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The Interval Vectors of χ^2 Sequence Space Defined by Musielak Orlicz Function

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Abstract: In this paper we introduce χ^2 of four dimensional interval vectors some theorems on four dimensional interval numbers and some definitions which are the natural combination of the definition of interval vectors of χ^2 of Musielak Orlicz function also some inclusion relations are studied.

Keywords: Analytic sequence, Museialk-Orlicz function, double sequences, chi sequences, interval vector, sequence space, solidity, sequence algebra.

1 Introduction

Throughout w, χ and $\wedge$ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex double sequences (x_{mn}), where $m,n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich[1]. Later on it was investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Tripathy et al., [6,7,8,9,10,11,12,13,14,15,16,17], Turkmenoglu [18], Raj [19,20,21,22,23,24,25] and many others.

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence (S_{mn}) is convergent, where

$$S_{mn} = \sum_{i.j=1}^{m,n} x_{ij} (m,n = 1,2,3,...)$$

A double sequence $x = (x_{mn})$ is said to be double analytic if

$$sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty$$

The vector space of all double analytic sequences are usually denoted by $\wedge^2$. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{\frac{1}{m+n}} \to 0 \ \ as \ \ m,n \to \infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by $\wedge^2$ and Γ^2 is a metric space with the metric

$$d(x,y) = sup_{m,n} \{|x_{mn} - y_{mn}|^{\frac{1}{m+n}} : m,n : 1,2,3,...,\}, \quad (1)$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Γ^2. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{mn})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \delta_{ij}$ for all $m,n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots \\ \vdots \\ 0 & 0 & \ldots & 1 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots \end{pmatrix}$$

with 1 in the $(m,n)^{th}$ position and zero otherwise. A double sequence $x = (x_{mn})$ is called double gai sequence

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if $((m+n)!|x_{mn}|)^{\frac{1}{m+n}} \to 0$. as $m,n \to \infty$. The double gai sequences will be denoted by χ^2.

2 Definition and Preliminaries

Definition 1.[26] *An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \le M(x)+M(y)$, then this function is called modulus function. An Orlicz function M is said to satisfy Δ_2-condition for all values u, if there exists $K > 0$ such that $M(2u) \le KM(u), u \ge 0$.*

Lemma 1. *Let M be an Orlicz function which satisfies Δ_2- condition and let $0 < \delta < 1$. Then for each $t \ge \delta$, we have $M(t) < K\delta^{-1}M(2)$ for some constant $K > 0$.*

Definition 2. *A sequence space E is said to be solid or normal if $(\alpha_{mn}x_{mn}) \in E$ whenever $(x_{mn}) \in E$ and for all sequences of scalars (α_{mn}) with $|\alpha_{mn}| \le 1$, for all $m,n \in \mathbb{N}$.*

Definition 3. *A sequence space E is said to be monotone if it contains the canonical per-images of all its step spaces.*

Definition 4. *For a subspace Ψ of a linear space is said to be sequence algebra if $x,y \in \Psi$ implies that $x.y = (x_{mn}y_{mn}) \in \Psi$, see Kamptan and Gupta [28].*

Definition 5.[27] *Let $n \in \mathbb{N}$ and X be a real vector space of dimension m, where $n \le m$. A real valued function $d_p(x_1,\ldots x_n) = \|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ on X satisfying the following four conditions:*

(i)$\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p = 0$ if and only if $d_1(x_1,0),\ldots,d_n(x_n,0)$ are linearly dependent,
(ii)$\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ is invariant under permutation,
(iii)$\|(\alpha d_1(x_1,0),\ldots,\alpha d_n(x_n,0))\|_p = |\alpha|\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p, \alpha \in \mathbb{R}$,
(iv)$d_p((x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)) = (d_X(x_1,x_2,\ldots,x_n)^p + d_Y(y_1,y_2,\ldots,y_n)^p)^{\frac{1}{p}}$ for $1 \le p < \infty$; (or)
(v)$d((x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)) := sup\{d_X(x_1,x_2,\ldots,x_n), d_Y(y_1,y_2,\ldots,y_n\}$, for $(x_1,x_2,\ldots,x_n) \in X, y_1,y_2,\ldots,y_n \in Y$

