lambda})\mu_{(s-1)} + \mu_XE(\varepsilon^2) + 2\mu_X^3 \\
 & - 2\mu_X^3(1-\alpha(1-e^{-\lambda})) - \alpha^2(1-e^{-\lambda})\mu_XE(X_{t+s-1}^2) \\
 & - \alpha(1-e^{-\lambda})\mu_X^2 + \alpha^2(1-e^{-\lambda})\mu_X^2 - \mu_XE(\varepsilon^2) \\
 & - 2\alpha(1-e^{-\lambda})(1-\alpha(1-e^{-\lambda}))\mu_X^3 \\
 = & \alpha^2(1-e^{-\lambda})\mu_{(s-1,s-1)} + [\alpha(1-e^{-\lambda}) - \alpha^2(1-e^{-\lambda}) \\
 & - 2\alpha^2(1-e^{-\lambda})\mu_X + 2\alpha\mu_X - 2\alpha e^{-\lambda}\mu_X \\
 & + 2\alpha^2 e^{-\lambda}(1-e^{-\lambda})\mu_X - 2\mu_X\alpha(1-e^{-\lambda})]\mu_{(s-1)} \\
 & + \mu_XE(\varepsilon^2) + 2\mu_X^3 - 2\mu_X^3 + 2\alpha(1-e^{-\lambda})\mu_X^3 \\
 & - \alpha^2(1-e^{-\lambda})\mu_XE(X_{t+s-1}^2) - \alpha(1-e^{-\lambda})\mu_X^2 \\
 & + \alpha^2(1-e^{-\lambda})\mu_X^2 - \mu_XE(\varepsilon^2) - 2\alpha\mu_X^3 \\
 & + 2\alpha^2(1-e^{-\lambda})\mu_X^3 + 2\alpha e^{-\lambda}\mu_X^3 - 2\alpha^2 e^{-\lambda}(1-e^{-\lambda})\mu_X^3 \\
 = & \alpha^2(1-e^{-\lambda})[\mu_{(s-1,s-1)} - 2\mu_X\mu_{(s-1)} - \mu_XE(X_{t+s-1}^2) \\
 & + 2\mu_X^3] + \alpha(1-e^{-\lambda})(1-\alpha)[\mu_{(s-1)} - \mu_X^2] \\
 & + 2\alpha^2\lambda e^{-\lambda}[\mu_{(s-1)} - \mu_X^2] \\
 = & \alpha^2(1-e^{-\lambda})C_3(s-1,s-1) + [\alpha(1-\alpha)(1-e^{-\lambda}) \\
 & + 2\alpha^2\lambda e^{-\lambda}]C_2(s-1) \\
 = & \alpha^2(1-e^{-\lambda})C_3(s-1,s-1) + [\alpha(1-\alpha)(1-e^{-\lambda}) \\
 & + 2\alpha^2\lambda e^{-\lambda}](\alpha(1-e^{-\lambda}))^{s-1}C_2(0)
 \end{aligned}$$

For simplicity, take $g = \alpha(1-\alpha)(1-e^{-\lambda}) + 2\alpha^2\lambda e^{-\lambda}$ and by using the iterations, we get

$$\begin{aligned}
 C_3(s,s) &= (\alpha^2(1-e^{-\lambda}))^s C_3(0,0) + gC_2(0) \times \\
 & \sum_{i=0}^{s-1} (\alpha(1-e^{-\lambda}))^{s-(i+1)} \cdot (\alpha^2(1-e^{-\lambda}))^i \\
 & = (\alpha^2(1-e^{-\lambda}))^s C_3(0,0) + gC_2(0) \times \\
 & \frac{(\alpha(1-e^{-\lambda}))^s - (\alpha^2(1-e^{-\lambda}))^s}{\alpha(1-e^{-\lambda}) - \alpha^2(1-e^{-\lambda})}.
 \end{aligned}$$

$\mu_{(s,\tau)}$ is calculated as

$$\begin{aligned}
 \mu_{(s,\tau)} &= E(X_t X_{t+s} X_{t+\tau}) = E(X_t X_{t+s} (\alpha_{t+\tau} \circ X_{t+\tau-1} + \varepsilon_{t+\tau})) \\
 &= \alpha(1-e^{-\lambda})E(X_t X_{t+s} X_{t+\tau-1}) + E(X_t X_{t+s})\mu_\varepsilon \\
 &= \alpha(1-e^{-\lambda})\mu_{(s,\tau-1)} + \mu_{(s)}\mu_\varepsilon = (\alpha(1-e^{-\lambda}))^{\tau-s} \mu_{(s,s)} \\
 &+ \mu_{(s)}\mu_\varepsilon \frac{1 - (\alpha(1-e^{-\lambda}))^{\tau-s}}{1 - \alpha(1-e^{-\lambda})}
 \end{aligned}$$

$$\begin{aligned}
 &= (\alpha(1-e^{-\lambda}))^{\tau-s} \mu_{(s,s)} + \mu_{(s)}\mu_X [1 - (\alpha(1-e^{-\lambda}))^{\tau-s}] \\
 &= (\alpha(1-e^{-\lambda}))^{\tau-s} (\mu_{(s,s)} - \mu_{(s)}\mu_X) + \mu_{(s)}\mu_X,
 \end{aligned}$$

and then,

$$\begin{aligned}
 C_3(s,\tau) &= \mu_{(s,\tau)} - \mu_X\mu_{(s)} - \mu_X\mu_{(\tau)} - \mu_XE(X_{t+\tau}X_{t+s}) + 2\mu_X^3 \\
 &= \alpha(1-e^{-\lambda})\mu_{(s,\tau-1)} + \mu_{(s)}\mu_\varepsilon - \mu_X\mu_{(s)} \\
 &- \mu_X[\alpha(1-e^{-\lambda})\mu_{(\tau-1)} + \mu_X\mu_\varepsilon] \\
 &- \mu_XE[(\alpha \circ X_{t+\tau-1})X_{t+s}] + 2\mu_X^3 \\
 &= \alpha(1-e^{-\lambda})\mu_{(s,\tau-1)} + \mu_{(s)}(1-\alpha(1-e^{-\lambda}))\mu_X \\
 &- \mu_X\mu_{(s)} - \alpha(1-e^{-\lambda})\mu_X\mu_{(\tau-1)} - \mu_X^3(1-\alpha(1-e^{-\lambda})) \\
 &- \alpha(1-e^{-\lambda})\mu_XE(X_{t+\tau-1}X_{t+s}) - \mu_X^2\mu_\varepsilon + 2\mu_X^3 \\
 &= \alpha(1-e^{-\lambda})\mu_{(s,\tau-1)} - \alpha(1-e^{-\lambda})\mu_{(s)}\mu_X \\
 &- \alpha(1-e^{-\lambda})\mu_X\mu_{(\tau-1)} + \alpha(1-e^{-\lambda})\mu_X^3 \\
 &- \alpha(1-e^{-\lambda})\mu_XE(X_{t+\tau-1}X_{t+s}) + \alpha(1-e^{-\lambda})\mu_X^3 \\
 &= \alpha(1-e^{-\lambda})[\mu_{(s,\tau-1)} - \mu_X\mu_{(s)} - \mu_X\mu_{(\tau-1)} \\
 &- \mu_XE(X_{t+\tau-1}X_{t+s}) + 2\mu_X^3] \\
 &= \alpha(1-e^{-\lambda})C_3(s,\tau-1) \\
 &= (\alpha(1-e^{-\lambda}))^{\tau-s} C_3(s,s).
 \end{aligned}$$

