Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/100316

Statistical Properties of the Nonlinear Time-dependent Interaction between a Three-level Atom and Optical Fields

Sameh T. Korashy[1,*], *Hilal Al Bayatti*[2] *and T. M. El-Shahat*[3]

[1]Department of Mathematics, Faculty of Science, Sohag University, 82524, Sohag, Egypt
[2]Department of Computer Science, Applied Science University, P.O. Box 5055, Kingdom of Bahrain
[3]Department of Mathematics, Faculty of Science, Al-Azhar University, 71524, Assiut, Egypt

Received: 7 Jun. 2021, Revised: 21 Aug. 2021, Accepted: 23 Aug. 2021
Published online: 1 Nov. 2021

Abstract: Some properties through a three-level Λ-type atom interacting with a two-mode deformed bosonic field are investigated. We study this system in the presence of detuning parameter and Kerr nonlinearity. Also, the coupling parameter is modulated to be time-dependent. The exact solution of this model is given using the Schrődinger equation when the atom and the field are initially prepared in superposition state and coherent state, respectively. We employed the results to calculate some aspects such as atomic population inversion, purity and Mandel Q-parameter. The results show that the time-dependent coupling parameter and the detuning parameter are quantum control parameters.

Keywords: Three-level atom, Kerr-medium, Purity, Mandel Q-parameter

1 Introduction

In quantum optics, Jaynes-Cummings model (JCM) [1] is a well-known and an important model. It is an exactly solvable model which clearly describes the interaction between a two-level atom and a single-mode of a quantized radiation field, when the Rotating-Wave Approximation (RWA) is considered. Much attention has been paid to generalize the JCM in different directions such as multi-photon transition, multi-level atoms, intensity-dependent coupling, multi-atoms interaction, multi-mode fields, Stark shift and Kerr nonlinearity, which have been recently investigated [2,3,4,5,6,7,8,9,10,11,12,13,14,15].

Numerous efforts have been devoted to the analytical solutions of multi-level atoms interacting with the cavity field problems. One of the interesting example is the system of three-level atom different configurations (Λ, V, and Ξ) and one- or two-mode field [2,7,16,17,18,19,20]. Many studies have addressed the atom-field entanglement and geometric phase in such systems [2,7,16,18,19,20,21]. Several studies on a three-level atom in motion which interacts with a single-mode field in an optical cavity in an intensity-dependent coupling regime have been conducted [22].

* Corresponding author e-mail: skorashe@yahoo.com

Another example of multi-level atoms interacting with the cavity field problems is the system of four-level atom. Several systems of the four-level atom configurations such as Ξ, $\Diamond$, N, Λ, Y, λ, double-Λ, etc. have been introduced and some of their aspects have been visualized [23,24,25,26,27,28,29]. For instance, the author in [25], has considered the quantum mutual entropy of a single four-level atom strongly coupled to a cavity field and driven by a laser field. In addition Abdel-Aty et al. in [26] have explored an intensity-dependent coupling regime that consists of a λ-type four-level atom interacting with a single-mode quantized field . Some physical properties of the atom-field entangled have been investigated, as well. More recently, we have proposed new features of a non-linear time-dependent two two-level atoms in [30,31].

A lot of researches focus on studying five-level atomic system in different areas of quantum optics. The optical switching by controlling the double dark resonance in an N-tripod five-level atom is explored in [32]. Colossal Kerr nonlinearity based on electromagnetically-induced transparency in a five-level double-ladder atomic system has been studied in [33]. The dynamical properties of a five-level fan type atom with a triple ground state interacting with one-mode electromagnetic cavity field in the presence of Kerr-like medium have been analyzed in [34].

In the recent years, much attention has been given to the properties of the three- and four-level atomic systems when time-dependent coupling with the field is considered [35,36,37,38,39]. More recently, the entanglement dynamics of three and four level atomic system under Stark effect and Kerr-like medium have been investigated in [40].

In this paper, we introduce a model that consists of a three-level Λ-type atom interacting with two-mode deformed bosonic field by considering the coupling parameter to be time-dependent. The present paper is arranged as follows: In Sec. 2, we introduce the model and its solution under certain approximation similar to that of the Rotating-Wave Approximation (RWA) at any time $t > 0$, In Sec. 3, we investigate the atomic inversion and the dynamical properties for different regimes. Numerical results for the purity and Mandel Q-parameter are presented in Sec. 4 and Sec. 5, respectively. Finally, The main results and conclusion are presented in Sec. 6.