is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|d_1(x_1,0),\ldots,(d_n,0)\|_E = sup(|det(d_{mn}(x_{mn},0))|)$$

$$= sup\left(\begin{vmatrix} d_{11}(x_{11},0) & d_{12}(x_{12},0) & \ldots & d_{1n}(x_{1n},0) \\ d_{21}(x_{21},0) & d_{22}(x_{22},0) & \ldots & d_{2n}(x_{2n},0) \\ & \vdots & & \\ d_{n1}(x_{n1},0) & d_{n2}(x_{n2},0) & \ldots & d_{nn}(x_{nn},0) \end{vmatrix}\right)$$

where $x_i = (x_{i1},\ldots,x_{in}) \in \mathbb{R}^n$ for each $i = 1,2,\ldots,n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric is said to be $p-$ Banach metric space.

An interval number $\tilde{x}$ is a closed subset of the real numbers and denoted as $\tilde{x} = [x_{pq},x_{rs}]$, where $x_{pq} \le x_{rs}$ and x_{pq},x_{rs} both are real numbers. let us denote the set of all real valued closed intervals by $R^2(I_4)$. The set of all interval numbers $R^2(I_4)$ is a metric space with the metric

$$d(\tilde{x},\tilde{y}) = max\{inf\{|x_{pq}-y_{pq}|,|x_{rs}-y_{rs}|\} \le 1\}.$$

Let us define transformation $f : N \times N \to R^2(I_4) \times R^2(I_4)$ by $(m,n) \to f(mn) = (\tilde{x}_{mn})$. Then $(\tilde{x}_{mn})$ is called the sequence of interval numbers. The $\tilde{x}_{mn}$ is called the $(m,n)^{th}$ term of sequence $(\tilde{x}_{mn})$.

Definition 6. *Let M be an sequence of Musielak Orlicz functions and a sequence $(\tilde{x}_{mn})$ of $(R^2(I_4),d)$ is said to be convergent to the interval number $\tilde{0}$ and we denote it by writing*

$$\left[\chi^2_M, \|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]$$
$$= \lim_{m,n\to\infty}\left\{\begin{matrix}\left[M\left(((m+n)!|\tilde{x}_{mn},\tilde{0}|)^{(\frac{1}{m})+n},\right.\right. \\ \left.\left.\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p)\right)\right] = 0\end{matrix}\right\}.$$

Thus

$$= \lim_{m,n\to\infty}\left\{\begin{matrix}[M(((m+n)!|\tilde{x}_{mn},\tilde{0}|)^{(\frac{1}{m})+n}, \\ \|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p)] = 0\end{matrix}\right\}$$
$$\Leftrightarrow \lim_{m,n\to\infty}\left\{\begin{matrix}[M(((p+q)!|\tilde{x}_{pq},\tilde{0}|)^{(\frac{1}{p})+q}, \\ \|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p))] = 0\end{matrix}\right\}.$$

and

$$\lim_{m,n\to\infty}\left\{\begin{matrix}[M(((r+s)!|\tilde{x}_{rs},\tilde{0}|)^{(\frac{1}{r})+s}, \\ \|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p))] = 0\end{matrix}\right\}$$

A four dimensional interval vector is an ordered 4-tuple of intervals,

$$\tilde{x} = (\tilde{x}_{11},\tilde{x}_{12},\tilde{x}_{21},\tilde{x}_{22}) = ([x_{11pq},x_{12pq}],[x_{21rs},x_{22rs}]).$$

If the absolute value of each element of $\tilde{x}$ is zero, then $\tilde{x}$ is called zero interval vector and is denoted by $\tilde{\theta} = (\tilde{0},\tilde{0},\tilde{0},\tilde{0}) = ([0,0],[0,0])$.

