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Bayesian Approximation for a Mixture Weibull and Lomax Distributions based on Progressive Type-II Censoring Scheme

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Abstract: The present study deals with the classical and Bayesian estimation of the progressive Type-II censored samples under assumption that the lifetimes follow the mixture of Weibull and Lomax distributions. We consider the maximum likelihood estimation (MLE) and Bayes estimator of the parameters under symmetric square error (SE), the asymmetric linear exponential (LINEX) and general entropy (GE) loss functions, by using the approximation forms of Lindley approximation and Tierney and Kadane (T-K) approximation. Simulation study is used to illustrate the discussed methodology.

Keywords: Mixture model, Progressive Type-II censoring, Maximum likelihood estimation, Bayesian estimation, Lindley approximation, T-K approximation.

1 Introduction

In many life test studies, units are lost or removed from the experiment before the failure occurs. However, in many situations, the removal of units prior to failure is pre-planned in order to provide saving in terms of time and cost associated with testing. In survival analysis Type-I and Type- II censoring schemes are the two most popular censoring scheme, but using the type of censoring they do not allow removal of units at points other than the termination point of an experiment. Progressively Type-II censoring which is considered to be a generalization Type-II censoring is an important method of obtaining data in such lifetime studies. This type of censoring allows the experiment before its end, thus resulting in saving in cost as well as experimental time (For more detail see Balakrishnan and Aggarwala [1], Balakrishnan et al. [2], Balakrishnan and Hossain [3] and Balakrishnan [4]). The estimation of parameters from different lifetime distributions based on progressive Type-II censored samples are studied by several authors (Krishna and Kumar [5], Rastogi and Tripathi [6], Singh et al. [7], Mahmoud et al. [8], Pandey and Kumari [9] and Rashad et al. [10]).

In recent years, mixtures of distributions have gained great interest in the role of the analysis, since mixtures of distributions play a vital role in many practical applications. In life testing, reliability and quality problems, mixed failure populations are sometimes encountered. Mixture distributions include a finite or infinite number of components that can analyse different distributional types, that can describe different features of data. Finite mixtures of distributions have been used as models through out history of modern statistics. For more details about finite mixtures of distributions [see Everitt and Hand [11], Titterington et al. [12], McLachlan and Basford [13], Lindsay [14], McLachlan and Peel [15]]. Also, mixtures of distributions have been considered extensively by several researchers using both classical and Bayesian techniques, [for example Elsherpieny [16], Shawky and Bakoban [17], Abu-Zinadah [18], Afify [19], Erisoglu et al. [20], Feroze and Aslam [21], Daniyal and Rajab [22], Elshahat and Mahmoud [23], Abushal and Al-Zaydi [24], and Mahmoud et al. [25]].

The Weibull distribution has been widely used in modeling of lifetime event data; this is due to the variety of shapes of the probability density function (pdf) based on its parameters. The Weibull distribution has been shown to be useful for modeling and analyzing lifetime data in the applied engineering sciences (Murthy et al. [26]). The Lomax distribution,

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sometimes called Pareto of the second kind, has a considerable importance in the field of life testing because of its uses to fit business failure data (Lomax [27]).

A random variable X is said to have a mixture of two components Weibull and Lomax distributions if its cumulative distribution function (cdf) is given by

$$F(x) = \sum_{j=1}^{2} p_j F_j(x) \quad , \qquad j = 1,2 \tag{1}$$

$$F_1(x) = 1 - e^{-\alpha_1 x^{\theta_1}} \qquad F_2(x) = 1 - (1+\theta_2 x)^{-\alpha_2}$$

where $x > 0, (\alpha_j > 0, \theta_j > 0)$, $j = 1,\ 2$. The mixing proportions p_j are such that $0 \le p_j \le 1$, $\sum_{j=1}^{2} p_j = 1$.

The corresponding probability density function (pdf) and reliability function, respectively are given by

$$f(x) = \sum_{j=1}^{2} p_j f_j(x) \quad , \qquad j = 1,2 \tag{2}$$

$$f_1(x) = \alpha_1 \theta_1 x^{\theta_1 - 1} e^{-\alpha_1 x^{\theta_1}} \qquad f_2(x) = \alpha_2 \theta_2 (1+\theta_2 x)^{-(\alpha_2+1)}$$

And

$$R(x) = \sum_{j=1}^{2} p_j R_j(x) \quad , \qquad j = 1,2 \tag{3}$$

$$R_1(x) = e^{-\alpha_1 x^{\theta_1}} \qquad R_2(x) = (1+\theta_2 x)^{-\alpha_2}$$

This paper is arranged as follows: In Section 2 we describe the classical estimation with maximum likelihood estimator (MLE) of parameters. Different approximation methods to evaluate Bayes estimators are discussed in Section 3. Monte Cario simulation results are presented Section 4. Finally, Section 5 presents concluding remarks.

2 Maximum Likelihood Estimation

Suppose that units from population with pdf (2), are placed on a life-testing and m failures are going to be observed. After units have failed each item can be attributed to the appropriate subpopulation. Suppose m units have failed during the interval $(0, x_m)$: r_1 units from the first subpopulation and r_2 units from the second subpopulation, such that $m = r_1 + r_2$. At the time of the first failure, $\mathscr{R}_1^*$of the remaining $n-1$ surviving units are randomly removed from the experiment. At the time of the second failure, $\mathscr{R}_2^*$of the remaining $n - \mathscr{R}_1^* - 2$ units are randomly removed from the experiment. This experiment terminates at the time when the m^{th} failure is observed and the remaining $\mathscr{R}_m^* = n - m - \mathscr{R}_1^* - \cdots - \mathscr{R}_{m-1}^*$ surviving units are all removed. Assume also that x_{ij} denote the failure time of the j^{th} unit belonging to the i^{th} subpopulation and $x_{ij} \le x_m$; $j = 1,2,\ldots,r_i$; $i = 1,2$, where x_m denotes the failure time of the m^{th} unit. Based on the observed sample $x_1 < \cdots < x_m$ from a progressive Type-II censoring scheme, $(\mathscr{R}_1^*, \ldots, \mathscr{R}_m^*)$, the likelihood function is given by

$$L(\alpha_1, \alpha_2, \theta_1, \theta_2, p|\underline{x}) \ = C\left[\prod_{j=1}^{r_1} p_1 f_1(x_{1j})\right]\left[\prod_{j=1}^{r_2} p_2 f_2(x_{2j})\right]\prod_{j=1}^{m}[1-F(x_j)]^{\mathscr{R}_j^*} \tag{4}$$

where $C = n(n-1-\mathscr{R}_1^*)\ldots(n-\mathscr{R}_1^* - \cdots - \mathscr{R}_{m-1}^* - m + 1)$, $R(x_j) = 1 - F(x_j)$. Substituting (2) and (3) into (4), the likelihood function can be written as

$$L(\alpha_1, \alpha_2, \theta_1, \theta_2, p|\underline{x}) \ = C\left[\prod_{j=1}^{r_1} p_1 \alpha_1 \theta_1 x_{1j}^{\theta_1 - 1} e^{-\alpha_1 x_{1j}^{\theta_1}}\right]\left[\prod_{j=1}^{r_2} p_2 \alpha_2 \theta_2 (1+\theta_2 x_{2j})^{-(\alpha_2+1)}\right] \times \prod_{j=1}^{m}[R(x_j)]^{\mathscr{R}_j^*} \tag{5}$$

where $R(x_j) = p_1 e^{-\alpha_1 x_j^{\theta_1}} + p_2(1+\theta_2 x_j)^{-\alpha_2}$, $p_1 = p$, $p_2 = 1-p$.

Assuming that the parameters θ_1 and θ_2 are known, the likelihood function (5) reduces to

$$L(\alpha_1, \alpha_2, p|\underline{x}) \propto \prod_{i=1}^{2}(p_i \alpha_i)^{r_i} \times e^{-\alpha_1 \sum_{j=1}^{r_1} x_{1j}^{\theta_1}} \prod_{j=1}^{r_2}(1+\theta_2 x_{2j})^{-\alpha_2} \times \prod_{j=1}^{m}[R(x_j)]^{\mathscr{R}_j^*} \tag{6}$$

Thus, the log-likelihood function can be expressed as

$$\ln L(\alpha_1,\alpha_2,p|\underline{x}) \propto \sum_{i=1}^{2}\{r_i \ln p_i + r_i \ln \alpha_i\} - \alpha_1 \sum_{j=1}^{r_1} x_{1j}^{\theta_1} - \alpha_2 \sum_{j=1}^{r_2} \ln(1+\theta_2 x_{2j}) + \sum_{j=1}^{m} \mathscr{R}_j^* \ln[R(x_j)] \tag{7}$$

Taking derivatives with respect to α_1, α_2 and p in Equation (7), we obtain the following

$$\frac{\partial \ln L(\alpha_1,\alpha_2,p|\underline{x})}{\partial \alpha_1} = \frac{r_1}{\alpha_1} - \sum_{j=1}^{r_1} x_{1j}^{\theta_1} + \sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{R(x_j)}\left(\frac{\partial R(x_j)}{\partial \alpha_1}\right)$$

$$\frac{\partial \ln L(\alpha_1,\alpha_2,p|\underline{x})}{\partial \alpha_2} = \frac{r_2}{\alpha_2} - \sum_{j=1}^{r_2} \ln(1+\theta_2 x_{2j}) + \sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{R(x_j)}\left(\frac{\partial R(x_j)}{\partial \alpha_2}\right)$$

$$\frac{\partial \ln L(\alpha_1,\alpha_2,p|\underline{x})}{\partial p} = \frac{r_1}{p_1} - \frac{r_2}{p_2} + \sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{R(x_j)}\left(\frac{\partial R(x_j)}{\partial p}\right)$$

where,

$$\frac{\partial R(x_j)}{\partial \alpha_1} = -\, p_1\, e^{-\alpha_1 x_j^{\theta_1}}\, x_j^{\theta_1}$$

$$\frac{\partial R(x_j)}{\partial \alpha_2} = -\, p_2 (1+\theta_2 x_j)^{-\alpha_2} \ln(1+\theta_2 x_j)$$

$$\frac{\partial R(x_j)}{\partial p} = e^{-\alpha_1 x_j^{\theta_1}} - (1+\theta_2 x_j)^{-\alpha_2}$$

The maximum likelihood estimators of the three parameters are obtained by solving the above equations simultaneously after equating to zero.

