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# On the Determination of Spacelike Ruled and Developable Surfaces in Minkowski 3-Space

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**Abstract:** In this paper, we introduce a method for determination of spacelike ruled surface from the coordinates and the first derivatives of the base curve by making use of dual vector calculus. Consequently, we discuss the method for spacelike developable ruled surfaces, and obtain a linear differential equation of first order. Finally, this method is demonstrated through several examples.

**Keywords:** Distribution parameter, Plucker coordinates, E. Study's map.
**MSC (2010)**: Primary 53A35. Secondary 53B30, 57R45, 58C27, 58K05.

## 1 Introduction

In spatial kinematics, the motion of an oriented line over a curve performs ruled surface. This oriented line is said to be ruling (generator), and each curve that meets all the rulings is called a directrix (base curve). The theory of ruled surfaces has been highlighted by researchers as well mathematicians because it has many applications such as screw systems, displacement analysis of spatial mechanisms, and computer aided design (CAD). Therefore, much researchers have investigated and gained numerous properties of the ruled surfaces [1-8]. Developable ruled surface is a special ruled surface, where whole points of the same ruling have a common tangent plane. The rulings are principal curvature lines with vanishing normal curvature and the Gaussian curvature vanishing at whole surface points. As we all known, the inner metric of a surface locates the Gaussian curvature, therefore all the angles, and the lengths on the surface remain invariant under bending. This feature is what makes ruled and developable surfaces are important in manufacturing. Hence, both ruled and developable surfaces have been paid attention in engineering, architecture, and design, etc. [8-10].

One of the most convenient methods to research the motion of oriented line space seems to organize a correlation among this space, dual numbers and dual vector calculus. Dual numbers were first introduced by W. Clifford after him E. Study utilized it as a tool for his study on the differential line geometry and kinematics. He devoted special care to the exemplification of oriented lines by dual unit vectors and defined the mapping that is known by his name. The E. Study map states that: The set of all oriented lines in Euclidean 3-space $\mathbb{E}^3$ is directly linked to the set of points on the dual unit sphere in the dual 3-space $\mathbb{D}^3$ [1, 5, 8]. Thus, the differential geometry of ruled surfaces based on the E. Study map has revised the curvature theory of a line trajectory and exposed the essential curvature functions that describe the form of ruled surface (See for example [11-14]).

Kose introduced a new method for determination of developable ruled surfaces by using dual vector calculus [15]. He showed that a developable ruled surface can be gained from coordinates and the first derivatives of the base curve. And also, Yildz et al. applied this method by using orthotomic concept [16]. In the course of time, the extensions of this method have been presented in the dual Lorentzian 3-space $\mathbb{D}^3_1$ by [17-20].

However, to the best of the authors' knowledge, no literature exists regarding the ruled surface can be constructed from coordinates and the first derivatives of the base curve. Thus, the present study hopes to serve such a need. This work is organized as follows: In section

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2, we give some of the basic concepts dealing with the dual Lorentzian 3-space $\mathbb{D}_1^3$. In section 3, we offer a method for determination of spacelike ruled surface from the coordinates and the first derivatives of the base curve by using dual vector calculus. Then, as a special case, we discuss the method for spacelike developable ruled surfaces, and obtain a linear differential equation of first order. We illustrate the method by giving some representative examples with their figures.

## 2 Basic concepts

We start with basic concepts on the Minkowski 3–space $\mathbb{E}_1^3$, the theory of dual numbers, dual Lorentzian vectors and E. Study map [1-3, 21-24]. A dual number $A$ is a number $a+\varepsilon a^*$, where $a$, $a^*$ in $\mathbb{R}$ and $\varepsilon$ is a dual unit with the property that $\varepsilon^2=0$. Then the set

$$\mathbb{D}^3=\{A:=\mathbf{a}+\varepsilon\mathbf{a}^*=(A_1,A_2,A_3)\},$$

together with the Lorentzian scalar product

$$<\mathbf{A},\mathbf{A}>=-A_1^2+A_2^2+A_3^2,$$

forms the so called dual Lorentzian 3-space $\mathbb{D}_1^3$. Thus, a point $A=(A_1,A_2,A_3)^t$ has dual coordinates $A_i=(a_i+\varepsilon a_i^*)\in\mathbb{D}$. If $A$ is spacelike or timelike dual vector the norm $\|\mathbf{A}\|$ of $A$ is defined by