4 Spectral and Bispectral Density Functions

The non-normalized spectral density function $f_{XX}(\omega)$ of ZTPINAR(1) is calculated as (see [14])

$$f_{XX}(\omega) = \frac{1 - \alpha^2(1 - e^{-\lambda})^2}{2\pi(1 + \alpha^2(1 - e^{-\lambda})^2 - 2\alpha(1 - e^{-\lambda}) \cos \omega)} \tag{5}$$

The normalized spectral density function $g_{XX}(\omega)$ is calculated as

$$g_{XX}(\omega) = \frac{\lambda e^\lambda (e^\lambda - \lambda - 1)(1 - \alpha^2(1 - e^{-\lambda})^2)}{2\pi(e^\lambda - 1)^2(1 + \alpha^2(1 - e^{-\lambda})^2 - 2\alpha(1 - e^{-\lambda}) \cos \omega)} \tag{6}$$

Theorem 2. The bispectral density function $f_{XXX}(\omega_1, \omega_2)$ of ZTPINAR(1) is calculated as

$$\begin{aligned}
 f_{XXX}(\omega_1, \omega_2) &= \frac{1}{(2\pi)^2} [C_3(0,0) + C_3(0,0) \times \\
 &\{H_1(-\omega_1) + H_1(-\omega_2) + H_1(\omega_1 + \omega_2)\} \\
 &+ (C_3(0,0) - \frac{gC_2(0)}{\alpha(1-e^{-\lambda}) - \alpha^2(1-e^{-\lambda})}) \times \\
 &\{H_2(\omega_1) + H_2(\omega_2) + H_2(-\omega_1 - \omega_2)\} \\
 &+ (\frac{gC_2(0)}{\alpha(1-e^{-\lambda}) - \alpha^2(1-e^{-\lambda})}) \times
 \end{aligned}$$

$$\begin{aligned}
& \{H_1(\omega_1) + H_1(\omega_2) + F_1(-\omega_1 - \omega_2)\} \\
& + (C_3(0,0) - \frac{gC_2(0)}{\alpha(1-e^{-\lambda}) - \alpha^2(1-e^{-\lambda})}) \times \\
& \{H_2(-\omega_1 - \omega_2)H_1(-\omega_2) + H_2(-\omega_1 - \omega_2)H_1(-\omega_1) \\
& + H_2(\omega_2)H_1(-\omega_2) + H_2(\omega_2)H_1(-\omega_1) \\
& + H_2(\omega_1)H_1(\omega_1 + \omega_2) + H_2(\omega_2)H_1(\omega_1 + \omega_2)\} \\
& + (\frac{gC_2(0)}{\alpha(1-e^{-\lambda}) - \alpha^2(1-e^{-\lambda})}) \\
& \times \{H_1(-\omega_1 - \omega_2)H_1(-\omega_2) + H_1(-\omega_1 - \omega_2)H_1(-\omega_1) \\
& + H_1(\omega_1)H_1(-\omega_2) + H_1(\omega_2)H_1(-\omega_1) \\
& + H_1(\omega_1)H_1(\omega_1 + \omega_2) + H_1(\omega_2)H_1(\omega_1 + \omega_2)\}, \quad (7)
\end{aligned}$$

where

$$H_1(\omega_k) = \frac{\alpha(1-e^{-\lambda})e^{i\omega_k}}{1-\alpha(1-e^{-\lambda})e^{i\omega_k}}$$

and

$$H_2(\omega_k) = \frac{\alpha^2(1-e^{-\lambda})e^{i\omega_k}}{1-(\alpha^2(1-e^{-\lambda})e^{i\omega_k})}, \quad k = 1, 2.$$

Proof. We can write $f_{XXX}(\omega_1, \omega_2)$ as (see [14]):

$$\begin{aligned}
f_{XXX}(\omega_1, \omega_2) &= \frac{1}{(2\pi)^2} \left[\sum_{t_1=0}^{\infty} \sum_{t_2=t_1}^{\infty} C_3(t_1, t_2) e^{-i(t_1\omega_1 + t_2\omega_2)} \right. \\
&+ \sum_{t_2=0}^{\infty} \sum_{t_1=t_2+1}^{\infty} C_3(t_2, t_1) e^{-i(t_1\omega_1 + t_2\omega_2)} \\
&+ \sum_{t_1=0}^{\infty} \sum_{t_2=-\infty}^{-1} C_3(-t_2, t_1 - t_2) e^{-i(t_1\omega_1 + t_2\omega_2)} \\
&+ \sum_{t_1=-\infty}^{-1} \sum_{t_2=-\infty}^{t_1-1} C_3(t_1 - t_2, -t_2) e^{-i(t_1\omega_1 + t_2\omega_2)} \\
&+ \sum_{t_2=-\infty}^{-1} \sum_{t_1=-\infty}^{-t_2} C_3(t_2 - t_1, -t_1) e^{-i(t_1\omega_1 + t_2\omega_2)} \\
&+ \left. \sum_{t_1=-\infty}^{-1} \sum_{t_2=0}^{\infty} C_3(-t_1, t_2 - t_1) e^{-i(t_1\omega_1 + t_2\omega_2)} \right]
\end{aligned}$$

using the symmetry properties of the third-order cumulants (see [9]) in the equation above, then

$$\begin{aligned}
f_{XXX}(\omega_1, \omega_2) &= \frac{1}{(2\pi)^2} [C_3(0,0) \\
&+ \sum_{\tau=1}^{\infty} C_3(0, \tau) \{e^{-i\tau\omega_1} + e^{-i\tau\omega_2} + e^{i\tau(\omega_1 + \omega_2)}\} \\
&+ \sum_{\tau=1}^{\infty} C_3(\tau, \tau) \{e^{i\tau\omega_1} + e^{i\tau\omega_2} + e^{-i\tau(\omega_1 + \omega_2)}\} \\
&+ \sum_{s=1}^{\infty} \sum_{\tau=1}^{\infty} C_3(s, s + \tau) \{e^{-is\omega_1 - i(s+\tau)\omega_2} \\
&+ e^{-is\omega_2 - i(s+\tau)\omega_1} + e^{is\omega_1 - i\tau\omega_2} e^{is\omega_2 - i\tau\omega_1} \\
&+ e^{i\tau\omega_1 + i(s+\tau)\omega_2} + e^{i\tau\omega_2 + i(s+\tau)\omega_1}\}
\end{aligned}$$