2 Physical model

The considered model is a time-dependent regime consisting of a moving three-level (Λ-type) atom with the energy levels $\omega_1 > \omega_2 > \omega_3$, and interacting with a two-mode deformed bosonic field of frequency Ω_j in an optical cavity surrounded by Kerr nonlinearity in the presence of detuning parameters. The transitions $|1\rangle \longleftrightarrow |2\rangle$, and $|1\rangle \longleftrightarrow |3\rangle$ are allowed while the transition $|2\rangle \longleftrightarrow |3\rangle$ is forbidden as shown in Fig. 1. The interaction Hamiltonian in the Rotating-Wave Approximation (RWA) of the introduced physical system ($\hbar = 1$):

$$\begin{aligned}\hat{H}_I = {} & f_1(t)(\hat{a}_q e^{i\Delta_1 t}\hat{\sigma}_{12} + \hat{a}_q^{\dagger} e^{-i\Delta_1 t}\hat{\sigma}_{21}) + f_2(t)(\hat{b}_q\, e^{i\Delta_2 t}\hat{\sigma}_{13} + \hat{b}_q^{\dagger}\, e^{-i\Delta_2 t}\hat{\sigma}_{31}) \\ & + \chi_1 \hat{a}_q^{\dagger 2}\hat{a}_q^{2} + \chi_2\, b_q^{\dagger 2}\, \hat{b}_q^{2}. \end{aligned} \quad (1)$$

In which, the operators $\hat{\sigma}_{ii} = |i\rangle\langle j|$ are the atomic raising or lowering operator, the operators $\hat{a}_q^{\dagger}(\hat{a}_q)$ are the field creation and annihilation operators of the deformed field mode, respectively, $f_i(t),\ i = 1,2$, are the atom-field coupling parameters, and χ_j is the third-order nonlinearity of the Kerr-medium. The detuning parameters, Δ_1 and Δ_2, are given by

$$\begin{aligned}\Delta_1 &= \omega_1 - \omega_2 - \Omega_1, \\ \Delta_2 &= \omega_1 - \omega_3 - \Omega_2, \end{aligned} \quad (2)$$

$\Omega_j,\ j = 1,2$ is the frequency of the field mode. We consider $f_1(t) = f_2(t) = f(t) = \lambda_j \cos(\mu t) = \frac{\lambda_j}{2}(e^{i\mu t} + e^{-i\mu t})$, where $\lambda_j,\ \mu,\ j = 1,2$ are arbitrary constants. There are two exponential terms in the Hamiltonian: One is rapidly oscillating terms $e^{\pm i(\Delta_j+\mu)t}$ and the other is slowly varying terms $e^{\pm i(\Delta_j-\mu)t}$. In this case, if we neglect the rapidly varying terms compared with the slowly varying terms, then the interaction Hamiltonian can be rewritten in the following manner

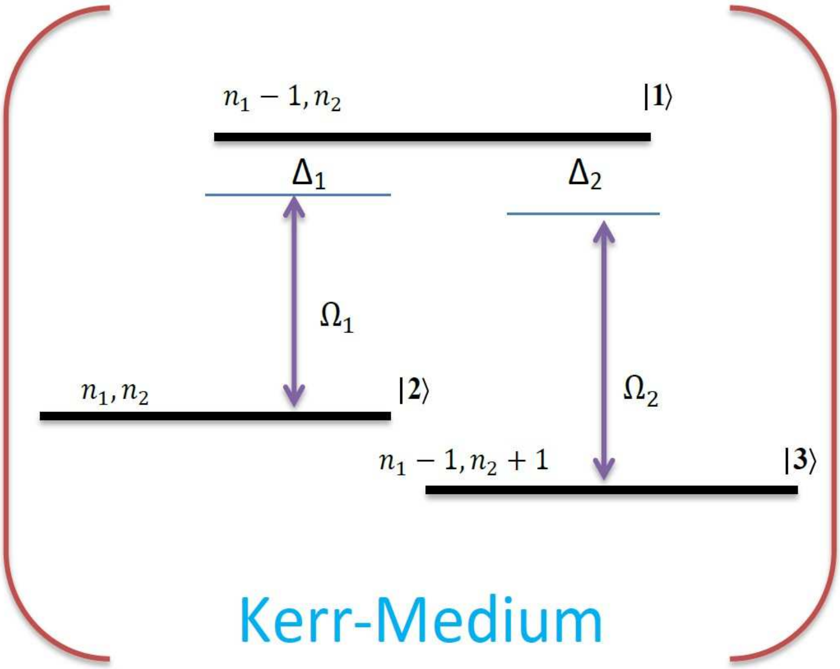

Fig. 1: Schematic diagram of a three-level Λ-type atom interacting with a two-mode deformed field.