Let $R^2(I_4)$ be the set of all four dimensional interval vector. The scalar multiplication and addition of four vectors in $R^2(I_4)$ are defined as follows:

$$\alpha\tilde{x} = (\alpha\tilde{x}_{11},\alpha\tilde{x}_{12},\alpha\tilde{x}_{21}\alpha\tilde{x}_{22})$$
$$= \begin{cases}([x_{11pq},x_{12pq}],[x_{21rs},x_{22rs}]), & if\alpha \ge 0 \\ ([x_{12pq},x_{11pq}],[x_{22rs},x_{21rs}]), & if\alpha < 0\end{cases}$$

$$\tilde{x}+\tilde{y} = (\tilde{x}_{11},\tilde{x}_{12},\tilde{x}_{21},\tilde{x}_{22}) + (\tilde{y}_{11},\tilde{y}_{12},\tilde{y}_{21},\tilde{y}_{22})$$
$$= \begin{pmatrix}[x_{11pq}+y_{11pq},x_{12pq}+y_{12pq}], \\ [x_{21rs}+y_{21rs},x_{22rs}+y_{22rs}],\end{pmatrix}$$

Now, we introduce a distance of four vectors in $R^2(I_4)$, which is defined as

$$d(\tilde{x},\tilde{y}) = max\left\{ inf \left\{ \begin{matrix} |x_{11pq} - y_{11pq}|, |x_{12rs} - y_{12rs}|, \\ |x_{21pq} - y_{21pq}|, |x_{22rs} - y_{22rs}|, \end{matrix} \right\} \leq 1 \right\}$$

where
$\tilde{x} = (\tilde{x}_{11},\tilde{x}_{12},\tilde{x}_{21},\tilde{x}_{22}), \tilde{y} = (\tilde{y}_{11},\tilde{y}_{12},\tilde{y}_{21},\tilde{y}_{22}) \in R^2(I_4)$.

Definition 7. *Two non-negative sequences of interval vectors $x = (\tilde{x}_{mn})$ and $y = (\tilde{y}_{mn})$ are asymptotically equivalent $\tilde{\theta}$ if*

$$lim_{mn} \frac{\tilde{x}_{mn}}{\tilde{y}_{mn}} = \tilde{\theta} = (\tilde{0},\tilde{0},\tilde{0},\tilde{0}) = ([0,0],[0,0])$$

and is denoted by $\tilde{x} \equiv \tilde{\theta}$.

3 Main Results

Theorem 1. *The set of all four dimensional interval vectors $R^2(I_4) \times R^2(I_4)$ forms a metric space with respect to the metric $d(\tilde{x},\tilde{y})$ defined above.*

Proof./ Easy to prove. Therefore omit the proof.

Let us define transformation $f : N \times N \to R^2(I_4) \times R^2(I_4)$ by $(m,n) \to f(mn) = (\tilde{x}_{mn})$. Then $(\tilde{x}_{mn})$ is called the sequence of four dimensional interval numbers.

Theorem 2./ *The space $(R^2(I_4) \times R^2(I_4), d)$ is a complete metric space.*

*Proof.*Let $(\tilde{x}_{mn})$ be any Cauchy sequence of $(R^2(I_4) \times R^2(I_4), d)$, then there exists a $k_0, l_0 \in \mathbb{N}$ such that

$$d(\tilde{x},\tilde{y}) = max\left\{ inf \left\{ \begin{matrix} |x^{mn}_{11pq} - y^{mn}_{11pq}|, |x^{mn}_{12rs} - y^{mn}_{12rs}|, \\ |x^{mn}_{21pq} - y^{mn}_{21pq}|, |x^{mn}_{22rs} - y^{mn}_{22rs}| \end{matrix} \right\} \leq 1 \right\}$$
$< \varepsilon ... *, \forall m,n \geq k_0, l_0$.