Using expressions of $C_3(0, \tau)$, $C_3(\tau, \tau)$ and $C_3(s, s + \tau) = (\alpha(1-e^{-\lambda}))^\tau C_3(s, s)$ given by Theorem 1, we get

$$\begin{aligned}
f_{XXX}(\omega_1, \omega_2) &= \frac{1}{(2\pi)^2} [C_3(0,0) + \sum_{\tau=1}^{\infty} (\alpha(1-e^{-\lambda}))^\tau \times \\
&C_3(0,0) \{e^{-i\tau\omega_1} + e^{-i\tau\omega_2} + e^{i\tau(\omega_1 + \omega_2)}\} \\
&+ \sum_{\tau=1}^{\infty} [(\alpha^2(1-e^{-\lambda}))^\tau C_3(0,0) + gC_2(0) \times \\
&\frac{(\alpha(1-e^{-\lambda}))^\tau - (\alpha^2(1-e^{-\lambda}))^\tau}{\alpha(1-e^{-\lambda}) - \alpha^2(1-e^{-\lambda})}] \\
&\{e^{i\tau\omega_1} + e^{i\tau\omega_2} + e^{-i\tau(\omega_1 + \omega_2)}\} \\
&+ \sum_{s=1}^{\infty} \sum_{\tau=1}^{\infty} (\alpha(1-e^{-\lambda}))^\tau [(\alpha^2(1-e^{-\lambda}))^s C_3(0,0) \\
&+ gC_2(0) \frac{(\alpha(1-e^{-\lambda}))^s - (\alpha^2(1-e^{-\lambda}))^s}{\alpha(1-e^{-\lambda}) - \alpha^2(1-e^{-\lambda})}] \times \\
&\{e^{-is\omega_1 - i(s+\tau)\omega_2} + e^{-is\omega_2 - i(s+\tau)\omega_1} + e^{is\omega_1 - i\tau\omega_2} \\
&+ e^{is\omega_2 - i\tau\omega_1} + e^{i\tau\omega_1 + i(s+\tau)\omega_2} + e^{i\tau\omega_2 + i(s+\tau)\omega_1}\},
\end{aligned}$$

All these summations can be evaluated as follows, for example

$$\begin{aligned}
\sum_{\tau=1}^{\infty} (\alpha(1-e^{-\lambda}))^\tau e^{-i\tau\omega_1} &= \sum_{\tau=1}^{\infty} (\alpha(1-e^{-\lambda})e^{-i\omega_1})^\tau \\
&= \frac{\alpha(1-e^{-\lambda})e^{-i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_1}},
\end{aligned}$$

so after some calculations and computations for all summations, we have

$$\begin{aligned}
 f_{XXX}(\omega_1, \omega_2) &= \frac{1}{(2\pi)^2} [C_3(0,0) + C_3(0,0) \times \\
 &\left\{ \frac{\alpha(1-e^{-\lambda})e^{-i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_1}} + \frac{\alpha(1-e^{-\lambda})e^{-i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_2}} \right. \\
 &\left. + \frac{\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}} \right\} \\
 &+ (C_3(0,0) - \frac{gC_2(0)}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})}) \times \\
 &\left\{ \frac{\alpha^2(1-e^{-\lambda})e^{i\omega_1}}{1-\alpha^2(1-e^{-\lambda})e^{i\omega_1}} + \frac{\alpha^2(1-e^{-\lambda})e^{i\omega_2}}{1-\alpha^2(1-e^{-\lambda})e^{i\omega_2}} \right. \\
 &\left. + \frac{\alpha^2(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}{1-\alpha^2(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}} \right. \\
 &\left. + \frac{gC_2(0)}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})} \left\{ \frac{\alpha(1-e^{-\lambda})e^{i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{i\omega_1}} \right. \right. \\
 &\left. \left. + \frac{\alpha(1-e^{-\lambda})e^{i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{i\omega_2}} \right. \right. \\
 &\left. \left. + \frac{\alpha(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}} \right\} + (C_3(0,0) \right. \\
 &\left. - \frac{gC_2(0)}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})}) \times \\
 &\left\{ \frac{\alpha^2(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}{1-\alpha^2(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}} \frac{\alpha(1-e^{-\lambda})e^{-i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_2}} \right. \\
 &\left. + \frac{\alpha^2(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}{1-\alpha^2(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}} \times \right. \\
 &\frac{\alpha(1-e^{-\lambda})e^{-i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_1}} + \frac{\alpha^2(1-e^{-\lambda})e^{i\omega_1}}{1-\alpha^2(1-e^{-\lambda})e^{i\omega_1}} \times \\
 &\frac{\alpha(1-e^{-\lambda})e^{-i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_2}} + \frac{\alpha^2(1-e^{-\lambda})e^{i\omega_2}}{1-\alpha^2(1-e^{-\lambda})e^{i\omega_2}} \times \\
 &\frac{\alpha(1-e^{-\lambda})e^{-i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_1}} + \frac{\alpha^2(1-e^{-\lambda})e^{i\omega_1}}{1-\alpha^2(1-e^{-\lambda})e^{i\omega_1}} \times \\
 &\frac{\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}} + \frac{\alpha^2(1-e^{-\lambda})e^{i\omega_2}}{1-\alpha^2(1-e^{-\lambda})e^{i\omega_2}} \times \\
 &\left. \left. \frac{\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}} \right\} + \frac{gC_2(0)}{\alpha(1-e^{-\lambda})-\alpha^2(1-e^{-\lambda})} \times \right. \\
 &\left. \left\{ \frac{\alpha(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}} \times \right. \right. \\
 &\frac{\alpha(1-e^{-\lambda})e^{-i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_2}} + \frac{\alpha(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{-i(\omega_1+\omega_2)}} \times \\
 &\frac{\alpha(1-e^{-\lambda})e^{-i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_1}} + \frac{\alpha(1-e^{-\lambda})e^{i\omega_1}}{1-\alpha e^{i\omega_1}} \times \\
 &\frac{\alpha(1-e^{-\lambda})e^{-i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_2}} + \frac{\alpha(1-e^{-\lambda})e^{i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{i\omega_2}} \times \\
 &\frac{\alpha(1-e^{-\lambda})e^{-i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{-i\omega_1}} + \frac{\alpha(1-e^{-\lambda})e^{i\omega_1}}{1-\alpha(1-e^{-\lambda})e^{i\omega_1}} \times \\
 &\frac{\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}} + \frac{\alpha(1-e^{-\lambda})e^{i\omega_2}}{1-\alpha(1-e^{-\lambda})e^{i\omega_2}} \times \\
 &\left. \left. \frac{\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}}{1-\alpha(1-e^{-\lambda})e^{i(\omega_1+\omega_2)}} \right\} \right],
 \end{aligned}$$

by taking

$$H_1(\omega_k) = \frac{\alpha(1-e^{-\lambda})e^{i\omega_k}}{1-\alpha(1-e^{-\lambda})e^{i\omega_k}}$$

and

$$H_2(\omega_k) = \frac{\alpha^2(1-e^{-\lambda})e^{i\omega_k}}{1-(\alpha^2(1-e^{-\lambda})e^{i\omega_k})}, \quad k = 1, 2.,$$

the proof is complete.