$$\begin{aligned}\|\mathbf{A}\| &= \sqrt{|<\mathbf{A},\mathbf{A}>|}=\sqrt{|<\mathbf{a},\mathbf{a}>|}\\ &+\varepsilon\frac{1}{2\sqrt{|<\mathbf{a},\mathbf{a}>|}}\frac{<\mathbf{a},\mathbf{a}>}{|<\mathbf{a},\mathbf{a}>|}.2<\mathbf{a},\mathbf{a}^*>\\ &=\|\mathbf{a}\|+\varepsilon\frac{1}{\|\mathbf{a}\|}\frac{<\mathbf{a},\mathbf{a}>}{|<\mathbf{a},\mathbf{a}>|}<\mathbf{a},\mathbf{a}^*>.\end{aligned}$$

If $\mathbf{a}$ is spacelike, we have

$$\|\mathbf{A}\|=\|\mathbf{a}\|+\varepsilon\frac{1}{\|\mathbf{a}\|}<\mathbf{a},\mathbf{a}^*>=\|\mathbf{a}\|\left(1+\varepsilon\frac{1}{\|\mathbf{a}\|^2}<\mathbf{a},\mathbf{a}^*>\right).$$

If $\mathbf{a}$ is timelike, we have

$$\|\mathbf{A}\|=\|\mathbf{a}\|-\varepsilon\frac{1}{\|\mathbf{a}\|}<\mathbf{a},\mathbf{a}^*>=\|\mathbf{a}\|\left(1-\varepsilon\frac{1}{\|\mathbf{a}\|^2}<\mathbf{a},\mathbf{a}^*>\right).$$

Therefore, $\mathbf{A}$ is called a spacelike (resp. timelike) dual unit vector if $<A,A>=1$(resp. $<A,A>=-1$). The hyperbolic and Lorentzian dual unit spheres, respectively, are

$$\mathbb{H}_+^2=\left\{\mathbf{A}\in\mathbb{D}_1^3 \mid -A_1^2+A_2^2+A_3^2=-1\right\},$$

and

$$\mathbb{S}_1^2=\left\{\mathbf{A}\in\mathbb{D}_1^3 \mid -A_1^2+A_2^2+A_3^2=1\right\}.$$

**Theorem 1**. There is a one-to-one correspondence between spacelike (resp. timelike) oriented lines in Minkowski 3-space $\mathbb{E}_1^3$ and ordered pairs of vectors $(\mathbf{a},\mathbf{a}^*)\in\mathbb{E}_1^3\times\mathbb{E}_1^3$ such that

$$\|\mathbf{A}\|^2=\pm1\Longleftrightarrow\|\mathbf{a}\|^2=\pm1,\ <\mathbf{a},\mathbf{a}^*>=0, \qquad (1)$$

where $a$, and $a^*$ are called the normed Plücker coordinates of the line.

Via Theorem 1 we have the following map (E. Study's map): The dual unit spheres are shaped as a pair of conjugate hyperboloids. The ring shaped hyperboloid represents the set of spacelike lines, the common asymptotic cone represents the set of null (lightlike) lines, and the oval shaped hyperboloid forms the set of timelike lines, opposite points of each hyperboloid perform the pair of obverse vectors on a line (see Fig. 1). Applying to E. Study map, a differentiable curve on $\mathbb{H}_+^2$ corresponds to a timelike ruled surface in $\mathbb{E}_1^3$. Similarly the dual curve on $\mathbb{S}_1^2$ corresponds to a spacelike or timelike ruled surface in $\mathbb{E}_1^3$.