From this inequality, we can write that

$$max\left\{ |x^{mn}_{11pq} - y^{mn}_{11pq}|, |x^{mn}_{12rs} - y^{mn}_{12rs}| \right\} < \varepsilon$$

and

$$max\left\{ |x^{mn}_{21pq} - y^{mn}_{21pq}|, |x^{mn}_{22rs} - y^{mn}_{22rs}| \right\} < \varepsilon$$

Therefore the sequence $(x_{11pq}),(x_{12rs}),(x_{21pq})$ and (x_{22rs}) are Cauchy sequence in $(R^2(I_4) \times R^2(I_4), d)$. But $(R^2(I_4) \times R^2(I_4), d)$ is complete. Hence we can write

$$\lim_{m,n\to\infty} (x^{mn}_{11pq}) = \tilde{\theta}, \lim_{m,n\to\infty} (x^{mn}_{12pq}) = \tilde{\theta},$$

$$\lim_{m,n\to\infty} (x^{mn}_{21rs}) = \tilde{\theta}, \lim_{m,n\to\infty} (x^{mn}_{22rs}) = \tilde{\theta}.$$

where $\tilde{\theta} = (\tilde{0},\tilde{0},\tilde{0},\tilde{0}) = ([0,0],[0,0])$. If we take the limit for $m,n \to \infty$ in $(*)$, then we get $d(\tilde{x},\tilde{y})$ for all $m,n \geq k_0, l_0$. This completes the proof.

Some sequence spaces of interval vectors:

Let $w^2(R^2(I_4) \times R^2(I_4))$ denote the set of all sequences of four dimensional interval vectors of $(R^2(I_4) \times R^2(I_4))$. Since the set $(R^2(I_4) \times R^2(I_4))$ is a quasi vector space, the set $w^2(R^2(I_4) \times R^2(I_4))$ be regarded as a quasi vector space. Now we define the following sequence spaces of Musielak Orlicz of gai and Musielak Orlicz of analytic sequence of four dimensional interval vectors;

$$\left[\chi^{2(R^2(I_4)\times R^2(I_4))}, \|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M$$
$$= \lim_{m,n\to\infty} \left\{ \begin{matrix} \Big[M\left(((m+n)!|\tilde{x}_{mn}|)^{(\frac{1}{m})+n},\right. \\ \|(d(x_1,0),d(x_2,0),\ldots, \\ \left. d(x_{n-1},0))\|_p)\right) \Big] = \tilde{\theta} \end{matrix} \right\},$$

$$\left[\wedge^{2(R^2(I_4)\times R^2(I_4))}, \|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M$$
$$= \sup_{m,n} \left\{ \begin{matrix} \Big[M\left((|\tilde{x}_{mn}|)^{(\frac{1}{m})+n},\right. \\ \|(d(x_1,0),d(x_2,0),\ldots, \\ \left. d(x_{n-1},0))\|_p)\right) \Big] < \infty \end{matrix} \right\}.$$

Therefore the space $\chi^{2(R^2(I_4)\times R^2(I_4))}$ and $\wedge^{2(R^2(I_4)\times R^2(I_4))}$ are subspaces of $w^2(R^2(I_4) \times R^2(I_4))$.

Theorem 3.

$$\left[\chi^{2(R^2(I_4)\times R^2(I_4))}, \|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M$$

$$\subseteq \left[\wedge^{2(R^2(I_4)\times R^2(I_4))}, \|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M$$

and the inclusion is strict.

Proof. If we take any

$$\tilde{x} \in \left[\chi^{2(R^2(I_4)\times R^2(I_4))}, \|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

Now, let

$$\tilde{x} = (\tilde{x}_{mn}) = \left(\frac{1}{(m+n)!mn}\right)^{m+n}, \left(\frac{1}{(m+n)!} + \frac{1}{(m+n)!mn}\right)^{m+n}$$

$$\left[\left(\frac{1}{(m+n)!}\right)^{m+n} - \left(\frac{1}{(m+n)!mn}\right)^{m+n}, \left(\frac{2}{(m+n)!}\right)^{m+n} + \left(\frac{2}{(m+n)!mn}\right)\right],$$

$$m,n \in \mathbb{N} \notin \left[\chi^{2(R^2(I_4)\times R^2(I_4))}, \|(d(x_1,0),d(x_2,0),\ldots, d(x_{n-1},0))\|_p\right]_M.$$