$$\hat{H}_I = \frac{\lambda_1}{2}(\hat{a}_q e^{i\delta_1 t}\hat{\sigma}_{12} + \hat{a}_q^{\dagger} e^{-i\delta_1 t}\hat{\sigma}_{21}) + \frac{\lambda_2}{2}(\hat{b}_q e^{i\delta_2 t}\hat{\sigma}_{13} + \hat{b}_q^{\dagger} e^{-i\delta_2 t}\hat{\sigma}_{31}) + \chi_1 \hat{a}_q^{\dagger 2}\hat{a}_q^2 + \chi_2 \hat{b}_q^{\dagger 2}\hat{b}_q^2, \tag{3}$$

where

$$\delta_1 = \Delta_1 - \mu, \; \delta_2 = \Delta_2 - \mu. \tag{4}$$

We assume that the wave function of the atom-field at any time $t > 0$ can be expressed as

$$|\Psi(t)\rangle = \sum_{n_1,n_2=0}^{\infty} [G_1(n_1-1,n_2,t)\,|1,n_1-1,n_2\rangle + G_2(n_1,n_2,t)\,|2,n_1,n_2\rangle + G_3(n_1-1,n_2+1,t)\,|3,n_1-1,n_2+1\rangle]. \tag{5}$$

To reach this goal, suppose that the atom-field initial state is

$$|\Psi(0)\rangle = \sum_{n_1,n_2=0}^{\infty} [\vartheta_1\,|n_1-1,n_2\rangle + \vartheta_2\,|n_1,n_2\rangle + \vartheta_3\,|n_1-1,n_2+1\rangle] \otimes |\alpha_1,\alpha_2\rangle_q, \tag{6}$$

where

$$\vartheta_1 = \cos(\theta)\cos(\phi), \; \vartheta_2 = \cos(\theta)\sin(\phi), \; \vartheta_3 = \sin(\theta). \tag{7}$$

The deformed bosonic coherent states $|\alpha_1\rangle_q$, $|\alpha_2\rangle_q$ are coherent states that are constructed using a formally analogous scheme as the one allowing the construction of the Glauber coherent states starting from the Heisenberg-Weyl algebra. These states are defined as the eigenstate of the annihilation operators of a q-deformed bosonic field $\hat{a}_q$ and $\hat{b}_q$

$$\begin{aligned}|\alpha_1\rangle_q &= Q_1|n_1\rangle,\ Q_1 = N_1\sum_{n_1=0}^{\infty}\frac{\alpha_1^{n_1}}{\sqrt{[n_1]_q!}},\ N_1=[\exp_q[\alpha_1^2]]^{-\frac{1}{2}},\\ |\alpha_2\rangle_q &= Q_2|n_2\rangle,\ Q_2 = N_2\sum_{n_2=0}^{\infty}\frac{\alpha_2^{n_2}}{\sqrt{[n_2]_q!}},\ N_2=[\exp_q[\alpha_2^2]]^{-\frac{1}{2}},\end{aligned} \tag{8}$$

where, we have considered $q\in\mathbb{R}$, and the deformed $\exp_q$ is defined as

$$\exp_q[y]=\sum_{n=0}^{\infty}\frac{y^n}{[n]_q!}. \tag{9}$$

The function $\exp_q$ is a deformation version of the usual exponential function. They become coincident when q is the identity. Notice that $\exp_q[x]\exp_q[y]\neq\exp_q[x+y]$ and $[\exp_q[y]]^b\neq\exp_q[by]$, i.e. we have a non-extensive exponential which can be found in many physical problems [41]. In this paper we assume that the deformation can be achieved using the following function [42]:

$$[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}}, \tag{10}$$

where q is the deformation parameter. By taking the classical $q\to 1$ limit, the deformed algebra reduces to the classical SU(2) algebra. The q-factorial is defined as