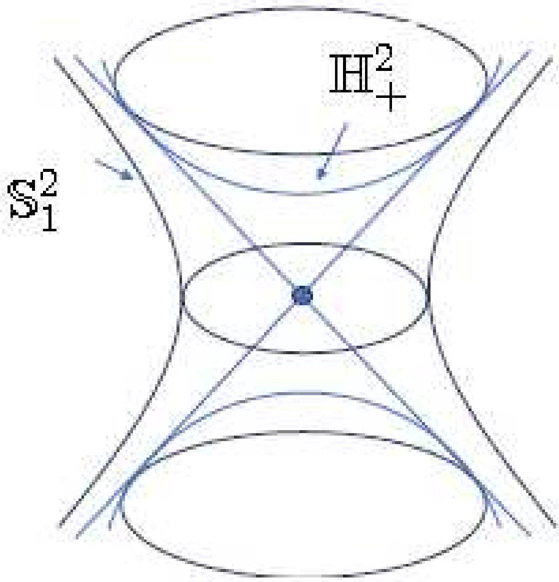


**Fig. 1:** The dual hyperbolic and dual Lorentaian unit spheres

## 3 Spacelike ruled and developable surfaces

In the Minkowski 3-space $\mathbb{E}_1^3$, a continuously moving of a spacelike line generates a ruled surface which can be timelike or spacelike ruled surface. It will be assumed a spacelike ruled surface in our study. If we consider a point on this line, the trajectory of the point generates a base curve in $\mathbb{E}_1^3$ while the line generates the spacelike ruled surface. Then, we can develop a procedure to construct spacelike ruled and developable surfaces from coordinates and the first derivatives of the base curve by using the E. Study map as follows: Let $\mathbf{b}(t)$ be a regular curve in the Minkowski 3-space $\mathbb{E}_1^3$ defined on $I \subseteq \mathbb{R}$, and $\mathbf{a}(t)$ be a spacelike unit vector of an oriented line in $\mathbb{E}_1^3$. Then a spacelike ruled surface $M$ can be defined as follows:

$$M : \mathbf{r}(t,v) = \mathbf{b}(t) + v\mathbf{a}(t), v \in \mathbb{R}, \tag{2}$$

where $\mathbf{b}(t)$ is its base (directrix) curve, and $t$ is the motion parameter. The E. Study map admits us to revision Eq. (2) using the dual vector function as

$$M : \mathbf{A}(t) = \mathbf{a}(t) + \varepsilon\mathbf{b}(t) \times \mathbf{a}(t) = \mathbf{a}(t) + \varepsilon\mathbf{a}^*(t). \tag{3}$$

Since $\|\mathbf{a}\|^2 = 1$, the spacelike dual vector $\mathbf{A}$ as well has unit magnitude as is seen from the following computation

$$\begin{aligned}\|\mathbf{A}\|^2 &= <\mathbf{a}+\varepsilon\mathbf{b}\times\mathbf{a}, \mathbf{a}+\varepsilon\mathbf{b}\times\mathbf{a}> \\ &= \|\mathbf{a}\|^2 + 2\varepsilon <\mathbf{a}, \mathbf{b}\times\mathbf{a}> + \varepsilon^2 <\mathbf{b}\times\mathbf{a}, \mathbf{b}\times\mathbf{a}> \\ &= \|\mathbf{a}\|^2 = 1.\end{aligned}$$

The dual arc-length of $\mathbf{A}(t)$ is

$$dS = ds + \varepsilon ds^* = \left\|\mathbf{A}'\right\| dt = \left\|\mathbf{a}'\right\| \left(1 + \varepsilon \frac{<\mathbf{a}', \mathbf{a}^{*'}>}{\|\mathbf{a}'\|^2}\right) dt.$$

Hence, the distribution parameter is given by

$$\lambda(t) := \frac{ds^*}{ds} = \frac{<\mathbf{a}', \mathbf{a}^{*'}>}{\|\mathbf{a}'\|^2}. \tag{4}$$

In Eq. (4): (i) if $\lambda(t) = 0$, then $M$ is a spacelike developable ruled surface (ii) if $\mathbf{a}' = \mathbf{0}$, then $M$ is a spacelike cylinder. Here and what follows a dash denotes derivatives with respect to $t$.