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Example 1. Let

$$\tilde{x}_{mn}=\left[\left(\frac{(-1)^{mn}}{(m+n)!}\right)^{m+n},\left(\frac{2}{(m+n)!}\right)^{m+n}+\left(\frac{1}{(m+n)!mn}\right)^{m+n}\right],$$

$$\left[\left(\frac{1}{(m+n)!}\right)^{m+n}-\left(\frac{1}{(m+n)!mn}\right)^{m+n},\left(\frac{2}{(m+n)!}\right)^{m+n}+\left(\frac{1}{(m+n)!mn}\right)^{m+n}\right],$$

$$m,n\in\mathbb{N}\in\left[\wedge^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

but not in

$$\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

Theorem 4. *The spaces*

$$\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

and

$$\left[\wedge^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

are complete metric space with the metric

$$d(\tilde{x},\tilde{y})=max\left\{inf\left\{\begin{matrix}|x_{11pq}-y_{11pq}|,|x_{12rs}-y_{12rs}|,\\|x_{21pq}-y_{21pq}|,|x_{22rs}-y_{22rs}|\end{matrix}\right\}\leq 1\right\}$$

where

$$x=(\tilde{x}_{mn}),\quad y=(\tilde{y}_{mn})\in\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

and

$$\left[\wedge^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

Proof. It is routine verification. Therefore omit the proof.

Theorem 5. *The spaces*

$$\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

and

$$\left[\wedge^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

are interval vector metric spaces with the metric

$$d(\tilde{x},\tilde{y})=max\left\{inf\left\{|x_{11pq}|,|x_{12rs}|,|x_{21pq}|,|x_{22rs}|\right\}\leq 1\right\},$$

$$\tilde{x}=(\tilde{x}_{11},\tilde{x}_{12},\tilde{x}_{21},\tilde{x}_{22})\in R^2(I_4)$$

where

$$x=(\tilde{x}_{mn})\in\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

and

$$\left[\wedge^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

Proof. Now consider

$$\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

Other space is proved by same manner. it is obvious that

(1)$d(\tilde{x},\tilde{\theta})\geq 0$ and $d(\tilde{x},\tilde{\theta})=0$ if and only if $\tilde{x}=\tilde{\theta}$.

(2)$d(\tilde{x},\tilde{y})$

$$=max\left\{inf\left\{\begin{matrix}|x_{11pq}-y_{11pq}|,|x_{12rs}-y_{12rs}|,\\|x_{21pq}-y_{21pq}|,|x_{22rs}-y_{22rs}|\end{matrix}\right\}\leq 1\right\}$$

$$\leq max\left\{inf\left\{\begin{matrix}|x_{11pq},\tilde{\theta}|,|x_{12rs},\tilde{\theta}|,\\|x_{21pq},\tilde{\theta}|,|x_{22rs},\tilde{\theta}|\end{matrix}\right\}\leq 1\right\}$$

$$\leq max\left\{inf\left\{\begin{matrix}|y_{11pq},\tilde{\theta}|,|y_{12rs},\tilde{\theta}|,\\|y_{21pq},\tilde{\theta}|,|y_{22rs},\tilde{\theta}|\end{matrix}\right\}\leq 1\right\}$$

$$d(\tilde{x},\tilde{y})=d(\tilde{x},\tilde{\theta})+d(\tilde{y},\tilde{\theta}).$$

(3)$d(\alpha\tilde{x},\tilde{\theta})$

$$=max\left\{inf\left\{\begin{matrix}|\alpha x_{11pq},\tilde{\theta}|,|\alpha x_{12rs},\tilde{\theta}|,\\|\alpha x_{21pq},\tilde{\theta}|,|\alpha x_{22rs},\tilde{\theta}|\end{matrix}\right\}\leq 1\right\}$$

$$\Rightarrow d(\alpha\tilde{x},\tilde{\theta})=|\alpha|d(\tilde{x},\tilde{\theta}).$$

Hence $x=(\tilde{x}_{mn})$ is metric on

$$\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

Theorem 6. *The spaces*

$$\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

is solid and monotone.