The normalized bispectral density function $g_{XXX}(\omega_1, \omega_2)$ is calculated as

$$g_{XXX}(\omega_1, \omega_2) = \frac{f_{XXX}(\omega_1, \omega_2)}{\sqrt{f_{XX}(\omega_1)f_{XX}(\omega_2)f_{XX}(\omega_1 + \omega_2)}}, \quad (8)$$

where $f_{XXX}(\omega_1, \omega_2)$ and $f_{XX}(\omega_1)$ are defined in (5) and (7).

The bispectral density function provides us useful information about the non-linearity of the process. For continuous non- Gaussian time series the modulus of the normalized bispectrum is flat. The bispectral density function is a complex valued function takes the form

$$f_{XXX}(\omega_1, \omega_2) = r(\omega_1, \omega_2) + iq(\omega_1, \omega_2),$$

The modulus and phase of the bispectral density function are given respectively, by

$$|f_{XXX}(\omega_1, \omega_2)| = \sqrt{r^2(\omega_1, \omega_2) + q^2(\omega_1, \omega_2)}, \quad (9)$$

$$phase = \tan^{-1}\left(\frac{q(\omega_1, \omega_2)}{r(\omega_1, \omega_2)}\right). \quad (10)$$

Fig.1 illustrate the simulated series of the ZTPINAR(1) model at $\lambda = 1$ and $\alpha = 0.35$. From Fig.1, we conclude that the process is stationary and values of the simulated series are non negative integers, so the plot satisfy the definition of the ZTPINAR(1) model. The theoretical spectrum $f_{XX}(\omega)$, theoretical bispectrum $f_{XXX}(\omega_1, \omega_2)$ and normalized bispectrum modulus $g_{XXX}(\omega_1, \omega_2)$ are respectively computed by setting $\lambda = 1$ and $\alpha = 0.35$ in (5), (7) and (8), and they are represented by Fig.2, Fig.3 and Fig.4, respectively.

5 Estimation of Spectrum and Bispectrum

Estimates of the spectral, bispectral and normalized bispectral density functions are calculated using the smoothed periodogram method with different lag windows (as Daniell, Tukey Hamming and Parzen lag window) and simulated series $\{X_t, t = 1, 2, \dots, 500\}$ from the ZTPINAR(1) model that defined by (3). Generally if X_1, X_2, \dots, X_N be a realizations of a real valued third order stationary process $\{X_t\}$ with mean μ , autocovariance

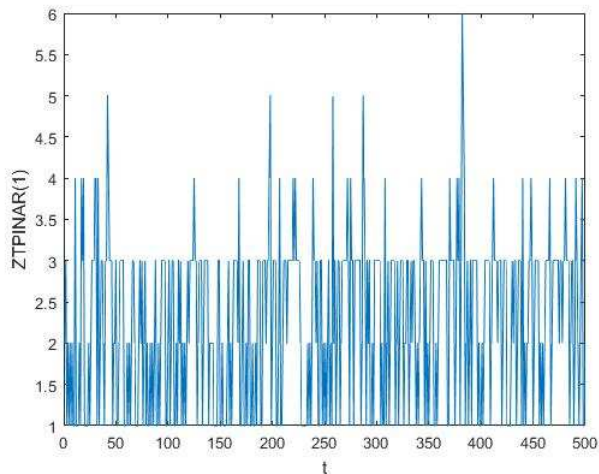


Fig. 1: The simulated series of the ZTPINAR(1) model at $\lambda = 1$ and $\alpha = 0.35$.

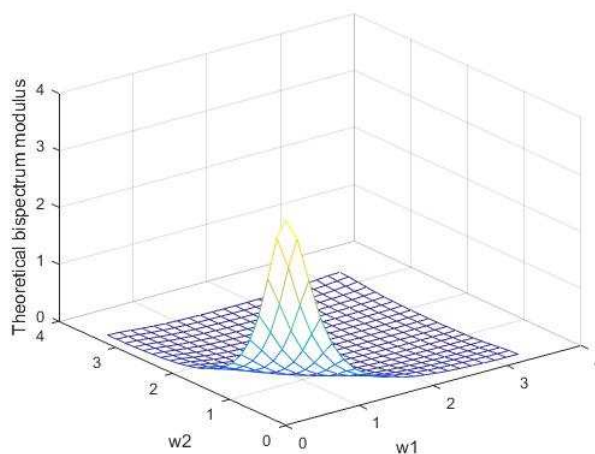


Fig. 3: The theoretical bispectral of the ZTPINAR(1) model at $\lambda = 1$ and $\alpha = 0.35$.

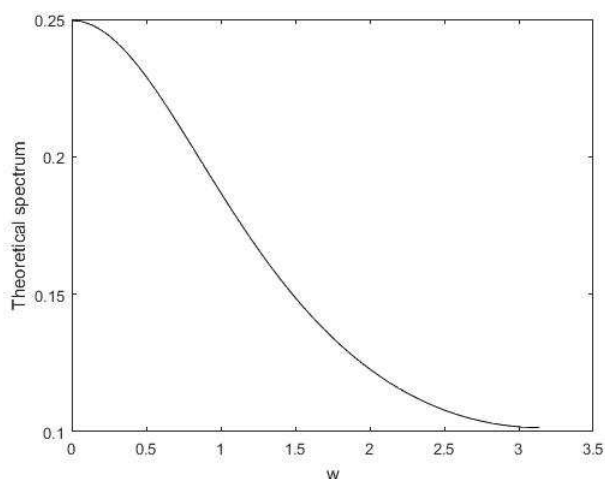


Fig. 2: The theoretical spectrum of the ZTPINAR(1) model at $\lambda = 1$ and $\alpha = 0.35$.

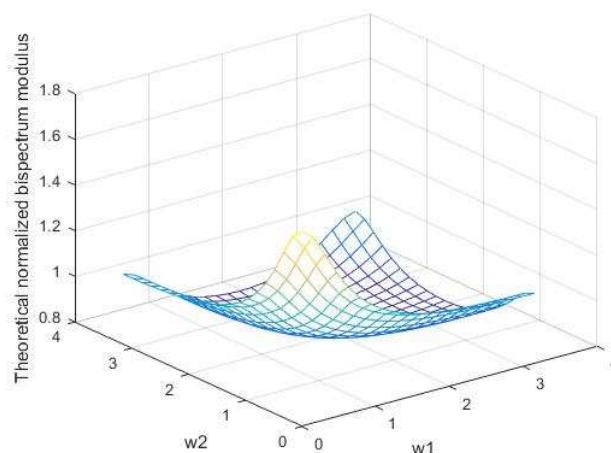


Fig. 4: The theoretical normalized bispectral of the ZTPINAR(1) model at $\lambda = 1$ and $\alpha = 0.35$.