The dual coordinates $A_i = (a_i + \varepsilon a_i^*)$ of an arbitrary point $\mathbf{A}$ of $\mathbb{S}_1^2$, fastened at the origin, may be represented as:

$$\mathbf{A} = (\sinh\Phi, \cosh\Phi\cos\Omega, \cosh\Phi\sin\Omega), \tag{5}$$

where $\Phi = \phi + \varepsilon\phi^*$, and $\Omega = \omega + \varepsilon\omega^*$ are dual hyperbolic and spacelike angles with $\phi^*$, $\phi$, $\omega^* \in \mathbb{R}$, and $0 \le \omega \le 2\pi$, respectively. Furthermore, let $\mathbf{A} = \mathbf{A}(t)$, $t \in \mathbb{R}$ correspond to a spacelike ruled surface $M$ . So, the dual arc-length of $\mathbf{A}(t)$ is given by

$$dS := ds + \varepsilon ds^* = \sqrt{\Psi'^2\cosh^2\Phi - \Phi'^2}dt. \tag{6}$$

According to the real and the dual parts of Eq. (6), respectively, we get:

$$ds = \sqrt{\omega'^2\cosh^2\phi - \phi'^2}dt,$$

and

$$ds^* = \frac{-\phi'\phi^{*'} + \phi^*\omega'^2\sinh\phi\cosh\phi + \omega'\omega^{*'}\cosh^2\phi}{\sqrt{\omega'^2\cosh^2\phi - \phi'^2}}.$$

Thus

$$\lambda(t) = \frac{-\phi'\phi^{*'} + \phi^*\omega'^2\sinh\phi\cosh\phi + \omega'\omega^{*'}\cosh^2\phi}{\omega'^2\cosh^2\phi - \phi'^2}. \tag{7}$$

Since $\varepsilon \neq 0$, and $\varepsilon^2 = \varepsilon^3 = ... = 0$, the Plucker coordinates of $\mathbf{A}$ are:

$$\left.\begin{array}{l} a_1 = \sinh\phi,\ a_1^* = \phi^*\cosh\phi, \\ a_2 = \cosh\phi\cos\omega,\ \ a_2^* = \phi^*\sinh\phi\cos\omega - \omega^*\sin\omega\cosh\phi, \\ a_3 = \cosh\phi\sin\omega,\ \ a_3^* = \phi^*\sinh\phi\sin\omega + \omega^*\cos\omega\cosh\phi. \end{array}\right\} \tag{8}$$

Now, it seems natural to pose the next question: when we are given a curve $\mathbf{b}(t) = (b_1(t), b_2(t), b_3(t))$ can we find a spacelike ruled surface such that its base curve is the curve $\mathbf{b}(t)$?. The answer is affirmative and can be declared as follows: Since $\mathbf{a}^* = \mathbf{b}\times\mathbf{a}$ we have the system of linear equations in $b_i$ for i=1, 2, 3 ($b_{is}$ are the coordinates of $\mathbf{b}$):

$$\left.\begin{array}{r} -b_2\cosh\phi\sin\omega + b_3\cosh\phi\cos\omega = a_1^*, \\ -b_1\cosh\phi\sin\omega + b_3\sinh\phi = a_2^*, \\ b_1\cosh\phi\cos\omega - b_2\sinh\phi = a_3^*. \end{array}\right\} \tag{9}$$

The matrix of coefficients of unknowns $b_1$, $b_2$, and $b_3$ is the skew-adjoint matrix

$$\begin{pmatrix} 0 & -\cosh\phi\sin\omega & \cosh\phi\cos\omega \\ -\cosh\phi\sin\omega & 0 & \sinh\phi \\ \cosh\phi\cos\omega & -\sinh\phi & 0 \end{pmatrix},$$

and thus its rank is 2 with $\phi \neq 0$, and $\omega \neq 2\pi k$ ($k$ is an integer). The rank of the augmented matrix

$$\begin{pmatrix} 0 & -\cosh\phi\sin\omega & \cosh\phi\cos\omega & x_1^* \\ -\cosh\phi\sin\omega & 0 & \sinh\phi & x_2^* \\ \cosh\phi\cos\omega & -\sinh\phi & 0 & x_2^* \end{pmatrix},$$

is also 2. Thereby, this system has infinite solutions given by

$$\begin{array}{c} b_2 = (b_1 - \omega^*)\coth\phi\cos\omega - \phi^*\sin\omega, \\ b_3 = (b_1 - \omega^*)\coth\phi\sin\omega + \phi^*\cos\omega, \\ b_1 = b_1(\phi(t), \omega(t)). \end{array} \tag{10}$$