$$[n]_q! = [n]_q[n-1]_q......[1]_q;\ [0]_q! = 1. \tag{11}$$

Now, by substituting $|\Psi(t)\rangle$ from Eq. (5) and $\hat{H}_I$ from Eq. (3) in the time-dependent Schrődinger equation $i\frac{\partial}{\partial t}|\Psi(t)\rangle=\hat{H}_I$ $|\Psi(t)\rangle$, one may arrive at the following coupled differential equations for the atomic probability amplitudes

$$i\frac{d}{dt}\begin{pmatrix}G_1\\G_2\\G_3\end{pmatrix}=\begin{pmatrix}\bar{\alpha}_1 & v_1e^{i\delta_1 t} & v_2e^{i\delta_2 t}\\ v_1e^{-i\delta_1 t} & \bar{\alpha}_2 & 0\\ v_2e^{-i\delta_2 t} & 0 & \bar{\alpha}_3\end{pmatrix}\begin{pmatrix}G_1\\G_2\\G_3\end{pmatrix}. \tag{12}$$

Where,

$$\begin{aligned}\bar{\alpha}_1 &= \chi_1[n_1-1]_q[n_1-2]_q+\chi_2[n_2]_q[n_2-1]_q,\\ \bar{\alpha}_2 &= \chi_1[n_1]_q[n_1-1]_q+\chi_2[n_2]_q[n_2-1]_q,\\ \bar{\alpha}_3 &= \chi_1[n_1-1]_q[n_1-2]_q+\chi_2[n_2]_q[n_2+1]_q,\\ v_1 &= \frac{\lambda_1}{2}\sqrt{[n_1]_q},\ v_2=\frac{\lambda_2}{2}\sqrt{[n_2+1]_q}.\end{aligned} \tag{13}$$

The solution of Eqs. (12) is given, as follows:

$$\begin{aligned}G_1 &= -\frac{1}{v_1}\sum_{j=1}^{3}B_j(\bar{\alpha}_2+\xi_j)e^{i(\xi_j+\delta_1)t},\\ G_2 &= \sum_{j=1}^{3}B_je^{i\xi_j t},\\ G_3 &= \frac{1}{v_1v_2}\sum_{j=1}^{3}B_j[(\delta_1+\bar{\alpha}_1+\xi_j)(\bar{\alpha}_2+\xi_j)-v_1^2]e^{i(\xi_j+\delta_1-\delta_2)t}.\end{aligned} \tag{14}$$

By applying these initial conditions for atom as well as field and using (14), the B_j coefficients are read as

$$B_j=\frac{v_1v_2\vartheta_3-(\Gamma_2-\xi_1\xi_2)\vartheta_2+(\Gamma_3+\xi_1+\xi_2)(v_1\vartheta_1+\bar{\alpha}_2\vartheta_2)}{\xi_{jk}\ \xi_{jl}},\ j\neq k=1,\ 2, \tag{15}$$

where $\theta_1 = Q_1Q_2\vartheta_1,\ \theta_2 = Q_1Q_2\vartheta_2,\ \theta_3 = Q_1Q_2\vartheta_3$ and $\xi_{jk} = \xi_j - \xi_k, \xi_j,\ j = 1,\ 2$ are the roots of the following third-order algebraic equation

$$\xi^3 + h_1\xi^2 + h_2\xi + h_3 = 0, \tag{16}$$

where

$$h_1 = \Gamma_1 + \Gamma_3,\quad h_2 = \Gamma_2 + \Gamma_1\Gamma_3 - V_2^2, \tag{17}$$
$$h_3 = \Gamma_1\Gamma_2 - \bar{\alpha}_2V_2^2,\quad \Gamma_1 = \bar{\alpha}_3 + \delta_1 - \delta_2, \tag{18}$$
$$\Gamma_2 = \bar{\alpha}_2(\bar{\alpha}_1 + \delta_1) - V_1^2,\ \Gamma_3 = \delta_1 + \bar{\alpha}_1 + \bar{\alpha}_2.$$

The three roots of third-order Eq. (16) are given in the following form [43]

$$\xi_m = -\frac{1}{3}h_1 + \frac{2}{3}\sqrt{h_1^2 - 3h_2}\cos(\Phi + \frac{2}{3}(m-1)\pi),\ m = 1,2,3,$$
$$\Phi = \frac{1}{3}\arccos[\frac{9h_1h_2 - 2h_1^3 - 27h_3}{2(h_1^2 - 3h_2)^{2/3}}]. \tag{19}$$