3 Bayesian Estimation

In this section, we derive Bayes estimators of the parameters α_1, α_2 and p of the considered model based on progressively Type-II censoring samples. Assuming the following independent prior distributions for the parameters $\alpha_1 \sim Gamma(a_1,b_1)$, $\alpha_2 \sim Gamma(a_2,b_2)$, and $p \sim Beta(c,d)$ for the mixing parameters p, the joint prior distribution of α_1 , α_2 and p is:

$$\pi(\alpha_1,\alpha_2,p) = \pi_1(\alpha_1)\,\pi_2(\alpha_2)\,\pi_3(p) \tag{8}$$

where

$$\left.\begin{aligned} \pi_i(\alpha_i) &= \frac{b_i^{a_i}}{\Gamma(a_i)} \alpha_i^{a_i-1} e^{-b_i\alpha_i}, \quad \alpha_i > 0,\ a_i,b_i > 0; i=1,2. \\ \pi_3(p) &= \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)} p_1^{c-1} p_2^{d-1}, \qquad 0<p<1,\ c,d>0 \end{aligned}\right\} \tag{9}$$

From Equation (8) and (9), the joint prior distribution of α_1 , α_2 and p can be written as follows

$$\pi(\alpha_1,\alpha_2,p) \propto \left[\prod_{i=1}^{2} \alpha_i^{a_i-1} e^{-b_i\alpha_i}\right] \times p_1^{c-1} p_2^{d-1} \tag{10}$$

The joint posterior density function of α_1, α_2 and p; can be written in the form

$$P(\alpha_1,\alpha_2,p|\underline{x}) = \frac{L(\alpha_1,\alpha_2,p|\underline{x})\,\pi(\alpha_1,\alpha_2,p)}{\int_0^1\int_0^\infty\int_0^\infty L(\alpha_1,\alpha_2,p|\underline{x})\,\pi(\alpha_1,\alpha_2,p)\,d\alpha_1 d\alpha_2 dp} \tag{11}$$

The Bayes estimator under squared error loss function of any function $\phi(\alpha_1,\alpha_2,p)$ is the posterior mean which is given by:

$$\widehat{\phi}_{SE}(\alpha_1,\alpha_2,p)=E_{\alpha_1,\alpha_2,p|\underline{x}}[\phi(\alpha_1,\alpha_2,p)] = \frac{\int_0^1\int_0^\infty\int_0^\infty \phi(\alpha_1,\alpha_2,p)L(\alpha_1,\alpha_2,p|\underline{x})\pi(\alpha_1,\alpha_2,p)d\alpha_1 d\alpha_2 dp}{\int_0^1\int_0^\infty\int_0^\infty L(\alpha_1,\alpha_2,p|\underline{x})\pi(\alpha_1,\alpha_2,p)d\alpha_1 d\alpha_2 dp} \tag{12}$$

Also, the Bayes estimators of $\phi(\alpha_1,\alpha_2,p)$ using LINEX and general entropy loss function are

$$\widehat{\phi}_{LINEX}(\alpha_1,\alpha_2,p)=-\frac{1}{q}\ln\left[E_{\alpha_1,\alpha_2,p|\underline{x}}\left[e^{-q\phi(\alpha_1,\alpha_2,p)}\right]\right],\quad q\neq 0 \tag{13}$$

where

$$E_{\alpha_1,\alpha_2,p|\underline{x}}\left[e^{-q\phi(\alpha_1,\alpha_2,p)}\right]=\frac{\int_0^1\int_0^\infty\int_0^\infty e^{-q\phi(\alpha_1,\alpha_2,p)}L(\alpha_1,\alpha_2,p|\underline{x})\pi(\alpha_1,\alpha_2,p)d\alpha_1 d\alpha_2 dp}{\int_0^1\int_0^\infty\int_0^\infty L(\alpha_1,\alpha_2,p|\underline{x})\pi(\alpha_1,\alpha_2,p)d\alpha_1 d\alpha_2 dp} \tag{14}$$

and

$$\widehat{\phi}_{GE}(\alpha_1,\alpha_2,p)=\left[E_{\alpha_1,\alpha_2,p|\underline{x}}\left[\phi(\alpha_1,\alpha_2,p)^{-h}\right]\right]^{-\frac{1}{h}},\qquad h\neq 0 \tag{15}$$

where

$$E_{\alpha_1,\alpha_2,p|\underline{x}}\left[\phi(\alpha_1,\alpha_2,p)^{-h}\right]=\frac{\int_0^1\int_0^\infty\int_0^\infty \phi(\alpha_1,\alpha_2,p)^{-h}L(\alpha_1,\alpha_2,p|\underline{x})\pi(\alpha_1,\alpha_2,p)d\alpha_1 d\alpha_2 dp}{\int_0^1\int_0^\infty\int_0^\infty L(\alpha_1,\alpha_2,p|\underline{x})\pi(\alpha_1,\alpha_2,p)d\alpha_1 d\alpha_2 dp} \tag{16}$$

The integrals in Equations (12), (14) and (16) cannot be obtained in a closed form. Therefore, in this case, we propose approximation methods namely Lindley's approximation method and Tierney-Kadane approximation method that can be used to compute the Bayes estimators for the parameters.

3.1 Lindley's Approximation Method

We consider, in this subsection, the Lindley's approximation method to obtain the Bayes estimators. According to Lindley [28]. The approximate Bayes estimate of $\phi(\alpha_1,\alpha_2,p)$ is given by:

$$E[\phi(\alpha_1,\alpha_2,p)|\underline{x}]=\frac{\int_0^1\int_0^\infty\int_0^\infty \phi(\alpha_1,\alpha_2,p)\ e^{L(\alpha_1,\alpha_2,p)+\rho(\alpha_1,\alpha_2,p)}d\alpha_1 d\alpha_2 dp}{\int_0^1\int_0^\infty\int_0^\infty\ e^{L(\alpha_1,\alpha_2,p)+\rho(\alpha_1,\alpha_2,p)}d\alpha_1 d\alpha_2 dp} \tag{17}$$

Where $L(\alpha_1,\alpha_2,p)$ is the log-likelihood function is given by (7) and $\rho(\alpha_1,\alpha_2,p)=\ln\pi\ (\alpha_1,\alpha_2,p)$. In a three parameter case, we approximate this expectation as follows

$$\begin{aligned}
&\widehat{\phi}(\alpha_1,\alpha_2,p)=E[\phi(\alpha_1,\alpha_2,p)|\underline{x}]\\
&=\phi(\alpha_1,\alpha_2,p)\ +\left(\phi_{\alpha_1}\tau_{\alpha_1}+\phi_{\alpha_2}\tau_{\alpha_2}+\phi_p\tau_p+\phi_{\alpha_1\alpha_2}\sigma_{\alpha_1\alpha_2}+\phi_{\alpha_1 p}\sigma_{\alpha_1 p}+\phi_{\alpha_2 p}\sigma_{\alpha_2 p}+\frac{1}{2}(\phi_{\alpha_1\alpha_1}\sigma_{\alpha_1\alpha_1}+\phi_{\alpha_2\alpha_2}\sigma_{\alpha_2\alpha_2}+\phi_{pp}\sigma_{pp})\right)\\
&+\frac{1}{2}[\psi_1(\phi_{\alpha_1}\sigma_{\alpha_1\alpha_1}+\phi_{\alpha_2}\sigma_{\alpha_1\alpha_2}+\phi_p\sigma_{\alpha_1 p})+\psi_2(\phi_{\alpha_1}\sigma_{\alpha_2\alpha_1}+\phi_{\alpha_2}\sigma_{\alpha_2\alpha_2}+\phi_p\sigma_{\alpha_2 p})+\psi_3(\phi_{\alpha_1}\sigma_{p\alpha_1}+\phi_{\alpha_2}\sigma_{p\alpha_2}+\phi_p\sigma_{pp})]
\end{aligned} \tag{18}$$

where, $\tau_i=\rho_{\alpha_1}\sigma_{i\alpha_1}+\rho_{\alpha_2}\sigma_{i\alpha_2}+\rho_p\sigma_{ip}$; $i=1,2,3$, where, $1,2,3$ refer for α_1,α_2,p respectively,

$$\psi_1=\sigma_{\alpha_1\alpha_1}L_{\alpha_1\alpha_1\alpha_1}+2(\sigma_{\alpha_1\alpha_2}L_{\alpha_1\alpha_2\alpha_1}+\sigma_{\alpha_1 p}L_{\alpha_1 p\alpha_1}+\sigma_{\alpha_2 p}L_{\alpha_2 p\alpha_1})+\sigma_{\alpha_2\alpha_2}L_{\alpha_2\alpha_2\alpha_1}+\sigma_{pp}L_{pp\alpha_1}$$