Since $b_1(t)$ can be arbitrarily, then we may take $b_1(t) = \omega^*(t)$. In this situation, Eqs. (10) reduces to

$$b_1 = \omega^*,\ b_2 = -\phi^*\sin\omega, b_3 = \phi^*\cos\omega. \tag{11}$$

From Eq. (11), we have

$$\phi^*(t) = \mp\sqrt{b_2^2 + b_3^2},\ \tan\omega = -\frac{b_2}{b_3}. \tag{12}$$

It is to be noted that $\phi^*(t)$ has two values; when we employ the minus sign we get the reciprocal of the spacelike ruled surface obtained by using the plus sign. Through the paper we choice lower sign. Into Eq. (4) we substitute from Eqs. (11), (12) and obtain:

$$\mathbf{r}(t,v) = (b_1, b_2, b_3) + v(\sinh\phi, \frac{b_3}{\sqrt{b_2^2+b_3^2}}\cosh\phi, -\frac{b_2}{\sqrt{b_2^2+b_3^2}}\cosh\phi), \tag{13}$$

where $b_2^2 + b_3^2 \neq 0$, $v \in \mathbb{R}$, and $\phi(t)$ is arbitrary. Hence, we arrive to the following main theorem:

**Theorem 2**. Let $\mathbf{b}(t)$ be a regular curve in Minkowski 3-space $\mathbb{E}_1^3$. Then there exists the family of spacelike ruled surface represented by Eq. (13).

Now, as an application of our main results, we give the following examples.

**Example 1**. Let $\mathbf{b}(t) = (t, t^2, t^2)$ be a curve in Minkowski 3-space $\mathbb{E}_1^3$. Then, in view of Eq. (13), the family of spacelike ruled surface is

$$\mathbf{r}(t,v) = (t,t^2,t^2) + v(\sinh\phi, \frac{1}{\sqrt{2}}\cosh\phi, -\frac{1}{\sqrt{2}}\cosh\phi),\ v \in \mathbb{R}. \quad (14)$$

The distribution parameter is

$$\lambda(t) = \frac{2\sqrt{2}t}{\phi'}.$$

The parameter $\phi$ can control the shape of the surface. If we assume $\phi(t) = t$, then $\lambda(t) = 2\sqrt{2}t$ , and the spacelike ruled surface is shown in Figure. 2. If $\phi(t) = -t$, then $\lambda(t) = -2\sqrt{2}t$, and the surface is shown in Figure 3; domain $D = \{-1 \leq t \leq 1$, and $-3 \leq v \leq 3\}$.

**Example 2**. Let $\mathbf{b}(t) = (t,t,1)$ be a null curve in Minkowski 3-space $\mathbb{E}_1^3$. Likwise, we have:

$$\mathbf{r}(t,v) = (t,t,1) + v(\sinh\phi, \frac{1}{\sqrt{1+t^2}}\cosh\phi, -\frac{t}{\sqrt{1+t^2}}\cosh\phi),\ v \in \mathbb{R}. \quad (15)$$

and

$$\lambda(t) = \frac{-\phi' t\sqrt{1+t^2} + \frac{1}{\sqrt{1+t^2}}\sinh\phi\cosh\phi + \cosh^2\phi}{(1+t^2)\cosh^2\phi - t^2}.$$

If we take $\phi(t) = t$, then for -1 $\leq t \leq 1$, and $-3 \leq v \leq 3$, the spacelike ruled surface is shown in Fig. 4. For $\phi(t) = -t$, -1 $\leq t \leq 1$, and $-3 \leq v \leq 3$ the surface is shown in Figure 5.