Proof. Let

$$x=(\tilde{x}_{mn})\in\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

and $y=(\tilde{y}_{mn})$ be such that $d(\tilde{y},\tilde{\theta})\leq d(\tilde{x},\tilde{\theta})(i.e.,)$

$$max\left\{inf\left\{\begin{matrix}|y_{11pq},\tilde{\theta}|,|y_{12rs},\tilde{\theta}|,\\|y_{21pq},\tilde{\theta}|,|y_{22rs},\tilde{\theta}|\end{matrix}\right\}\leq 1\right\}$$

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$$\leq max\left\{inf\left\{\begin{matrix}|x_{11pq},\tilde{\theta}|,|x_{12rs},\tilde{\theta}|,\\|x_{21pq},\tilde{\theta}|,|x_{22rs},\tilde{\theta}|\end{matrix}\right\}\leq 1\right\}.$$

Thus we have obtain
$y_{11pq} \leq x_{11pq}, y_{12rs} \leq x_{12rs}, y_{21pq} \leq x_{21pq}, y_{22rs} \leq x_{22rs}, (i.e.,) y \leq x$. Therefore

$$y=(\tilde{y}_{mn})\in\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

Hence

$$\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

is solid.

A solid sequence space is always monotone.[see [28]] Hence

$$\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

is monotone.

Theorem 7. *The space*

$$\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

is sequence algebra.

*Proof.*Let

$$x=(\tilde{x}_{mn}),$$
$$y=(\tilde{y}_{mn})\in\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

then for $\varepsilon > 0$, we can find $r_1, r_2 \in \mathbb{N}\times\mathbb{N}$ such that $d(\tilde{x}_{mn},\tilde{\theta}) < \varepsilon$, for all $m,n \geq r_1$, and $d(\tilde{y}_{mn},\tilde{\theta}) < \varepsilon$, for all $m,n \geq r_2$,

$$max\left\{|x_{11pq}|,|x_{12rs}|,|x_{21pq}|,|x_{22rs}|\right\}<\varepsilon, for\ all\ \ m,n\geq r_1. \quad (2)$$

$$max\left\{|y_{11pq}|,|y_{12rs}|,|y_{21pq}|,|y_{22rs}|\right\}<\varepsilon, for\ all\ \ m,n\geq r_2. \quad (3)$$

Let $r_3 = maxr_1, r_2$, then for all $m,n \geq r_3$, we have

$$d(\tilde{x}_{mn}\otimes\tilde{y}_{mn},\tilde{\theta})$$
$$=max\left\{inf\left\{\begin{vmatrix}x_{11pq}\cdot y_{11pq},x_{11pq}\cdot y_{12rs},\\ x_{12rs}\cdot y_{11pq},x_{12rs}\cdot y_{12rs}\end{vmatrix}\right\}\leq 1\right\},$$

$$max\left\{inf\left\{\begin{matrix}|x_{21pq}\cdot y_{11pq},x_{21pq}\cdot y_{12rs},\\ x_{22rs}\cdot y_{11pq},x_{22rs}\cdot y_{12rs}|\end{matrix}\right\}\leq 1\right\},$$

$$max\left\{inf\left\{\begin{matrix}|x_{11pq}\cdot y_{21pq},x_{11pq}\cdot y_{22rs},\\ x_{12rs}\cdot y_{21pq},x_{12rs}\cdot y_{22rs}|\end{matrix}\right\}\leq 1\right\},$$

$$max\left\{inf\left\{\begin{matrix}|x_{21pq}\cdot y_{21pq},x_{21pq}\cdot y_{22rs},\\ x_{22rs}\cdot y_{21pq},x_{22rs}\cdot y_{22rs}|\end{matrix}\right\}\leq 1\right\}<\varepsilon^2$$

by (2) and (3). Hence

$$(x\otimes y)=(\tilde{x}_{mn}\otimes\tilde{y}_{mn})\in\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

Hence

$$\left[\chi^{2(R^2(I_4)\times R^2(I_4))},\|(d(x_1,0),d(x_2,0),\ldots,d(x_{n-1},0))\|_p\right]_M.$$

is sequence algebra.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this research paper.

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