$C_2(s)$ and third cumulant $C_3(s_1, s_2)$. The smoothed spectral and bispectral density functions are respectively given by (see [26])

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{2\pi} \sum_{s=-(N-1)}^{N-1} \lambda(s) \hat{C}_2(s) e^{-is\omega} \\ &= \frac{1}{2\pi} \sum_{s=-(N-1)}^{N-1} \lambda(s) \hat{C}_2(s) \cos\omega s, \end{aligned} \quad (11)$$

$$\begin{aligned} \hat{f}(\omega_1, \omega_2) &= \\ \frac{1}{4\pi^2} \sum_{s_1=-(N-1)}^{N-1} \sum_{s_2=-(N-1)}^{N-1} \lambda(s_1, s_2) \hat{C}_3(s_1, s_2) e^{-is_1\omega_1 - is_2\omega_2}, \end{aligned} \quad (12)$$

where $\hat{C}_2(s)$ and $\hat{C}_3(s_1, s_2)$ the natural estimators for estimators for $C_2(s)$ and $C_3(s_1, s_2)$ are, respectively, given by

$$\hat{C}_2(s) = \frac{1}{N-s} \sum_{t=1}^{N-|s|} (X_t - \bar{X})(X_{t+|s|} - \bar{X}), \quad (13)$$

$$\bar{X} = \frac{1}{N} \sum_{t=1}^N X_t,$$

$$\hat{C}_3(s_1, s_2) = \frac{1}{N} \sum_{t=1}^{N-\gamma} (X_t - \bar{X})(X_{t+s_1} - \bar{X})(X_{t+s_2} - \bar{X}) \quad (14)$$

where $s_1, s_2 \geq 0, \gamma = \max(0, s_1, s_2), s = 0, \pm 1, \pm 2, \dots, \pm(N - 1), -\pi \leq \omega_1, \omega_2 \leq \pi, \lambda(\cdot)$ is one-dimensional lag window and $\lambda(s_1, s_2)$ is two-dimensional lag window.

The normalized bispectrum is estimated by

$$\hat{g}(\omega_1, \omega_2) = \frac{\hat{f}(\omega_1, \omega_2)}{\sqrt{\hat{f}(\omega_1)\hat{f}(\omega_2)\hat{f}(\omega_1 + \omega_2)}} \quad (15)$$

To compare the spectral estimates with different lag windows With each other and with the theoretical spectrum, we used the sample mean square errors criterion for measuring the accuracy of $\hat{f}(\omega)$ as an estimate of $f_{XX}(\omega)$. This sample mean square error (M.S.E) is defined as

$$M.S.E = \frac{1}{k} \sum_i^k (\hat{f}(\omega_i) - f_{XX}(\omega_i))^2$$

where $\hat{f}(\omega_i)$ and $f_{XX}(\omega_i)$ are given by (11) and (6) respectively. k is the number of frequencies ω_i . As for the bispectral case, the M.S.E is defined as

$$M.S.E = \frac{1}{K} \sum_{i=1}^k \sum_{j=1}^k (|\hat{f}(\omega_i, \omega_j)| - |f_{XX}(\omega_i, \omega_j)|)^2,$$

where $|\hat{f}(\omega_i, \omega_j)|$ is the modulus of the bispectral density estimate and $|f_{XX}(\omega_i, \omega_j)|$ is the theoretical bispectral modulus. k is the number of frequencies ω_i or ω_j and $K = k^2$ is the total number of these frequencies (ω_i, ω_j) .

Firstly using the Daniell lag window, [28] introduced Daniell lag window as

$$\lambda(s) = \frac{\sin(\frac{s\pi}{M})}{\frac{s\pi}{M}} \quad (16)$$

where M is window parameter or number of frequencies used in smoothed. In this paper we choose M=7 for all different lag windows. The two dimensional lag window $\lambda(s_1, s_2)$, given by [29] is given by

$$\lambda(s_1, s_2) = \lambda(s_1)\lambda(s_2)\lambda(s_1 - s_2) \quad (17)$$

Fig.5 represent the theoretical spectrum and the estimated spectral density function using Daniell window with M=7 from (11) and (16). Fig.6 and Fig.7 represents the estimate of the bispectrum and normalized bispectrum modulus using Daniell window at M=7 as in (12), (17) (16) and (15).

Secondly using the Parzen lag window, [30] proposed the Parzen lag window

$$\lambda(s) = \begin{cases} 1 - 6s^2 + 6|s|^3 & |s| \leq \frac{1}{2} \\ 2(1 - |s|)^3 & \frac{1}{2} < |s| \leq 1, \\ 0 & |s| > 1 \end{cases} \quad (18)$$

and $\lambda(s_1, s_2)$ is given by (17). Fig.8 represent the theoretical spectrum and the estimated spectral density function using Parzen window with M=7 from (11) and (18). Fig.9 and Fig.10, represents the estimate of the bispectrum and normalized bispectrum modulus using Parzen window at M=7 as in (12), (17) (18) and (15).

Thirdly and finally using the Tukey Hamming window which reduced from [31] and given by

$$\lambda(s) = \begin{cases} 0.54 + 0.46 \cos(\frac{\pi s}{M}) & |s| \leq M \\ 0 & |s| > M \end{cases} \quad (19)$$

and $\lambda(s_1, s_2)$ is given by equation(17). Fig.11 represent the theoretical spectrum and the estimated spectral density function using Tukey Hamming window with M=7 from (11) and (19). Fig.12 and Fig.13, represents the estimate of the bispectrum and normalized bispectrum modulus using Tukey Hamming window at M=7 as in (12), (17) (19) and (15).

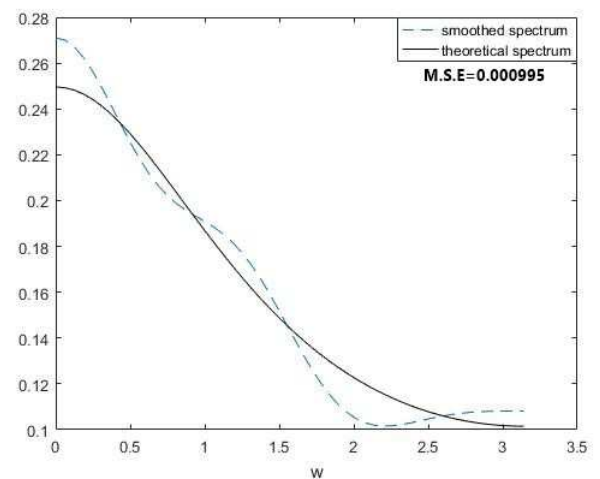


Fig. 5: Estimated spectrum using Daniell window at M=7 and theoretical spectrum.