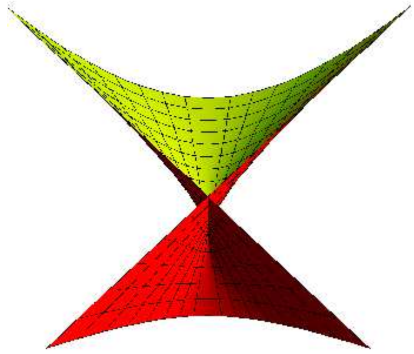

**Fig. 2:** Spacelike ruled surface.

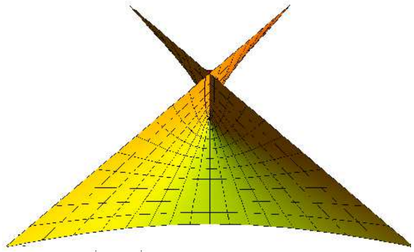

**Fig. 3:** Spacelike ruled surface.

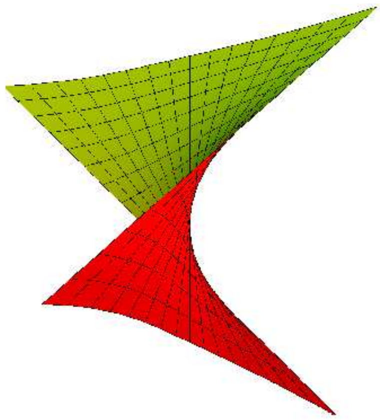

**Fig. 4:** Spacelike ruled surface.

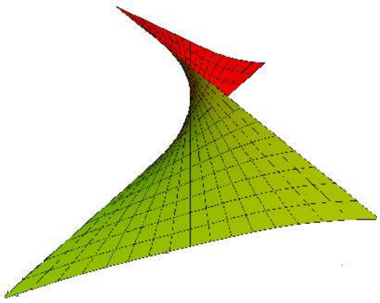

**Fig. 5:** Spacelike ruled surface.

Now, it seems natural to pose the following inquiry: Under what condition $\mathbf{r}(t,v)$ is a developable spacelike ruled surface in Minkowski 3-space $\mathbb{E}_1^3$ ?. The answer is positive and can be stated as follows: In fact, in view of Eq. (7), $\mathbf{r}(t,v)$ is developable if and only if $\lambda(t)=0$, that is,

$$-\phi'\phi^{*'}+\phi^{*}\omega'^{2}\sinh\phi\cosh\phi+\omega'\omega^{*'}\cosh^{2}\phi=0$$

or equivalently

$$(\tanh\phi)'-\frac{\phi^{*}\omega'^{2}}{\phi^{*'}}\tanh\phi-\frac{\omega'\omega^{*'}}{\phi^{*'}}=0. \quad (16)$$

If we setting

$$f(t)=\tanh\phi,\ G(t)=-\frac{\phi^{*}\omega'^{2}}{\phi^{*'}},\ H(t)=\frac{\omega'\omega^{*'}}{\phi^{*'}},$$

then we are lead to a linear differential equation of first order

$$\frac{df(t)}{dt}+G(t)f(t)-H(t)=0. \quad (17)$$

Now we need only to locate $\phi(t)$. The solution of (17) gives $\tanh\phi$. This solution contains an integral constant, thus we have infinitely numerous spacelike developable ruled surfaces such that each of them has a base curve $\mathbf{b}(t)$; from Eqs. (11) and, (12), we have

$$\omega^{*}=b_1,\ \phi^{*}=\sqrt{b_2^2+b_3^2},\ \tan\omega=-\frac{b_2}{b_3}. \quad (18)$$

**Example 3**. Based on the curve in example 1, it is easy to show that

$$\tan\omega=-1,\ \phi^{*}=\sqrt{2}t^{2},\ \omega^{*}=t.$$

and

$$\omega^{*'}=1,\ \omega'=0,\ \phi^{*'}=\sqrt{2}t^{2}.$$

we substitute these values into Eq. (17) and solve this differential equation

$$f(t)=\tanh\phi=c,\ c\in\mathbb{R}.$$

Since $\tanh\phi=c$, then we have:

$$\cosh\phi=\pm\frac{1}{\sqrt{1-c^2}},\ \sinh\phi=\pm\frac{c}{\sqrt{1-c^2}}. \quad (19)$$

If we select the plus sign, then the family of spacelike developable ruled surface is given by

$$\mathbf{r}(t,v)=(t,t^2,t^2)+\frac{v}{\sqrt{1-c^2}}(c,\frac{t}{\sqrt{2}}c,-\frac{t}{\sqrt{2}}c),\ v\in\mathbb{R}.$$

If $c=.5$, $-1\le t\le 1$, and $0\le v\le 1$, then we obtain a member in the family as shown in Figure 6. Figure 7 shows the surface with $c=-.5$, $-1\le t\le .1$, and $0\le v\le 1$.