At any time $t > 0$ the reduced density matrix of the atom is given by:

$$\hat{\rho}_A(t) = Tr_F(|\Psi(t)\rangle\langle\psi(t)|) = \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) & \rho_{13}(t) \\ \rho_{21}(t) & \rho_{22}(t) & \rho_{23}(t) \\ \rho_{31}(t) & \rho_{32}(t) & \rho_{33}(t) \end{pmatrix}, \tag{20}$$

where

$$\rho_{11}(t) = \sum_{n_1,n_2=0}^{\infty} G_1(n_1-1,n_2,t)G_1^*(n_1-1,n_2,t),$$
$$\rho_{22}(t) = \sum_{n_1,n_2=0}^{\infty} G_2(n_1,n_2,t)G_2^*(n_1,n_2,t),$$
$$\rho_{33}(t) = \sum_{n_1,n_2=0}^{\infty} G_3(n_1-1,n_2+1,t)G_3^*(n_1-1,n_2+1,t), \ldots..,$$
$$\rho_{il}(t) = \rho_{li}^*(t). \tag{21}$$

The reduced density operator of the field $\hat{\rho}_F(t)$ is given by

$$\hat{\rho}_F(t) = Tr_A(|\Psi(t)\rangle\langle\psi(t)|). \tag{22}$$

In the next sections, for simplicity, we consider that the constants $\lambda_i = \lambda$ have been taken to be real and the interaction time is the scaled time $\tau = \lambda t$.

3 Atomic Population Inversion

In fact, we can get information about the behavior of the atom-field interaction through the collapse and revival phenomenon. Thus, we shall study the dynamics of an important quantity, namely atomic population inversion. The atomic population inversion is defined as the difference between the exited state $|1\rangle$ and the sum of populations of the state $|2\rangle$ and state $|3\rangle$ which may be written, as follows [2,16,44]:

$$W(t) = \rho_{11}(t) - (\rho_{22}(t) + \rho_{33}(t)). \tag{23}$$

Now, for different values of the q-deformation parameter, we shall study the behavior of the atomic population inversion in the time-dependent/independent case for the classical (undeformed) $q \to 1$ limit and $q = 2$. This will be done