Table 1: Theoretical bispectrum modulus of ZTPINAR(1) with $\lambda = 1$ and $\alpha = 0.35$.

ω_2	0.00 π	0.05 π	0.10 π	0.15 π	0.20 π	0.25 π	0.30 π	0.35 π	0.40 π	0.45 π	0.50 π	0.55 π	0.60 π	0.65 π	0.70 π	0.75 π	0.80 π	0.85 π	0.90 π	0.95 π	π	
ω_1																						
0.00 π	3.604	3.221	2.433	1.719	1.215	0.884	0.667	0.521	0.420	0.348	0.296	0.258	0.229	0.206	0.189	0.176	0.166	0.159	0.155	0.152	0.151	
0.05 π	3.221	2.646	1.933	1.365	0.979	0.725	0.556	0.441	0.361	0.303	0.261	0.229	0.205	0.186	0.172	0.161	0.153	0.148	0.144	0.142	0.142	
0.10 π	2.433	1.933	1.411	1.010	0.736	0.554	0.432	0.347	0.287	0.244	0.212	0.187	0.169	0.155	0.144	0.136	0.130	0.126	0.124	0.123	0.124	
0.15 π	1.719	1.365	1.010	0.735	0.545	0.416	0.329	0.267	0.224	0.191	0.168	0.150	0.136	0.125	0.117	0.112	0.107	0.105	0.103	0.103	0.105	
0.20 π	1.215	0.979	0.736	0.545	0.410	0.317	0.253	0.208	0.176	0.152	0.134	0.120	0.110	0.102	0.096	0.092	0.089	0.087	0.087	0.087	0.089	
0.25 π	0.884	0.725	0.554	0.416	0.317	0.248	0.200	0.166	0.141	0.123	0.109	0.099	0.091	0.085	0.081	0.078	0.076	0.075	0.075	0.076	0.078	
0.30 π	0.667	0.556	0.432	0.329	0.253	0.200	0.163	0.136	0.117	0.103	0.092	0.084	0.077	0.073	0.069	0.067	0.066	0.065	0.066	0.067	0.069	
0.35 π	0.521	0.441	0.347	0.267	0.208	0.166	0.136	0.115	0.099	0.088	0.079	0.072	0.067	0.064	0.061	0.060	0.059	0.059	0.060	0.061	0.064	
0.40 π	0.420	0.361	0.287	0.224	0.176	0.141	0.117	0.099	0.086	0.077	0.070	0.064	0.060	0.057	0.055	0.054	0.054	0.054	0.055	0.057	0.060	
0.45 π	0.348	0.303	0.244	0.191	0.152	0.123	0.103	0.088	0.077	0.069	0.063	0.058	0.055	0.052	0.051	0.050	0.050	0.051	0.052	0.055	0.058	
0.50 π	0.296	0.261	0.212	0.168	0.134	0.109	0.092	0.079	0.070	0.063	0.057	0.054	0.051	0.049	0.048	0.048	0.048	0.049	0.051	0.054	0.057	
0.55 π	0.258	0.229	0.187	0.150	0.120	0.099	0.084	0.072	0.064	0.058	0.054	0.050	0.048	0.047	0.046	0.046	0.047	0.048	0.050	0.054	0.058	
0.60 π	0.229	0.205	0.169	0.136	0.110	0.091	0.077	0.067	0.060	0.055	0.051	0.048	0.046	0.045	0.045	0.045	0.046	0.048	0.051	0.055	0.060	
0.65 π	0.206	0.186	0.155	0.125	0.102	0.085	0.073	0.064	0.057	0.052	0.049	0.047	0.045	0.044	0.044	0.045	0.047	0.049	0.052	0.057	0.064	
0.70 π	0.189	0.172	0.144	0.117	0.096	0.081	0.069	0.061	0.055	0.051	0.048	0.046	0.045	0.044	0.045	0.046	0.048	0.051	0.055	0.061	0.069	
0.75 π	0.176	0.161	0.136	0.112	0.092	0.078	0.067	0.060	0.054	0.050	0.048	0.046	0.045	0.045	0.046	0.048	0.050	0.054	0.060	0.067	0.078	
0.80 π	0.166	0.153	0.130	0.107	0.089	0.076	0.066	0.059	0.054	0.050	0.048	0.047	0.046	0.047	0.048	0.050	0.054	0.059	0.066	0.076	0.089	
0.85 π	0.159	0.148	0.126	0.105	0.087	0.075	0.065	0.059	0.054	0.051	0.049	0.048	0.048	0.049	0.051	0.054	0.059	0.065	0.075	0.087	0.105	
0.90 π	0.155	0.144	0.124	0.103	0.087	0.075	0.066	0.060	0.055	0.052	0.051	0.050	0.051	0.052	0.055	0.060	0.066	0.075	0.087	0.103	0.124	
0.95 π	0.152	0.142	0.123	0.103	0.087	0.076	0.067	0.061	0.057	0.055	0.054	0.054	0.055	0.057	0.061	0.067	0.076	0.087	0.103	0.123	0.142	
π	0.151	0.142	0.124	0.105	0.089	0.078	0.069	0.064	0.060	0.058	0.057	0.058	0.060	0.064	0.069	0.078	0.089	0.105	0.124	0.142	0.151	

Table 2: Theoretical normalized bispectrum modulus of ZTPINAR(1) with $\lambda = 1$ and $\alpha = 0.35$.