$$\psi_2=\sigma_{\alpha_1\alpha_1}L_{\alpha_1\alpha_1\alpha_2}+2(\sigma_{\alpha_1\alpha_2}L_{\alpha_1\alpha_2\alpha_2}+\sigma_{\alpha_1 p}L_{\alpha_1 p\alpha_2}+\sigma_{\alpha_2 p}L_{\alpha_2 p\alpha_2})+\sigma_{\alpha_2\alpha_2}L_{\alpha_2\alpha_2\alpha_2}+\sigma_{pp}L_{pp\alpha_2}$$

$$\psi_3=\sigma_{\alpha_1\alpha_1}L_{\alpha_1\alpha_1 p}+2(\sigma_{\alpha_1\alpha_2}L_{\alpha_1\alpha_2 p}+\sigma_{\alpha_1 p}L_{\alpha_1 pp}+\sigma_{\alpha_2 p}L_{\alpha_2 pp})+\sigma_{\alpha_2\alpha_2}L_{\alpha_2\alpha_2 p}+\sigma_{pp}L_{ppp}$$

and $\rho_i = \dfrac{\partial \rho}{\partial \omega_i}, i = 1,2,3,$ $\omega_1 = \alpha_1$, $\omega_2 = \alpha_2$ and $\omega_3 = p$, $\phi_i = \dfrac{\partial \phi(\alpha_1, \alpha_2, p)}{\partial \omega_i}, \phi_{ij} = \dfrac{\partial^2 \phi(\alpha_1, \alpha_2, p)}{\partial \omega_i \partial \omega_j}, L_{ij} = \dfrac{\partial^2 L}{\partial \omega_i \partial \omega_j}, i, j =$ $1,2,3$ $L_{ijk} = \dfrac{\partial^3 L}{\partial \omega_i \partial \omega_j \partial \omega_k}, i, j, k = 1,2,3$ and σ_{ij} denotes the element in the inverse of matrix $(-L_{ij}), i, j = 1,2,3,$ all evaluated at the MLE of the parameters. The method requires that $(\widehat{\alpha}_1, \widehat{\alpha}_2, \hat{p})$ be the unique MLE of (α_1, α_2, p). We have, $\rho_{\alpha_1} = \dfrac{(a_1-1)}{\alpha_1} - b_1$, $\rho_{\alpha_2} = \dfrac{(a_2-1)}{\alpha_2} - b_2$ and $\rho_p = \dfrac{(c-1)}{p_1} - \dfrac{(d-1)}{p_2}$, and

$$L_{\alpha_1} = \frac{\partial L}{\partial \alpha_1} = \frac{r_1}{\alpha_1} - \sum_{j=1}^{r_1} x_{1j}^{\theta_1} + \sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{R(x_j)} \left(\frac{\partial R(x_j)}{\partial \alpha_1} \right)$$

$$L_{\alpha_2} = \frac{\partial L}{\partial \alpha_2} = \frac{r_2}{\alpha_2} - \sum_{j=1}^{r_2} \ln(1 + \theta_2 x_{2j}) + \sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{R(x_j)} \left(\frac{\partial R(x_j)}{\partial \alpha_2} \right)$$

$$L_p = \frac{\partial L}{\partial p} = \frac{r_1}{p_1} - \frac{r_2}{p_2} + \sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{R(x_j)} \left(\frac{\partial R(x_j)}{\partial p} \right)$$

$$L_{\alpha_1\alpha_1} = \frac{\partial^2 L}{\partial \alpha_1^2} = -\frac{r_1}{\alpha_1^2} + \sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{R(x_j)} \left\{ \frac{\partial^2 R(x_j)}{\partial \alpha_1^2} - \left(\frac{\partial R(x_j)}{\partial \alpha_1} \right)^2 \frac{1}{R(x_j)} \right\}$$

$$L_{\alpha_2\alpha_2} = \frac{\partial^2 L}{\partial \alpha_2^2} = -\frac{r_2}{\alpha_2^2} + \sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{R(x_j)} \left\{ \frac{\partial^2 R(x_j)}{\partial \alpha_2^2} - \left(\frac{\partial R(x_j)}{\partial \alpha_2} \right)^2 \frac{1}{R(x_j)} \right\}$$

$$L_{pp} = \frac{\partial^2 L}{\partial p^2} = -\frac{r_1}{p_1^2} - \frac{r_2}{p_2^2} - \sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{(R(x_j))^2} \left(\frac{\partial R(x_j)}{\partial p} \right)^2$$

$$L_{\alpha_1\alpha_2} = L_{\alpha_2\alpha_1} = \frac{\partial^2 L}{\partial \alpha_1 \partial \alpha_2} = -\sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{(R(x_j))^2} \left(\frac{\partial R(x_j)}{\partial \alpha_1} \right) \left(\frac{\partial R(x_j)}{\partial \alpha_2} \right)$$

$$L_{\alpha_1 p} = L_{p\alpha_1} = \frac{\partial^2 L}{\partial \alpha_1 \partial p} = \sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{R(x_j)} \left\{ \frac{\partial^2 R(x_j)}{\partial \alpha_1 \partial p} - \left(\frac{\partial R(x_j)}{\partial \alpha_1} \right) \left(\frac{\partial R(x_j)}{\partial p} \right) \frac{1}{R(x_j)} \right\}$$

$$L_{\alpha_2 p} = L_{p\alpha_2} = \frac{\partial^2 L}{\partial \alpha_2 \partial p} = \sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{R(x_j)} \left\{ \frac{\partial^2 R(x_j)}{\partial \alpha_2 \partial p} - \left(\frac{\partial R(x_j)}{\partial \alpha_2} \right) \left(\frac{\partial R(x_j)}{\partial p} \right) \frac{1}{R(x_j)} \right\}$$

$$L_{\alpha_1\alpha_1\alpha_1} = \frac{\partial^3 L}{\partial \alpha_1^3} = \frac{2r_1}{\alpha_1^3} + \sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{R(x_j)} \left\{ \frac{\partial^3 R(x_j)}{\partial \alpha_1^3} - 3\left(\frac{\partial R(x_j)}{\partial \alpha_1} \right) \left(\frac{\partial^2 R(x_j)}{\partial \alpha_1^2} \right) \frac{1}{R(x_j)} + 2\left(\frac{\partial R(x_j)}{\partial \alpha_1} \right)^3 \frac{1}{(R(x_j))^2} \right\}$$

$$L_{\alpha_2\alpha_2\alpha_2} = \frac{\partial^3 L}{\partial \alpha_2^3} = \frac{2r_2}{\alpha_2^3} + \sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{R(x_j)} \left\{ \frac{\partial^3 R(x_j)}{\partial \alpha_2^3} - 3\left(\frac{\partial R(x_j)}{\partial \alpha_2} \right) \left(\frac{\partial^2 R(x_j)}{\partial \alpha_2^2} \right) \frac{1}{R(x_j)} + 2\left(\frac{\partial R(x_j)}{\partial \alpha_2} \right)^3 \frac{1}{(R(x_j))^2} \right\}$$

$$L_{ppp} = \frac{\partial^3 L}{\partial p^3} = \frac{2r_1}{p_1^3} - \frac{2r_2}{p_2^3} + \sum_{j=1}^{m} \frac{2\mathscr{R}_j^*}{(R(x_j))^3} \left(\frac{\partial R(x_j)}{\partial p} \right)^3$$

$$L_{\alpha_1\alpha_1\alpha_2} = \frac{\partial^3 L}{\partial \alpha_1^2 \partial \alpha_2} = \sum_{j=1}^{m} \frac{\mathscr{R}_j^*}{R(x_j)} \left\{ 2\left(\frac{\partial R(x_j)}{\partial \alpha_2} \right) \left(\frac{\partial R(x_j)}{\partial \alpha_1} \right)^2 \frac{1}{(R(x_j))^2} - \left(\frac{\partial R(x_j)}{\partial \alpha_2} \right) \left(\frac{\partial^2 R(x_j)}{\partial \alpha_1^2} \right) \frac{1}{R(x_j)} \right\}$$

$$L_{\alpha_1\alpha_1 p}=\frac{\partial^3 L}{\partial\alpha_1^2\partial p}$$
$$=\sum_{j=1}^{m}\frac{\mathscr{R}_j^*}{R(x_j)}\left\{\left(\frac{\partial^3 R(x_j)}{\partial\alpha_1^2\partial p}\right)-\left(\frac{\partial R(x_j)}{\partial p}\right)\left(\frac{\partial^2 R(x_j)}{\partial\alpha_1^2}\right)\frac{1}{R(x_j)}-2\left(\frac{\partial R(x_j)}{\partial\alpha_1}\right)\left(\frac{\partial^2 R(x_j)}{\partial\alpha_1\partial p}\right)\frac{1}{R(x_j)}\right.$$
$$\left.+2\left(\frac{\partial R(x_j)}{\partial p}\right)\left(\frac{\partial R(x_j)}{\partial\alpha_1}\right)^2\frac{1}{(R(x_j))^2}\right\}$$

$$L_{\alpha_1\alpha_2\alpha_2}=\frac{\partial^3 L}{\partial\alpha_1\partial\alpha_2^2}=\sum_{j=1}^{m}\frac{\mathscr{R}_j^*}{R(x_j)}\left\{2\left(\frac{\partial R(x_j)}{\partial\alpha_1}\right)\left(\frac{\partial R(x_j)}{\partial\alpha_2}\right)^2\frac{1}{(R(x_j))^2}-\left(\frac{\partial R(x_j)}{\partial\alpha_1}\right)\left(\frac{\partial^2 R(x_j)}{\partial\alpha_2^2}\right)\frac{1}{R(x_j)}\right\}$$