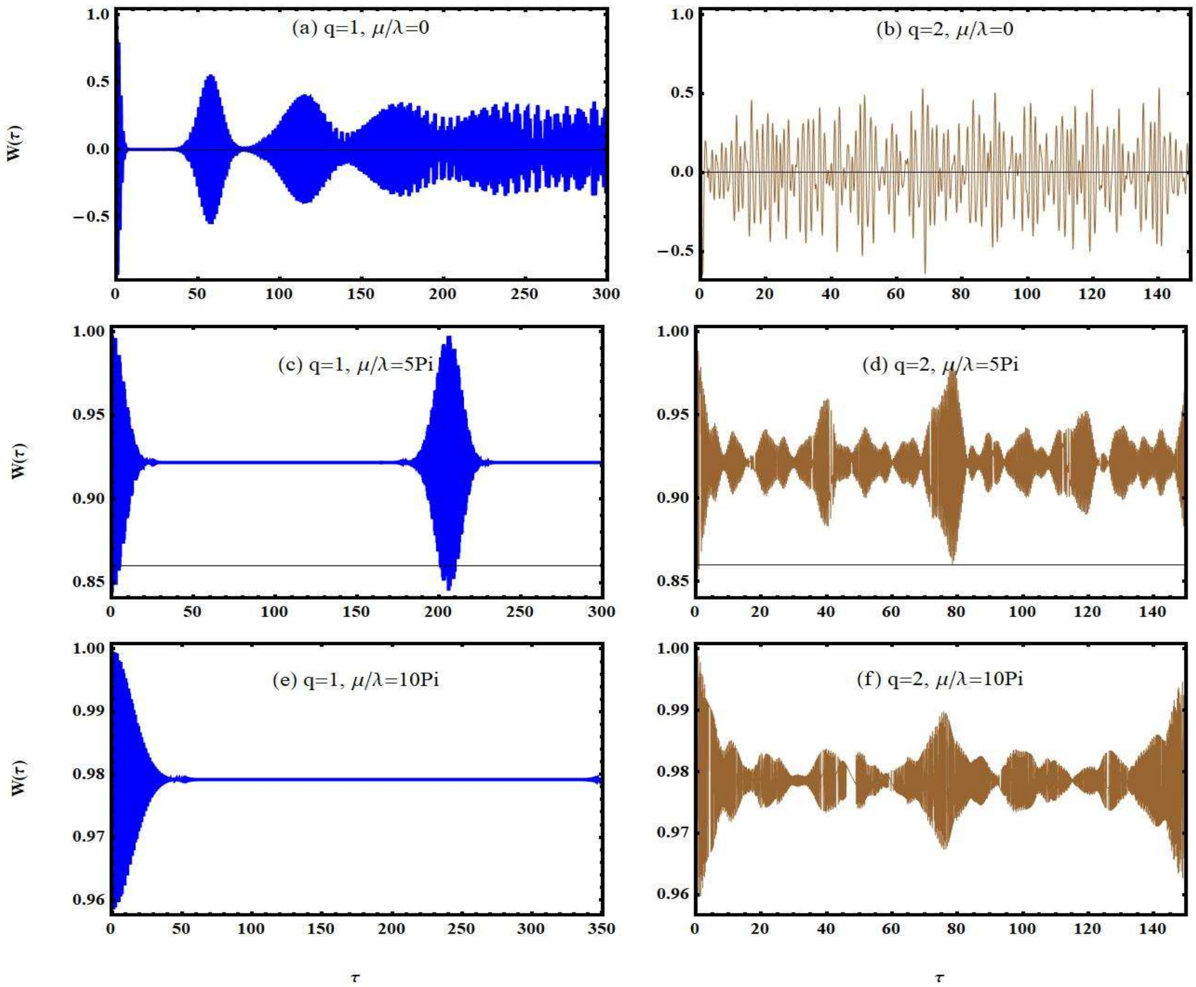

Fig. 2: Evolution of the atomic population inversion $W(\tau)$ for a three-level atom interacting with a two-mode deformed field for $q = 1$ (left plot), and for $q = 2$ (right plot) for the parameters $\alpha_1 = \alpha_2 = \sqrt{10}$, $\theta = \phi = 0$, $\Delta_1 = \Delta_2 = 0$, $\chi_1 = \chi_2 = 0$ and for: (a,b) $\mu/\lambda = 0$, (c,d) $\mu/\lambda = 5Pi$, (e,f) $\mu/\lambda = 10Pi$.

on the basis of the previous calculations. We examine the influence of the time-dependent coupling parameter, detuning parameters, Kerr-medium on the behavior of the atomic population inversion. The temporal evolution of the atomic inversion has been given in Figs. 2-4 for $\alpha_1 = \alpha_2 = \sqrt{10}$. The left plot for $q \to 1$ and the right plot for $q = 2$. In Fig. 2(a) and Fig. 2(b), we have considered the time-dependent coupling parameter $\mu/\lambda = 0$ in the absence of the detuning parameters and Kerr-medium ($\Delta_1/\lambda = \Delta_2/\lambda = \chi_1 = \chi_2 = 0$). The behavior of the atomic population inversion exhibits the collapse and revival phenomena. The number of oscillations in Fig. 2(b) is greater than that in Fig. 2(a). In Fig. 2(c) and Fig. 2(d), where the value of the time-dependent coupling parameter $\mu/\lambda = 5Pi$, the behavior of the atomic population inversion in Fig. 2(c) changes compared with Fig. 2(a).

The mean value of oscillations is shifted upward and the collapse interval is elongated, so we can consider the time-dependent coupling parameter as a quantum control parameter. When $\mu/\lambda = 10Pi$, the mean value of the collapses and the revivals is shifted upward compared with the previous cases (see Fig. 2(e) and Fig. 2(f)). Also, in Fig. 2(e), the behavior of the atomic population inversion has started only with a period of revival with a big amplitude followed by a long time-interval of collapse compared with the previous cases. The effect of the detuning parameters on the atomic population