ω_2	0.00 π	0.05 π	0.10 π	0.15 π	0.20 π	0.25 π	0.30 π	0.35 π	0.40 π	0.45 π	0.50 π	0.55 π	0.60 π	0.65 π	0.70 π	0.75 π	0.80 π	0.85 π	0.90 π	0.95 π	π	
ω_1																						
0.00 π	1.654	1.615	1.526	1.434	1.356	1.297	1.252	1.218	1.192	1.172	1.156	1.144	1.135	1.127	1.121	1.117	1.113	1.110	1.109	1.108	1.107	
0.05 π	1.615	1.551	1.461	1.377	1.309	1.257	1.218	1.188	1.165	1.147	1.133	1.122	1.114	1.107	1.102	1.098	1.095	1.093	1.091	1.090	1.090	
0.10 π	1.526	1.461	1.381	1.307	1.247	1.201	1.165	1.139	1.118	1.102	1.089	1.079	1.072	1.066	1.061	1.057	1.055	1.053	1.052	1.051	1.052	
0.15 π	1.434	1.377	1.307	1.241	1.187	1.145	1.113	1.089	1.070	1.055	1.043	1.034	1.027	1.021	1.017	1.014	1.011	1.010	1.009	1.009	1.010	
0.20 π	1.356	1.309	1.247	1.187	1.138	1.099	1.069	1.046	1.028	1.014	1.004	0.995	0.988	0.983	0.979	0.976	0.974	0.973	0.973	0.973	0.974	
0.25 π	1.297	1.257	1.201	1.145	1.099	1.063	1.034	1.013	0.996	0.982	0.972	0.964	0.958	0.953	0.949	0.947	0.945	0.944	0.944	0.945	0.947	
0.30 π	1.252	1.218	1.165	1.113	1.069	1.034	1.007	0.986	0.970	0.958	0.948	0.940	0.934	0.930	0.927	0.924	0.923	0.923	0.923	0.924	0.927	
0.35 π	1.218	1.188	1.139	1.089	1.046	1.013	0.986	0.966	0.951	0.939	0.929	0.922	0.916	0.912	0.909	0.907	0.906	0.906	0.907	0.909	0.912	
0.40 π	1.192	1.165	1.118	1.070	1.028	0.996	0.970	0.951	0.936	0.924	0.915	0.908	0.903	0.899	0.896	0.895	0.894	0.895	0.896	0.899	0.903	
0.45 π	1.172	1.147	1.102	1.055	1.014	0.982	0.958	0.939	0.924	0.913	0.904	0.897	0.892	0.889	0.887	0.886	0.886	0.887	0.889	0.892	0.897	
0.50 π	1.156	1.133	1.089	1.043	1.004	0.972	0.948	0.929	0.915	0.904	0.896	0.889	0.885	0.882	0.880	0.879	0.880	0.882	0.885	0.889	0.896	
0.55 π	1.144	1.122	1.079	1.034	0.995	0.964	0.940	0.922	0.908	0.897	0.889	0.883	0.879	0.877	0.875	0.875	0.877	0.879	0.883	0.889	0.897	
0.60 π	1.135	1.114	1.072	1.027	0.988	0.958	0.934	0.916	0.903	0.892	0.885	0.879	0.876	0.873	0.873	0.873	0.876	0.879	0.885	0.892	0.903	
0.65 π	1.127	1.107	1.066	1.021	0.983	0.953	0.930	0.912	0.899	0.889	0.882	0.877	0.873	0.872	0.872	0.873	0.877	0.882	0.889	0.899	0.912	
0.70 π	1.121	1.102	1.061	1.017	0.979	0.949	0.927	0.909	0.896	0.887	0.880	0.875	0.873	0.872	0.873	0.875	0.880	0.887	0.896	0.909	0.927	
0.75 π	1.117	1.098	1.057	1.014	0.976	0.947	0.924	0.907	0.895	0.886	0.879	0.875	0.873	0.873	0.875	0.879	0.886	0.895	0.907	0.924	0.947	
0.80 π	1.113	1.095	1.055	1.011	0.974	0.945	0.923	0.906	0.894	0.886	0.880	0.877	0.876	0.877	0.880	0.886	0.894	0.906	0.923	0.945	0.974	
0.85 π	1.110	1.093	1.053	1.010	0.973	0.944	0.923	0.906	0.895	0.887	0.882	0.879	0.879	0.882	0.887	0.895	0.906	0.923	0.944	0.973	1.010	
0.90 π	1.109	1.091	1.052	1.009	0.973	0.944	0.923	0.907	0.896	0.889	0.885	0.883	0.885	0.889	0.896	0.907	0.923	0.944	0.973	1.009	1.052	
0.95 π	1.108	1.090	1.051	1.009	0.973	0.945	0.924	0.909	0.899	0.892	0.889	0.889	0.892	0.899	0.909	0.924	0.945	0.973	1.009	1.051	1.090	
π	1.107	1.090	1.052	1.010	0.974	0.947	0.927	0.912	0.903	0.897	0.896	0.897	0.903	0.912	0.927	0.947	0.974	1.010	1.052	1.090	1.107	

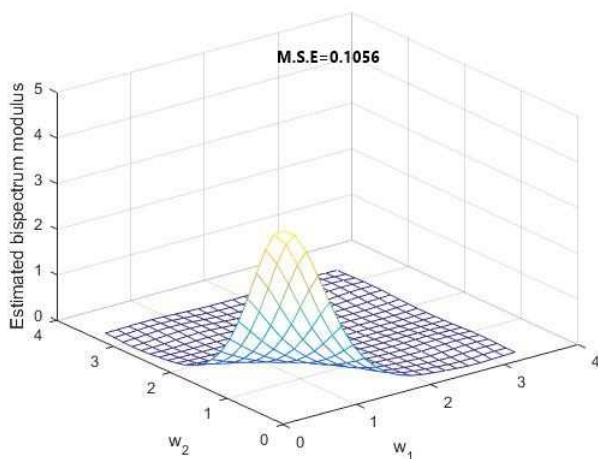


Fig. 6: Estimated bispectrum modulus using Daniell window at M=7.

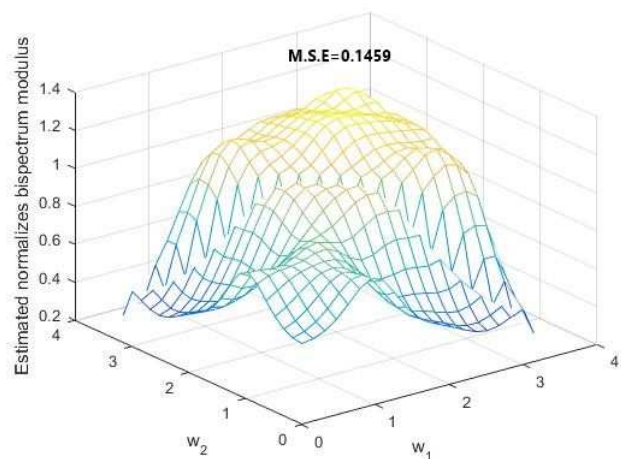


Fig. 7: Estimated normalized bispectrum modulus using Daniell window at M=7.

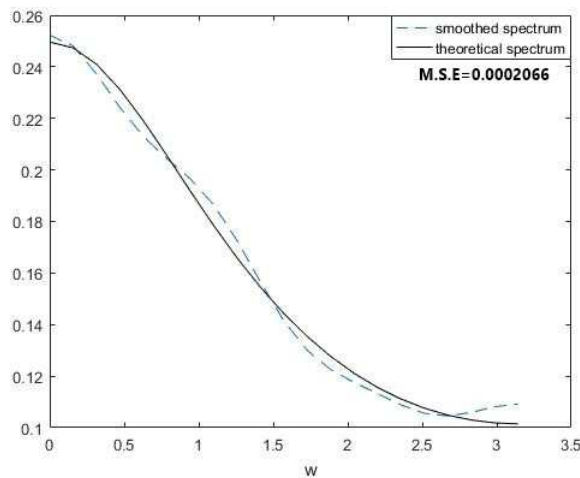


Fig. 8: Estimated spectrum using Parzen window at $M=7$ and theoretical spectrum.