$$L_{\alpha_1\alpha_2 p}=\frac{\partial^3 L}{\partial\alpha_1\partial\alpha_2\partial p}=\sum_{j=1}^{m}\frac{\mathscr{R}_j^*}{R(x_j)}$$
$$\times\left\{2\left(\frac{\partial R(x_j)}{\partial\alpha_1}\right)\left(\frac{\partial R(x_j)}{\partial\alpha_2}\right)\left(\frac{\partial R(x_j)}{\partial p}\right)\frac{1}{(R(x_j))^2}-\left(\frac{\partial R(x_j)}{\partial\alpha_2}\right)\left(\frac{\partial^2 R(x_j)}{\partial\alpha_1\partial p}\right)\frac{1}{R(x_j)}-\left(\frac{\partial R(x_j)}{\partial\alpha_1}\right)\left(\frac{\partial^2 R(x_j)}{\partial\alpha_2\partial p}\right)\frac{1}{R(x_j)}\right\}$$

$$L_{\alpha_2\alpha_2 p}=\frac{\partial^3 L}{\partial\alpha_2^2\partial p}=\sum_{j=1}^{m}\frac{\mathscr{R}_j^*}{R(x_j)}$$
$$\times\left\{\left(\frac{\partial^3 R(x_j)}{\partial\alpha_2^2\partial p}\right)-\left(\frac{\partial R(x_j)}{\partial p}\right)\left(\frac{\partial^2 R(x_j)}{\partial\alpha_2^2}\right)\frac{1}{R(x_j)}-2\left(\frac{\partial R(x_j)}{\partial\alpha_2}\right)\left(\frac{\partial^2 R(x_j)}{\partial\alpha_2\partial p}\right)\frac{1}{R(x_j)}+2\left(\frac{\partial R(x_j)}{\partial p}\right)\left(\frac{\partial R(x_j)}{\partial\alpha_2}\right)^2\frac{1}{(R(x_j))^2}\right\}$$

$$L_{pp\alpha_1}=\frac{\partial^3 L}{\partial\alpha_1\partial p^2}=\sum_{j=1}^{m}\frac{2\mathscr{R}_j^*}{R(x_j)}\left\{\left(\frac{\partial R(x_j)}{\partial\alpha_1}\right)\left(\frac{\partial R(x_j)}{\partial p}\right)^2\frac{1}{(R(x_j))^2}-\left(\frac{\partial R(x_j)}{\partial p}\right)\left(\frac{\partial^2 R(x_j)}{\partial p\partial\alpha_1}\right)\frac{1}{R(x_j)}\right\}$$

$$L_{\alpha_2 pp}=\frac{\partial^3 L}{\partial\alpha_2\partial p^2}=\sum_{j=1}^{m}\frac{2\mathscr{R}_j^*}{R(x_j)}\left\{\left(\frac{\partial R(x_j)}{\partial\alpha_2}\right)\left(\frac{\partial R(x_j)}{\partial p}\right)^2\frac{1}{(R(x_j))^2}-\left(\frac{\partial R(x_j)}{\partial p}\right)\left(\frac{\partial^2 R(x_j)}{\partial p\partial\alpha_2}\right)\frac{1}{R(x_j)}\right\}$$

and

$$\frac{\partial R(x_j)}{\partial\alpha_1}=-\,p_1\,e^{-\alpha_1 x_j^{\theta_1}}\;x_j^{\theta_1}$$

$$\frac{\partial R(x_j)}{\partial\alpha_2}=-\,p_2(1+\theta_2 x_j)^{-\alpha_2}\ln(1+\theta_2 x_j)$$

$$\frac{\partial R(x_j)}{\partial p}=e^{-\alpha_1 x_j^{\theta_1}}-(1+\theta_2 x_j)^{-\alpha_2}$$

$$\frac{\partial^2 R(x_j)}{\partial\alpha_1^2}=p_1\,e^{-\alpha_1 x_j^{\theta_1}}\;x_j^{2\theta_1}$$

$$\frac{\partial^2 R(x_j)}{\partial\alpha_2^2}=p_2(1+\theta_2 x_j)^{-\alpha_2}[\ln(1+\theta_2 x_j)]^2$$

$$\frac{\partial^2 R(x_j)}{\partial\alpha_1\partial p}=-\,e^{-\alpha_1 x_j^{\theta_1}}\,x_j^{\theta_1}$$

$$\frac{\partial^2 R(x_j)}{\partial\alpha_2\partial p}=(1+\theta_2 x_j)^{-\alpha_2}\ln(1+\theta_2 x_j)$$

$$\frac{\partial^3 R(x_j)}{\partial \alpha_1^3} = -\ p_1\ e^{-\alpha_1 x_j^{\theta_1}}\ x_j^{3\theta_1}$$

$$\frac{\partial^3 R(x_j)}{\partial \alpha_2^3} = -\ p_2 (1+\theta_2 x_j)^{-\alpha_2} [\ln(1+\theta_2 x_j)\ ]^3$$

$$\frac{\partial^3 R(x_j)}{\partial \alpha_1^2 \partial p} = e^{-\alpha_1 x_j^{\theta_1}}\ x_j^{2\theta_1}$$

$$\frac{\partial^3 R(x_j)}{\partial \alpha_2^2 \partial p} = -\ (1+\theta_2 x_j)^{-\alpha_2} [\ln(1+\theta_2 x_j)\ ]^2$$

Now, the Bayes estimators of α_1, α_2 and p are computed as follows:
If $\phi(\alpha_1, \alpha_2, p) = \alpha_1$, Thus, the approximate Bayes estimator of α_1 under square error loss function is given by

$$\widehat{\alpha}_{1SE\_Lindley} = \alpha_1 + \left(\phi_{\alpha_1}\tau_{\alpha_1} + \phi_{\alpha_2}\tau_{\alpha_2} + \phi_p\tau_p + \phi_{\alpha_1\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_{\alpha_1 p}\sigma_{\alpha_1 p} + \phi_{\alpha_2 p}\sigma_{\alpha_2 p} + \frac{1}{2}\left(\phi_{\alpha_1\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_{pp}\sigma_{pp}\right)\right)$$
$$+\frac{1}{2}\left[\psi_1\left(\phi_{\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_p\sigma_{\alpha_1 p}\right) + \psi_2\left(\phi_{\alpha_1}\sigma_{\alpha_2\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_p\sigma_{\alpha_2 p}\right) + \psi_3\left(\phi_{\alpha_1}\sigma_{p\alpha_1} + \phi_{\alpha_2}\sigma_{p\alpha_2} + \phi_p\sigma_{pp}\right)\right]$$

Similarly the Bayes estimator of α_2 *and* p, respectively

$$\widehat{\alpha}_{2SE\_Lindley} = \alpha_2 + \left(\phi_{\alpha_1}\tau_{\alpha_1} + \phi_{\alpha_2}\tau_{\alpha_2} + \phi_p\tau_p + \phi_{\alpha_1\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_{\alpha_1 p}\sigma_{\alpha_1 p} + \phi_{\alpha_2 p}\sigma_{\alpha_2 p} + \frac{1}{2}\left(\phi_{\alpha_1\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_{pp}\sigma_{pp}\right)\right)$$
$$+\frac{1}{2}\left[\psi_1\left(\phi_{\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_p\sigma_{\alpha_1 p}\right) + \psi_2\left(\phi_{\alpha_1}\sigma_{\alpha_2\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_p\sigma_{\alpha_2 p}\right) + \psi_3\left(\phi_{\alpha_1}\sigma_{p\alpha_1} + \phi_{\alpha_2}\sigma_{p\alpha_2} + \phi_p\sigma_{pp}\right)\right]$$

and

$$\hat{p}_{SE\_Lindley} = p + \left(\phi_{\alpha_1}\tau_{\alpha_1} + \phi_{\alpha_2}\tau_{\alpha_2} + \phi_p\tau_p + \phi_{\alpha_1\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_{\alpha_1 p}\sigma_{\alpha_1 p} + \phi_{\alpha_2 p}\sigma_{\alpha_2 p} + \frac{1}{2}\left(\phi_{\alpha_1\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_{pp}\sigma_{pp}\right)\right)$$
$$+\frac{1}{2}\left[\psi_1\left(\phi_{\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_p\sigma_{\alpha_1 p}\right) + \psi_2\left(\phi_{\alpha_1}\sigma_{\alpha_2\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_p\sigma_{\alpha_2 p}\right) + \psi_3\left(\phi_{\alpha_1}\sigma_{p\alpha_1} + \phi_{\alpha_2}\sigma_{p\alpha_2} + \phi_p\sigma_{pp}\right)\right]$$

If $\phi(\alpha_1, \alpha_2, p) = e^{-q\alpha_1}$, The approximate Bayes estimator of α_1 under LINEX loss function is given by

$$\hat{\alpha}_{1LINEX\_Lindley} =$$
$$-\frac{1}{q}\ln\left[e^{-q\alpha_1} + \left(\phi_{\alpha_1}\tau_{\alpha_1} + \phi_{\alpha_2}\tau_{\alpha_2} + \phi_p\tau_p + \phi_{\alpha_1\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_{\alpha_1 p}\sigma_{\alpha_1 p} + \phi_{\alpha_2 p}\sigma_{\alpha_2 p} + \frac{1}{2}\left(\phi_{\alpha_1\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_{pp}\sigma_{pp}\right)\right)\right.$$
$$\left.+\frac{1}{2}\left[\psi_1\left(\phi_{\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_p\sigma_{\alpha_1 p}\right) + \psi_2\left(\phi_{\alpha_1}\sigma_{\alpha_2\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_p\sigma_{\alpha_2 p}\right) + \psi_3\left(\phi_{\alpha_1}\sigma_{p\alpha_1} + \phi_{\alpha_2}\sigma_{p\alpha_2} + \phi_p\sigma_{pp}\right)\right]\right]$$