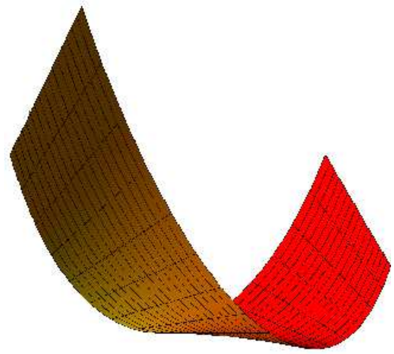

**Fig. 6:** Spacelike developable surface

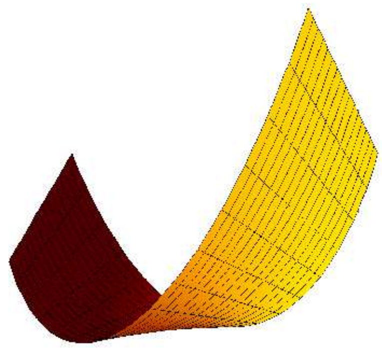

**Fig. 7:** Spacelike developable surface

**Example 4**. For the curve in example 2, we obtain

$$\begin{aligned}\omega^{*'}=1,\ \omega'=-\tfrac{1}{1+t^2},\ \phi^{*'}=\tfrac{t}{\sqrt{1+t^2}},\\ G(t)=-\tfrac{1}{t(1+t^2)},\ H(t)=-\tfrac{1}{t\sqrt{1+t^2}}.\end{aligned} \quad (20)$$

Then, combining Eqs (17), and (20), we have:

$$\frac{df(t)}{dt}-\frac{1}{t(1+t^2)}f(t)-\frac{1}{t\sqrt{1+t^2}}=0. \quad (21)$$

The solution of this differential equation gives

$$f(t)=\frac{ct-1}{\sqrt{1+t^2}},\ c\in\mathbb{R}.$$

Since $f(t)=\tanh\phi$, then we have:

$$\sinh\phi=\pm\frac{ct-1}{\sqrt{(1-c^2)t^2+2ct}},\ \cosh\phi=\pm\frac{\sqrt{1+t^2}}{\sqrt{(1-c^2)t^2+2ct}}. \quad (22)$$

According Eqs. (13) and Eq. (22), the family of the spacelike developable ruled surface is given by

$$\mathbf{r}(t,v)=(t,t,1)+\frac{v}{\sqrt{(1-c^2)t^2+2ct}}(ct-1,1,-t),\ v\in\mathbb{R}. \quad (23)$$

If $c = 1$, $1 \leq t \leq 3$, and $-3 \leq v \leq 3$, then we obtain a member in the family as shown in Figure 8. Figure 9 shows the surface with $c = -1$, $-3 \leq t \leq -1$, and $-3 \leq v \leq 3$.

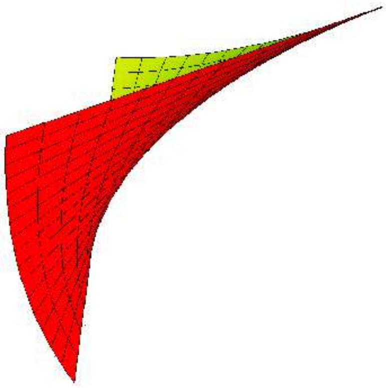

**Fig. 8:** Spacelike developable surface.

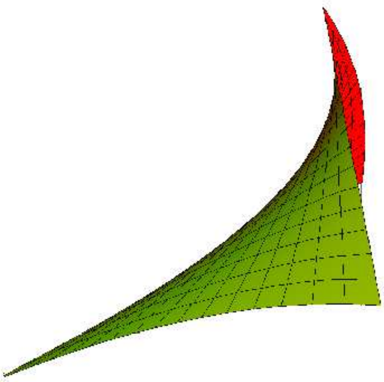

**Fig. 9:** Spacelike developable surface.

## 4 Conclusion

This work simply provided an approach to construct spacelike ruled and developable surfaces in a new form. Hopefully the outcomes will be meaningful to those studying general relativity theory.

## Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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