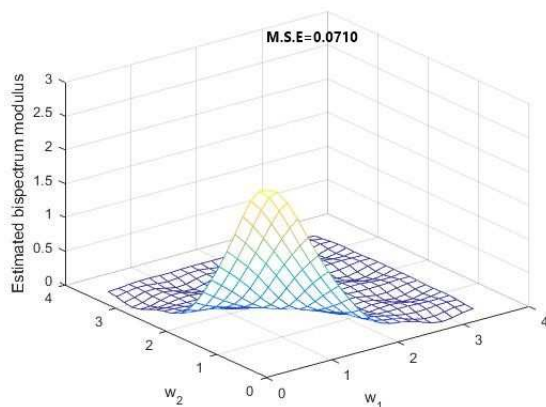


Fig. 9: Estimated bispectrum modulus using Parzen window at $M=7$.

–Depending on M.S.E which appears on each image, we find that

–When comparing the smoothed spectrum based on Daniell, Parzen, Tukey Hamming lag windows with each other and with the theoretical spectrum that calculated at $\lambda = 1$ and $\alpha = 0.35$, we conclude that using the Parzen window is the appropriate window among the other lag windows in Figures 5, 8 and 11, since the smoothed spectrum using Parzen window is closer to the theoretical spectrum than the smoothed spectrum based on Daniell or Tukey Hamming window.

–Also, Figures 6, 9 and 12 show that the parzen window is the appropriate window among the

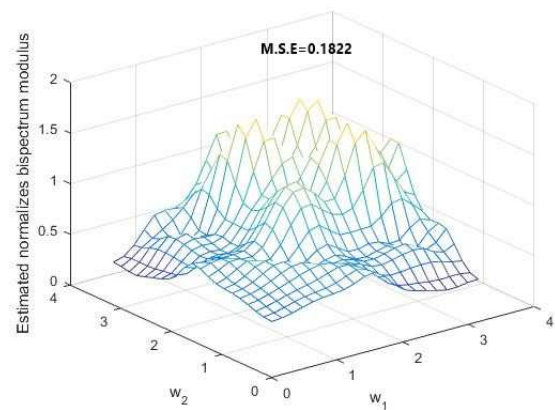


Fig. 10: Estimated normalized bispectrum modulus using Parzen window at $M=7$.

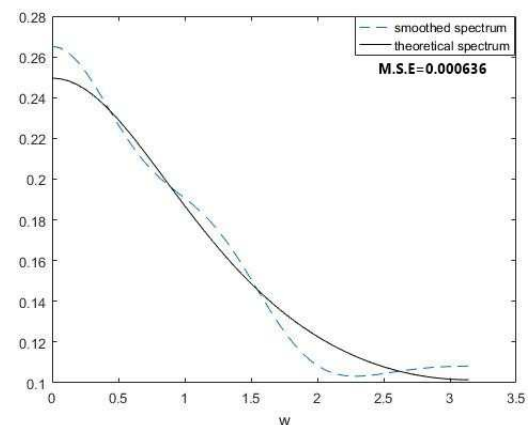


Fig. 11: Estimated spectrum using Tukey Hamming window at $M=7$ and theoretical spectrum.

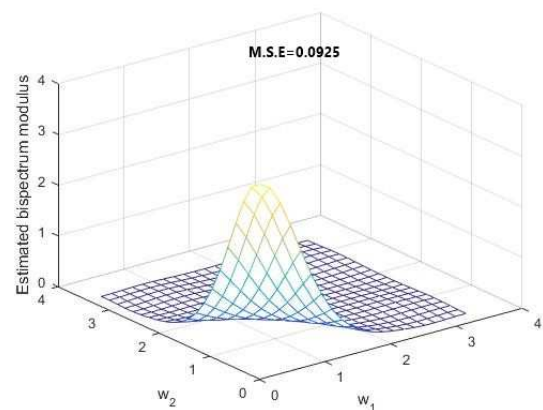


Fig. 12: Estimated bispectrum modulus using Tukey Hamming window at $M=7$.

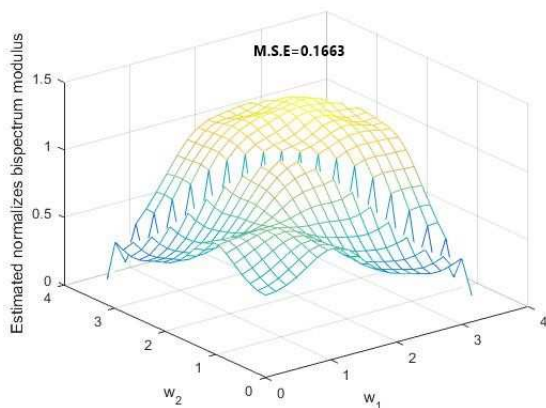


Fig. 13: Estimated normalized bispectrum modulus using Tukey Hamming window at $M=7$.

other lag windows for estimating the bispectrum modulus, since the smoothed bispectrum using Parzen window to the theoretical bispectrum modulus than the smoothed bispectrum based on Daniell or Tukey Hamming windows.

–Moreover, Figures 7, 10 and 13 show that the Daniell window is the appropriate window among the other lag windows for estimating the normalized bispectrum modulus, since the smoothed bispectrum using Daniell window to the theoretical bispectrum modulus than the smoothed bispectrum based on Daniell or Tukey Hamming windows.

–From Fig.3 and Fig.4 and Tabel 1 and Tabel 2, the normalized bispectrum modulus of the ZTPINAR(1) is more flat than the non-normalized bispectrum modulus, since the values of the normalized bispectrum modulus lies between (0.8,1.7) and the non-normalized bispectrum modulus lies between (0,4). This indicates that the test of linearity given by [32] can be used for integer valued time series models.

6 Conclusions

We studied in this paper, some higher order moments, cumulants, spectrum, bispectrum and normalized bispectrum density functions of the ZTPINAR(1) model. The spectrum, bispectrum and normalized bispectrum are estimated using a smoothed periodogram based on the different lag windows (Daniell, parzen and Tukey Hamming lag windows) and using a simulated series from this process. The normalized bispectrum modulus of the mentioned model is more flat than the non-normalized bispectrum modulus, so the test of linearity given by [32] can be used for integer valued time series models.

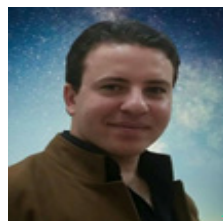
Conflict of Interest

The author declare that there is no conflict of interest regarding the publication of this article.

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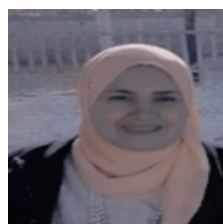
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