Also, similarly, the Bayes estimator of α_2 and p under LINEX loss function, respectively, are

$$\hat{\alpha}_{2LINEX\_Lindley} =$$
$$-\frac{1}{q}\ln\left[e^{-q\alpha_2} + \left(\phi_{\alpha_1}\tau_{\alpha_1} + \phi_{\alpha_2}\tau_{\alpha_2} + \phi_p\tau_p + \phi_{\alpha_1\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_{\alpha_1 p}\sigma_{\alpha_1 p} + \phi_{\alpha_2 p}\sigma_{\alpha_2 p} + \frac{1}{2}\left(\phi_{\alpha_1\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_{pp}\sigma_{pp}\right)\right)\right.$$
$$\left.+\frac{1}{2}\left[\psi_1\left(\phi_{\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_p\sigma_{\alpha_1 p}\right) + \psi_2\left(\phi_{\alpha_1}\sigma_{\alpha_2\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_p\sigma_{\alpha_2 p}\right) + \psi_3\left(\phi_{\alpha_1}\sigma_{p\alpha_1} + \phi_{\alpha_2}\sigma_{p\alpha_2} + \phi_p\sigma_{pp}\right)\right]\right]$$

and

$$\hat{p}_{LINEX\_Lindley} =$$
$$-\frac{1}{q}\ln\left[e^{-qp} + \left(\phi_{\alpha_1}\tau_{\alpha_1} + \phi_{\alpha_2}\tau_{\alpha_2} + \phi_p\tau_p + \phi_{\alpha_1\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_{\alpha_1 p}\sigma_{\alpha_1 p} + \phi_{\alpha_2 p}\sigma_{\alpha_2 p} + \frac{1}{2}\left(\phi_{\alpha_1\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_{pp}\sigma_{pp}\right)\right)\right.$$
$$\left.+\frac{1}{2}\left[\psi_1\left(\phi_{\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_p\sigma_{\alpha_1 p}\right) + \psi_2\left(\phi_{\alpha_1}\sigma_{\alpha_2\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_p\sigma_{\alpha_2 p}\right) + \psi_3\left(\phi_{\alpha_1}\sigma_{p\alpha_1} + \phi_{\alpha_2}\sigma_{p\alpha_2} + \phi_p\sigma_{pp}\right)\right]\right]$$

If $\phi(\vartheta) = \alpha_1^{-h}$, then the approximate Bayes estimator of α_1 under general entropy loss function is given by

$$\widehat{\alpha}_{1GE\_Lindley} = \left[\alpha_1^{-h} + \left(\phi_{\alpha_1}\tau_{\alpha_1} + \phi_{\alpha_2}\tau_{\alpha_2} + \phi_p\tau_p + \phi_{\alpha_1\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_{\alpha_1 p}\sigma_{\alpha_1 p} + \phi_{\alpha_2 p}\sigma_{\alpha_2 p} + \frac{1}{2}\left(\phi_{\alpha_1\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_{pp}\sigma_{pp}\right)\right)\right.$$

$$\left. + \frac{1}{2}\left[\psi_1\left(\phi_{\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_p\sigma_{\alpha_1 p}\right) + \psi_2\left(\phi_{\alpha_1}\sigma_{\alpha_2\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_p\sigma_{\alpha_2 p}\right) + \psi_3\left(\phi_{\alpha_1}\sigma_{p\alpha_1} + \phi_{\alpha_2}\sigma_{p\alpha_2} + \phi_p\sigma_{pp}\right)\right]\right]^{-\frac{1}{h}}$$

Also, similarly, the Bayes estimator of α_2 *and* p under general entropy loss function, respectively, are

$$\widehat{\alpha}_{2GE\_Lindley} = \left[\alpha_2^{-h} + \left(\phi_{\alpha_1}\tau_{\alpha_1} + \phi_{\alpha_2}\tau_{\alpha_2} + \phi_p\tau_p + \phi_{\alpha_1\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_{\alpha_1 p}\sigma_{\alpha_1 p} + \phi_{\alpha_2 p}\sigma_{\alpha_2 p} + \frac{1}{2}\left(\phi_{\alpha_1\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_{pp}\sigma_{pp}\right)\right)\right.$$

$$\left. + \frac{1}{2}\left[\psi_1\left(\phi_{\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_p\sigma_{\alpha_1 p}\right) + \psi_2\left(\phi_{\alpha_1}\sigma_{\alpha_2\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_p\sigma_{\alpha_2 p}\right) + \psi_3\left(\phi_{\alpha_1}\sigma_{p\alpha_1} + \phi_{\alpha_2}\sigma_{p\alpha_2} + \phi_p\sigma_{pp}\right)\right]\right]^{-\frac{1}{h}}$$

and

$$\hat{p}_{GE\_Lindley} = \left[p^{-h} + \left(\phi_{\alpha_1}\tau_{\alpha_1} + \phi_{\alpha_2}\tau_{\alpha_2} + \phi_p\tau_p + \phi_{\alpha_1\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_{\alpha_1 p}\sigma_{\alpha_1 p} + \phi_{\alpha_2 p}\sigma_{\alpha_2 p} + \frac{1}{2}\left(\phi_{\alpha_1\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_{pp}\sigma_{pp}\right)\right)\right.$$

$$\left. + \frac{1}{2}\left[\psi_1\left(\phi_{\alpha_1}\sigma_{\alpha_1\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_1\alpha_2} + \phi_p\sigma_{\alpha_1 p}\right) + \psi_2\left(\phi_{\alpha_1}\sigma_{\alpha_2\alpha_1} + \phi_{\alpha_2}\sigma_{\alpha_2\alpha_2} + \phi_p\sigma_{\alpha_2 p}\right) + \psi_3\left(\phi_{\alpha_1}\sigma_{p\alpha_1} + \phi_{\alpha_2}\sigma_{p\alpha_2} + \phi_p\sigma_{pp}\right)\right]\right]^{-\frac{1}{h}}$$

Noting the when value $h = -1$, the general entropy loss function is the same as the squared error loss function.

3.2 Tierney_ Kadane's approximation method

Here, we compute the Bayes estimators of α_1, α_2 and p by using the method given by Tierney and Kadane [29]. The Bayes estimator of $\phi(\alpha_1, \alpha_2, p)$ can be expressed as

$$E\left[\phi(\alpha_1, \alpha_2, p)|x\right] = \frac{\int_0^1\int_0^\infty\int_0^\infty \quad e^{n\delta_\phi^*(\alpha_1,\alpha_2,p)}d\alpha_1 d\alpha_2 dp}{\int_0^1\int_0^\infty\int_0^\infty \quad e^{n\delta(\alpha_1,\alpha_2,p)}d\alpha_1 d\alpha_2 dp} \tag{19}$$

We consider the following functions

$$\delta(\alpha_1, \alpha_2, p) = \frac{1}{n}\left[L(\alpha_1, \alpha_2, p|\underline{x}) + \rho(\alpha_1, \alpha_2, p)\right], \qquad \delta_\phi^*(\alpha_1, \alpha_2, p) = \delta(\alpha_1, \alpha_2, p) + \frac{1}{n}\ln\phi(\alpha_1, \alpha_2, p)$$

where, $L(\alpha_1, \alpha_2, p|x)$ is the log-likelihood function is given by (7) and $\rho(\alpha_1, \alpha_2, p) = \ln\pi\ (\alpha_1, \alpha_2, p)$, and also assuming that $(\widehat{\alpha}_{1\delta^*}, \widehat{\alpha}_{2\delta^*}, \hat{p}_{\delta^*})$ and $(\widehat{\alpha}_{1\delta}, \widehat{\alpha}_{2\delta}, \hat{p}_\delta)$ maximize functions $\delta_\phi^*(\alpha_1, \alpha_2, p)$ and $\delta(\alpha_1, \alpha_2, p)$, respectively

Following Tierney and Kadane [29], Equation (19) can be approximated in the following form

$$\widehat{\phi}(\alpha_1, \alpha_2, p) = \sqrt{\frac{|\Sigma_\phi^*|}{|\Sigma|}} exp\left[n\left\{\delta_\phi^*(\widehat{\alpha}_{1\delta^*}, \widehat{\alpha}_{2\delta^*}, \hat{p}_{\delta^*}) - \delta(\widehat{\alpha}_{1\delta}, \widehat{\alpha}_{2\delta}, \hat{p}_\delta)\right\}\right] \tag{20}$$

where $\sum_\phi^*$ and $\sum$ are the negatives of the inverse Hessians of $\delta_\phi^*(\alpha_1, \alpha_2, p)$ and $\delta(\alpha_1, \alpha_2, p)$ at $(\widehat{\alpha}_{1\delta^*}, \widehat{\alpha}_{2\delta^*}, \hat{p}_{\delta^*})$ and $(\widehat{\alpha}_{1\delta}, \widehat{\alpha}_{2\delta}, \hat{p}_\delta)$, respectively. We have

$$\delta(\alpha_1, \alpha_2, p) = \frac{1}{n}\left[(r_1 + c - 1)\ln p_1 + (r_1 + a_1 - 1)\ln\alpha_1 + (r_2 + d - 1)\ln p_2 + (r_2 + a_2 - 1)\ln\alpha_2\right.$$
$$\left. - \alpha_1\sum_{j=1}^{r_1} x_{1j}^{\theta_1} - \alpha_2\sum_{j=1}^{r_2}\ln\left(1 + \theta_2 x_{2j}\right) + \sum_{j=1}^{m}\mathscr{R}_j^*\ln\left[R(x_j)\right] - b_1\alpha_1 - b_2\alpha_2\right]$$

Then $(\widehat{\alpha}_{1\delta}, \widehat{\alpha}_{2\delta}, \hat{p}_\delta)$ are computed by solving the following non-linear equations

$$\frac{\partial\delta}{\partial\alpha_1}=\frac{1}{n}\left[\frac{(r_1+a_1-1)}{\alpha_1}-\sum_{j=1}^{r_1}x_{1j}^{\theta_1}+\sum_{j=1}^{m}\frac{\mathscr{R}_j^*}{R(x_j)}\left(\frac{\partial R(x_j)}{\partial\alpha_1}\right)-b_1\right]=0,$$

$$\frac{\partial\delta}{\partial\alpha_2}=\frac{1}{n}\left[\frac{(r_2+a_2-1)}{\alpha_2}-\sum_{j=1}^{r_2}\ln\left(1+\theta_2x_{2j}\right)+\sum_{j=1}^{m}\frac{\mathscr{R}_j^*}{R(x_j)}\left(\frac{\partial R(x_j)}{\partial\alpha_2}\right)-b_2\right]=0,$$

$$\frac{\partial\delta}{\partial p}=\frac{1}{n}\left[\frac{(r_1+c-1)}{p_1}-\frac{(r_2+d-1)}{p_2}+\sum_{j=1}^{m}\frac{\mathscr{R}_j^*}{R(x_j)}\left(\frac{\partial R(x_j)}{\partial p}\right)\right]=0$$

We further obtain

$$\delta_{\alpha_1\alpha_1}=\frac{\partial^2\delta}{\partial\alpha_1^2}=\frac{1}{n}\left[-\frac{(r_1+a_1-1)}{\alpha_1^2}+\sum_{j=1}^{m}\frac{\mathscr{R}_j^*}{R(x_j)}\left\{\frac{\partial^2R(x_j)}{\partial\alpha_1^2}-\left(\frac{\partial R(x_j)}{\partial\alpha_1}\right)^2\frac{1}{R(x_j)}\right\}\right]$$

$$\delta_{\alpha_2\alpha_2}=\frac{\partial^2\delta}{\partial\alpha_2^2}=\frac{1}{n}\left[-\frac{(r_2+a_2-1)}{\alpha_2^2}+\sum_{j=1}^{m}\frac{\mathscr{R}_j^*}{R(x_j)}\left\{\frac{\partial^2R(x_j)}{\partial\alpha_2^2}-\left(\frac{\partial R(x_j)}{\partial\alpha_2}\right)^2\frac{1}{R(x_j)}\right\}\right]$$

$$\delta_{pp}=\frac{\partial^2\delta}{\partial p^2}=\frac{1}{n}\left[-\frac{(r_1+c-1)}{p_1^2}-\frac{(r_2+d-1)}{p_2^2}-\sum_{j=1}^{m}\frac{\mathscr{R}_j^*}{(R(x_j))^2}\left(\frac{\partial R(x_j)}{\partial p}\right)^2\right]$$

$$\delta_{\alpha_1\alpha_2}=\delta_{\alpha_2\alpha_1}=\frac{\partial^2\delta}{\partial\alpha_1\partial\alpha_2}=\frac{1}{n}\left[-\sum_{j=1}^{m}\frac{\mathscr{R}_j^*}{(R(x_j))^2}\left(\frac{\partial R(x_j)}{\partial\alpha_1}\right)\left(\frac{\partial R(x_j)}{\partial\alpha_2}\right)\right]$$

$$\delta_{\alpha_1p}=\delta_{p\alpha_1}=\frac{\partial^2\delta}{\partial\alpha_1\partial p}=\frac{1}{n}\left[\sum_{j=1}^{m}\frac{\mathscr{R}_j^*}{R(x_j)}\left\{\frac{\partial^2R(x_j)}{\partial\alpha_1\partial p}-\left(\frac{\partial R(x_j)}{\partial\alpha_1}\right)\left(\frac{\partial R(x_j)}{\partial p}\right)\frac{1}{R(x_j)}\right\}\right]$$

$$\delta_{\alpha_2p}=\delta_{p\alpha_2}=\frac{\partial^2\delta}{\partial\alpha_2\partial p}=\frac{1}{n}\left[\sum_{j=1}^{m}\frac{\mathscr{R}_j^*}{R(x_j)}\left\{\frac{\partial^2R(x_j)}{\partial\alpha_2\partial p}-\left(\frac{\partial R(x_j)}{\partial\alpha_2}\right)\left(\frac{\partial R(x_j)}{\partial p}\right)\frac{1}{R(x_j)}\right\}\right]$$

where, the derivatives of $R(x_j)$ with respect to the three parameters are given previously in subsection 3.1.

The matrix $\sum$ takes the form:

$$\sum=\left(-\frac{\partial^2\delta}{\partial\omega_i\partial\omega_j}\right),\qquad i,j=1,2,3 \tag{21}$$

where $\omega_1=\alpha_1$, $\omega_2=\alpha_2$ and $\omega_3=p$.

Now, the Bayes estimators of α_1,α_2 and p are computed as follows:

If $\phi(\alpha_1,\alpha_2,p)=\alpha_1$ and consequently $\delta_\phi^*(\alpha_1,\alpha_2,p)$ becomes

$$\delta_{\alpha_1}^*(\alpha_1,\alpha_2,p)=\delta(\alpha_1,\alpha_2,p)+\frac{1}{n}\ln\alpha_1$$

and then $(\widehat{\alpha}_{1\delta^*},\widehat{\alpha}_{2\delta^*},\hat{p}_{\delta^*})$ are obtained by solving the following non-linear equations:

$$\frac{\partial\delta_{\alpha_1}^*}{\partial\alpha_1}=\frac{\partial\delta}{\partial\alpha_1}+\frac{1}{n\alpha_1}=0,\frac{\partial\delta_{\alpha_1}^*}{\partial\alpha_2}=\frac{\partial\delta}{\partial\alpha_2}=0,\frac{\partial\delta_{\alpha_1}^*}{\partial p}=\frac{\partial\delta}{\partial p}=0.$$

Using the following expressions

$$\delta_{\alpha_1\alpha_1}^*=\frac{\partial^2\delta_{\alpha_1}^*}{\partial\alpha_1^2}=\frac{\partial^2\delta}{\partial\alpha_1^2}-\frac{1}{n\alpha_1^2},\delta_{\alpha_1\alpha_2}^*=\delta_{\alpha_1\alpha_2},\delta_{\alpha_1p}^*=\delta_{\alpha_1p},\delta_{\alpha_2\alpha_2}^*=\delta_{\alpha_2\alpha_2},\ \delta_{\alpha_2p}^*=\delta_{\alpha_2p},\delta_{pp}^*=\delta_{pp}.$$

Hence,

$$\sum\nolimits_{\alpha_1}^*=\left(-\frac{\partial^2\delta_{\alpha_1}^*}{\partial\omega_i\partial\omega_j}\right),\qquad i,j=1,2,3. \tag{22}$$

where $\omega_1 = \alpha_1$, $\omega_2 = \alpha_2$ and $\omega_3 = p$.

Thus, the approximate Bayes estimator of α_1 under square error loss function is given by

$$\widehat{\alpha}_{1SE\_TK} = \sqrt{\frac{|\Sigma^*_{\alpha_1}|}{|\Sigma|}} exp\left[n\left\{\delta^*_{\alpha_1}\left(\widehat{\alpha}_{1\delta^*}, \widehat{\alpha}_{2\delta^*}, \hat{p}_{\delta^*}\right) - \delta\left(\widehat{\alpha}_{1\delta}, \widehat{\alpha}_{2\delta}, \hat{p}_{\delta}\right)\right\}\right]$$

Similarly the Bayes estimators of α_2 *and* p, respectively

$$\widehat{\alpha}_{2SE\_TK} = \sqrt{\frac{|\Sigma^*_{\alpha_2}|}{|\Sigma|}} exp\left[n\left\{\delta^*_{\alpha_2}\left(\widehat{\alpha}_{1\delta^*}, \widehat{\alpha}_{2\delta^*}, \hat{p}_{\delta^*}\right) - \delta\left(\widehat{\alpha}_{1\delta}, \widehat{\alpha}_{2\delta}, \hat{p}_{\delta}\right)\right\}\right]$$

and

$$\hat{p}_{SE\_TK} = \sqrt{\frac{|\Sigma^*_{p}|}{|\Sigma|}} exp\left[n\left\{\delta^*_{p}\left(\widehat{\alpha}_{1\delta^*}, \widehat{\alpha}_{2\delta^*}, \hat{p}_{\delta^*}\right) - \delta\left(\widehat{\alpha}_{1\delta}, \widehat{\alpha}_{2\delta}, \hat{p}_{\delta}\right)\right\}\right]$$

If $\phi\left(\alpha_1, \alpha_2, p\right) = e^{-q\alpha_1}$ and consequently $\delta^*_{\phi}\left(\alpha_1, \alpha_2, p\right)$ becomes

$$\delta^*_{\alpha_1}\left(\alpha_1, \alpha_2, p\right) = \delta\left(\alpha_1, \alpha_2, p\right) - \frac{1}{n} q\alpha_1$$

and then $\left(\widehat{\alpha}_{1\delta^*}, \widehat{\alpha}_{2\delta^*}, \hat{p}_{\delta^*}\right)$ are obtained by solving the following non-linear equations:

$$\frac{\partial \delta^*_{\alpha_1}}{\partial \alpha_1} = \frac{\partial \delta}{\partial \alpha_1} - \frac{1}{n} q = 0, \frac{\partial \delta^*_{\alpha_1}}{\partial \alpha_2} = \frac{\partial \delta}{\partial \alpha_2} = 0, \frac{\partial \delta^*_{\alpha_1}}{\partial p} = \frac{\partial \delta}{\partial p} = 0.$$

Again, one can obtain $\sum^*_{\alpha_1}$ from (22)

Where $\delta^*_{\alpha_1\alpha_1} = \frac{\partial^2 \delta^*_{\alpha_1}}{\partial \alpha_1^2} = \frac{\partial^2 \delta}{\partial \alpha_1^2}$, $\delta^*_{\alpha_1\alpha_2} = \delta_{\alpha_1\alpha_2}$, $\delta^*_{\alpha_1 p} = \delta_{\alpha_1 p}$, $\delta^*_{\alpha_2\alpha_2} = \delta_{\alpha_2\alpha_2}$, $\delta^*_{\alpha_2 p} = \delta_{\alpha_2 p}$, $\delta^*_{pp} = \delta_{pp}$.

The approximate Bayes estimator of α_1 under LINEX loss function is given by

$$\widehat{\alpha}_{1LINEX\_TK} = -\frac{1}{q}\ln\left[\sqrt{\frac{|\Sigma^*_{\alpha_1}|}{|\Sigma|}} exp\left[n\left\{\delta^*_{\alpha_1}\left(\widehat{\alpha}_{1\delta^*}, \widehat{\alpha}_{2\delta^*}, \hat{p}_{\delta^*}\right) - \delta\left(\widehat{\alpha}_{1\delta}, \widehat{\alpha}_{2\delta}, \hat{p}_{\delta}\right)\right\}\right]\right]$$

Also, similarly, the Bayes estimator of α_2 *and* p under LINEX loss function, respectively, are

$$\widehat{\alpha}_{2LINEX\_TK} = -\frac{1}{q}\ln\left[\sqrt{\frac{|\Sigma^*_{\alpha_2}|}{|\Sigma|}} exp\left[n\left\{\delta^*_{\alpha_2}\left(\widehat{\alpha}_{1\delta^*}, \widehat{\alpha}_{2\delta^*}, \hat{p}_{\delta^*}\right) - \delta\left(\widehat{\alpha}_{1\delta}, \widehat{\alpha}_{2\delta}, \hat{p}_{\delta}\right)\right\}\right]\right]$$

and

$$\hat{p}_{LINEX\_TK} = -\frac{1}{q}\ln\left[\sqrt{\frac{|\Sigma^*_{p}|}{|\Sigma|}} exp\left[n\left\{\delta^*_{p}\left(\widehat{\alpha}_{1\delta^*}, \widehat{\alpha}_{2\delta^*}, \hat{p}_{\delta^*}\right) - \delta\left(\widehat{\alpha}_{1\delta}, \widehat{\alpha}_{2\delta}, \hat{p}_{\delta}\right)\right\}\right]\right]$$

If $\phi\left(\alpha_1, \alpha_2, p\right) = \alpha_1^{-h}$ and consequently $\delta^*_{\phi}\left(\alpha_1, \alpha_2, p\right)$ becomes

$$\delta^*_{\alpha_1}\left(\alpha_1, \alpha_2, p\right) = \delta\left(\alpha_1, \alpha_2, p\right) - \frac{1}{n} h \ln\alpha_1$$

and then $\left(\widehat{\alpha}_{1\delta^*}, \widehat{\alpha}_{2\delta^*}, \hat{p}_{\delta^*}\right)$ are obtained by solving the following non-linear equations:

$$\frac{\partial \delta^*_{\alpha_1}}{\partial \alpha_1} = \frac{\partial \delta}{\partial \alpha_1} - \frac{h}{n\alpha_1} = 0, \frac{\partial \delta^*_{\alpha_1}}{\partial \alpha_2} = \frac{\partial \delta}{\partial \alpha_2} = 0, \frac{\partial \delta^*_{\alpha_1}}{\partial p} = \frac{\partial \delta}{\partial p} = 0.$$

And obtain $\sum^*_{\alpha_1}$ from (22)

Where $\delta^*_{\alpha_1\alpha_1} = \frac{\partial^2 \delta^*_{\alpha_1}}{\partial \alpha_1^2} = \frac{\partial^2 \delta}{\partial \alpha_1^2} + \frac{h}{n\alpha_1^2}$, $\delta^*_{\alpha_1\alpha_2} = \delta_{\alpha_1\alpha_2}$, $\delta^*_{\alpha_1 p} = \delta_{\alpha_1 p}$, $\delta^*_{\alpha_2\alpha_2} = \delta_{\alpha_2\alpha_2}$, $\delta^*_{\alpha_2 p} = \delta_{\alpha_2 p}$, $\delta^*_{pp} = \delta_{pp}$

The approximate Bayes estimator of α_1 under general entropy loss function is given by

$$\widehat{\alpha}_{1GE\_TK} = \left[\sqrt{\frac{|\Sigma^*_{\alpha_1}|}{|\Sigma|}} exp\left[n\left\{ \delta^*_{\alpha_1}\left(\widehat{\alpha}_{1\delta^*}, \widehat{\alpha}_{2\delta^*}, \hat{p}_{\delta^*}\right) - \delta\left(\widehat{\alpha}_{1\delta}, \widehat{\alpha}_{2\delta}, \hat{p}_{\delta}\right)\right\}\right]\right]^{-\frac{1}{h}}$$

Also, similarly, the Bayes estimator of α_2 *and* p under general entropy loss function, respectively, are

$$\widehat{\alpha}_{2GE\_TK} = \left[\sqrt{\frac{|\Sigma^*_{\alpha_2}|}{|\Sigma|}} exp\left[n\left\{ \delta^*_{\alpha_2}\left(\widehat{\alpha}_{1\delta^*}, \widehat{\alpha}_{2\delta^*}, \hat{p}_{\delta^*}\right) - \delta\left(\widehat{\alpha}_{1\delta}, \widehat{\alpha}_{2\delta}, \hat{p}_{\delta}\right)\right\}\right]\right]^{-\frac{1}{h}}$$

and

$$\hat{p}_{GE\_TK} = \left[\sqrt{\frac{|\Sigma^*_{p}|}{|\Sigma|}} exp\left[n\left\{ \delta^*_{p}\left(\widehat{\alpha}_{1\delta^*}, \widehat{\alpha}_{2\delta^*}, \hat{p}_{\delta^*}\right) - \delta\left(\widehat{\alpha}_{1\delta}, \widehat{\alpha}_{2\delta}, \hat{p}_{\delta}\right)\right\}\right]\right]^{-\frac{1}{h}}$$

Noting the when value $h = -1$, the general entropy loss function is the same as the squared error loss function.

4 Simulation Study

In this section, a Monte Carlo simulation study is carried out to examine the performance of proposed estimators. The samples were generated based on the algorithms of Balakrishnan and Sandhu [30], we generate progressively Type-II censored from a mixture of Weibull and Lomax distributions with parameters $\alpha_1 = 2$, $\alpha_2 = 3$, $p = 0.45$ along with $(\theta_1 = 0.5, \theta_2 = 1.5)$ with sample sizes (m) from a random sample of size (n) under censoring scheme $\mathscr{R}$. We estimate the parameters using Maximum likelihood estimation and Bayes estimation obtained by Lindley's approximation and T-K approximation. The following values are used for the hyper parameters $(a_1 = 0.2, a_2 = 0.12, b_1 = 0.35, b_2 = 0.15, c = 1.5$ and $d = 3.5)$ for informative prior. In case of non informative prior, we take $\{(a_1 = a_2 = b_1 = b_2 = 0), (c = d = 1)\}$. All results are based on 1000 replications and the average values and mean squared error (MSE) of estimates are given in Tables 1-6. All results are obtained using Mathematica 11.

5 Conclusion

The purpose of this paper is estimation of the mixture of Weibull and Lomax distributions based a progressive Type-II censoring scheme. The maximum likelihood estimators and Bayes estimators under assumption symmetric and asymmetric loss function using Lindley and T-K approaches are provided. It is observed from Tables (1-6), that the performance of Bayes estimators obtained under informative prior have less MSE as compared to the non-informative prior in all considered approximation techniques especially in case T-K approximation. Also, in most cases, notice that the MSE of Bayesian estimation are smaller than the maximum likelihood estimation, so Bayesian procedure is better than the maximum likelihood as expected. In most values, the MSE decreases when the sample size n and the effective sample m increase. We have observed that, in general, Bayes estimation, T-K approximation are quite good compared to the Lindley's approximation.

Conflict of Interest

The authors declare that they have no conflict of interest.

Table 1: Average estimates and corresponding MSE of the parameter α_1 based on informative prior

(n,m)	Scheme	MLE	Bayes_Lindley					Bayes_T-K				
			SE	LINEX		GE		SE	LINEX		GE	
				q=0.5	q=-0.5	h=0.5	h=-0.5		q=0.5	q=-0.5	h=0.5	h=-0.5
(30,15)	1*15	2.10370 (0.02124)	2.07103 (0.00647)	1.99367 (0.00097)	2.15749 (0.02707)	1.94826 (0.00356)	2.04363 (0.00262)	2.05496 (0.00321)	2.01396 (0.00146)	2.16272 (0.02670)	1.95922 (0.00181)	2.00473 (0.00018)
	(3,0,0)*5	2.05446 (0.02066)	2.01272 (0.00053)	1.94883 (0.00305)	2.09091 (0.00860)	1.91324 (0.00770)	1.96272 (0.00174)	2.00527 (0.00008)	1.97072 (0.00141)	2.08002 (0.00672)	1.90253 (0.00968)	1.98120 (0.00051)
(50,25)	1*25	2.08548 (0.01430)	2.09957 (0.01023)	2.03069 (0.00106)	2.13440 (0.01854)	2.00427 (0.00017)	2.05280 (0.00330)	2.06015 (0.00407)	2.03960 (0.00173)	2.13736 (0.01910)	2.00203 (0.00010)	2.03907 (0.00172)
	(5,0,0,0,0)*5	2.02428 (0.01215)	2.01311 (0.00028)	1.95913 (0.00221)	2.04432 (0.00219)	1.93642 (0.00414)	1.99358 (0.00013)	2.00087 (0.00012)	1.96782 (0.00112)	2.05751 (0.00353)	1.93766 (0.00396)	1.96771 (0.00124)

Table 2: Average estimates and corresponding MSE of the parameter α_2 based on informative prior

(n,m)	Scheme	MLE	Bayes_Lindley					Bayes_T-K				
			SE	LINEX		GE		SE	LINEX		GE	
				q=0.5	q=-0.5	h=0.5	h=-0.5		q=0.5	q=-0.5	h=0.5	h=-0.5
(30,15)	1*15	3.28540 (0.12803)	3.16843 (0.03173)	2.98187 (0.00755)	3.42717 (0.18590)	3.00105 (0.00303)	3.13117 (0.02129)	3.21069 (0.04470)	3.04936 (0.00357)	3.51270 (0.26352)	3.05492 (0.00344)	3.19540 (0.03887)
	(3,0,0)*5	3.18031 (0.11026)	3.13863 (0.02028)	2.90101 (0.01036)	3.2680 (0.07253)	2.91768 (0.00864)	3.05646 (0.00447)	3.14759 (0.02244)	2.93362 (0.00563)	3.34933 (0.12330)	2.92810 (0.01175)	3.08909 (0.00845)
(50,25)	1*25	3.24931 (0.09268)	3.21493 (0.04666)	3.08434 (0.00797)	3.35943 (0.13018)	3.11621 (0.01395)	3.18549 (0.03498)	3.25282 (0.06419)	3.16290 (0.02710)	3.50372 (0.25445)	3.14193 (0.02061)	3.21087 (0.04528)
	(5,0,0,0,0)*5	3.15628 (0.06280)	3.11301 (0.01343)	2.96850 (0.00248)	3.24258 (0.05932)	3.01517 (0.00205)	3.06138 (0.00410)	3.11917 (0.01461)	3.06169 (0.00395)	3.25255 (0.06459)	3.03723 (0.00197)	3.10091 (0.01071)

Table 3: Average estimates and corresponding MSE of the parameter p based on informative prior

(n,m)	Scheme	MLE	Bayes_Lindley					Bayes_T-K				
			SE	LINEX		GE		SE	LINEX		GE	
				q=0.5	q=-0.5	h=0.5	h=-0.5		q=0.5	q=-0.5	h=0.5	h=-0.5
(30,15)	1*15	0.44676 (0.00004)	0.43312 (0.00034)	0.43146 (0.00052)	0.43575 (0.00029)	0.42043 (0.00119)	0.42893 (0.00054)	0.451079 (7.6302× 10ˆ(-6))	0.44631 (0.00005)	0.45332 (0.00001)	0.43633 (0.00019)	0.44706 (9.7697× 10ˆ(-6))
	(3,0,0)*5	0.44612 (0.00010)	0.42816 (0.00049)	0.42554 (0.00061)	0.43013 (0.00041)	0.41513 (0.00123)	0.42396 (0.00069)	0.44942 (3.39695× 10ˆ(-6))	0.44435 (0.00009)	0.45266 (7.76503× 10ˆ(-6))	0.43564 (0.00021)	0.44607 (0.00002)
(50,25)	1*25	0.44676 (0.00002)	0.43841 (0.00015)	0.43761 (0.00017)	0.43966 (0.00012)	0.43033 (0.00040)	0.43562 (0.00022)	0.45752 (0.00006)	0.45450 (0.00003)	0.45886 (0.00008)	0.44767 (5.76842× 10ˆ(-6))	0.45452 (0.00002)
	(5,0,0,0,0)*5	0.44652 (0.00006)	0.43399 (0.00026)	0.43335 (0.00028)	0.43604 (0.00020)	0.42586 (0.00058)	0.43202 (0.00032)	0.45501 (0.00005)	0.45464 (0.00002)	0.45731 (0.00005)	0.44717 (8.57949× 10ˆ(-6))	0.45308 (9.8256× 10ˆ(-6))

Table 4: Average estimates and corresponding MSE of the parameter α_1 based on non-informative prior

(n,m)	Scheme	Bayes_Lindley					Bayes_T-K				
		SE	LINEX		GE		SE	LINEX		GE	
			q=0.5	q=-0.5	h=0.5	h=-0.5		q=0.5	q=-0.5	h=0.5	h=-0.5
(30,15)	1*15	2.10891 (0.01264)	2.03167 (0.00222)	2.18041 (0.03361)	1.98561 (0.00171)	2.07113 (0.00836)	2.06603 (0.00460)	2.04492 (0.00233)	2.18051 (0.03289)	1.96262 (0.00157)	2.0330 (0.00137)
	(3,0,0)*5	2.04835 (0.00284)	1.98782 (0.00059)	2.13663 (0.01917)	1.91543 (0.00751)	2.00687 (0.00024)	2.01697 (0.00041)	1.99618 (0.00029)	2.10682 (0.01221)	1.93701 (0.00414)	1.97497 (0.00084)
(50,25)	1*25	2.09409 (0.00907)	2.05493 (0.00321)	2.14780 (0.02209)	2.03432 (0.00133)	2.07697 (0.00628)	2.07592 (0.00591)	2.05164 (0.00271)	2.13849 (0.01927)	2.01451 (0.00037)	2.05592 (0.00322)
	(5,0,0,0,0)*5	2.02780 (0.00086)	1.98777 (0.00048)	2.08109 (0.00671)	1.96540 (0.00124)	2.00179 (0.00012)	2.02566 (0.00069)	1.97986 (0.00046)	2.0690 (0.00483)	1.96729 (0.00115)	1.99630 (0.00022)

Table 5: Average estimates and corresponding MSE of the parameter α_2 based on non-informative prior

(n,m)	Scheme	Bayes_Lindley					Bayes_T-K				
		SE	LINEX		GE		SE	LINEX		GE	
			q=0.5	q=-0.5	h=0.5	h=-0.5		q=0.5	q=-0.5	h=0.5	h=-0.5
(30,15)	1*15	3.34943 (0.12661)	3.1370 (0.02194)	3.51991 (0.36041)	3.13613 (0.02104)	3.28911 (0.08609)	3.23977 (0.05882)	3.14338 (0.02144)	3.51752 (0.26905)	3.11764 (0.01851)	3.22701 (0.05281)
	(3,0,0)*5	3.23382 (0.05557)	3.01537 (0.01354)	3.36736 (0.13743)	3.03872 (0.00202)	3.16901 (0.02976)	3.19635 (0.03949)	3.07076 (0.00555)	3.42015 (0.17761)	3.05902 (0.00418)	3.14854 (0.02341)
(50,25)	1*25	3.31846 (0.10245)	3.17203 (0.03015)	3.41361 (0.17326)	3.18914 (0.03654)	3.28254 (0.08039)	3.26755 (0.07206)	3.18493 (0.03456)	3.43765 (0.19188)	3.15515 (0.02458)	3.24292 (0.05916)
	(5,0,0,0,0)*5	3.15859 (0.02584)	3.05399 (0.00453)	3.29341 (0.08699)	3.05360 (0.00320)	3.13620 (0.01953)	3.16235 (0.02819)	3.08645 (0.00783)	3.43765 (0.09753)	3.04553 (0.00344)	3.13812 (0.01998)

Table 6: Average estimates and corresponding MSE of the parameter p based on non-informative prior

(n,m)	Scheme	Bayes_Lindley					Bayes_T-K				
		SE	LINEX		GE		SE	LINEX		GE	
			q=0.5	q=-0.5	h=0.5	h=-0.5		q=0.5	q=-0.5	h=0.5	h=-0.5
(30,15)	1*15	0.46123 (0.00020)	0.45414 (0.02468)	0.46388 (0.00027)	0.44753 (0.00010)	0.45737 (0.00011)	0.46900 (0.00037)	0.46695 (0.00029)	0.47362 (0.0006)	0.45482 (0.00002)	0.46596 (0.00026)
	(3,0,0)*5	0.45612 (0.00005)	0.45270 (0.00003)	0.45866 (0.00009)	0.44222 (0.00008)	0.45135 (0.00002)	0.46928 (0.00037)	0.46625 (0.00027)	0.47180 (0.00048)	0.45453 (0.00002)	0.46491 (0.00022)
(50,25)	1*25	0.45550 (0.00004)	0.45442 (0.00004)	0.45742 (0.00007)	0.44728 (0.00002)	0.45319 (0.00002)	0.46930 (0.00038)	0.46565 (0.00027)	0.47096 (0.00044)	0.45945 (0.00009)	0.46603 (0.00026)
	(5,0,0,0,0)*5	0.45171 (6.7339×10^{-6})	0.44970 (7.90426×10^{-6})	0.45299 (0.00001)	0.44275 (0.00006)	0.44817 (6.00968×10^{-6})	0.46651 (0.00028)	0.46560 (0.00025)	0.46961 (0.00039)	0.45817 (0.00007)	0.46498 (0.00022)

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