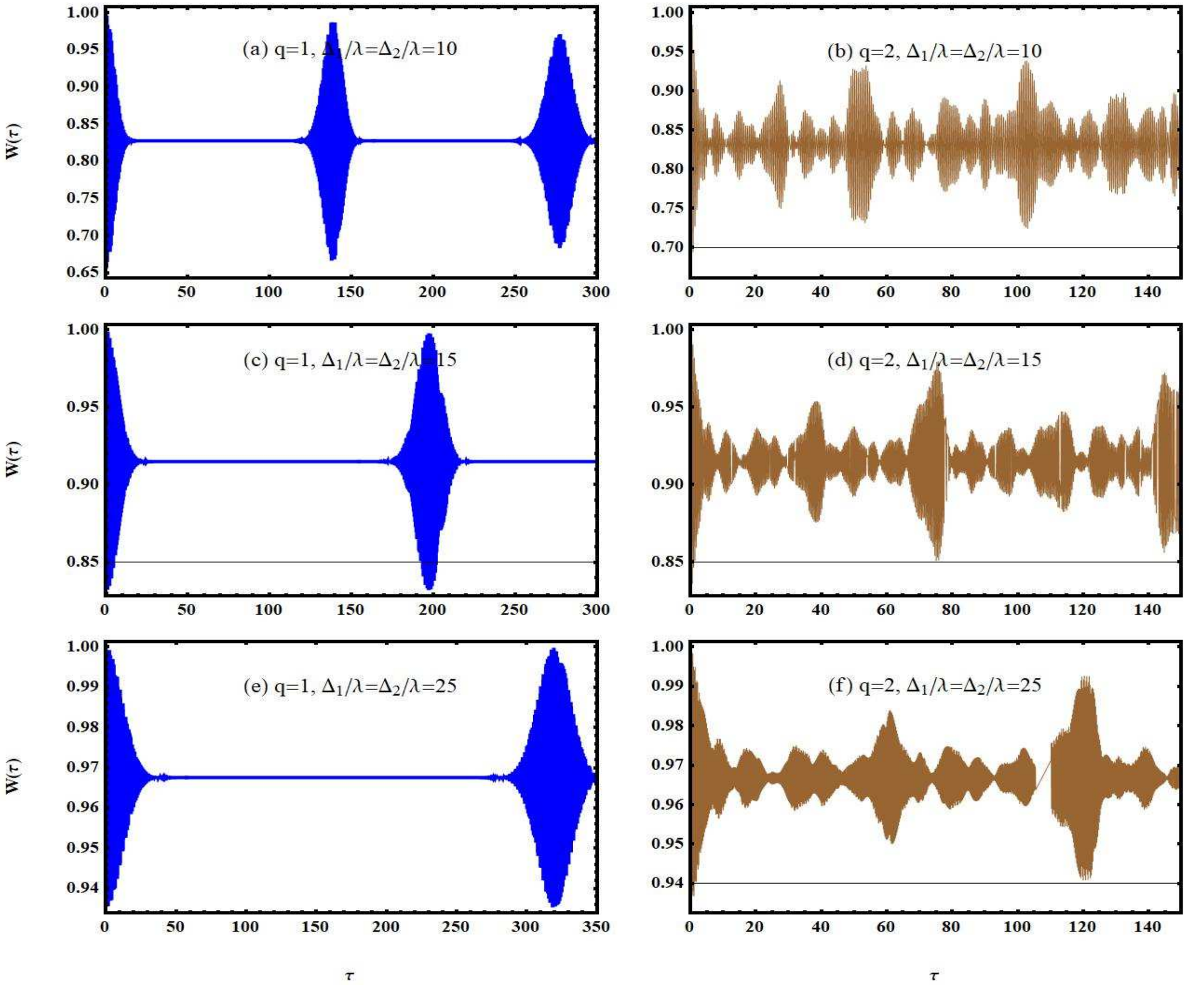

Fig. 3: Dynamics of the atomic population inversion $W(\tau)$ with the same conditions as stated in Fig. 2 but for $\mu/\lambda = 0$, and for: (a,b) $\Delta_1/\lambda = \Delta_2/\lambda = 10$, (c,d) $\Delta_1/\lambda = \Delta_2/\lambda = 15$, (e,f) $\Delta_1/\lambda = \Delta_2/\lambda = 25$.

inversion in the absence of the time-dependent coupling parameter ($\mu/\lambda = 0$) and in the absence of Kerr-medium appears in Fig. 3. In Fig. 3(a) and Fig. 3(b), when $\Delta_1/\lambda = \Delta_2/\lambda = 10$, the mean value of oscillations is shifted upward compared with that in Fig. 2(a) and Fig. 2(b), respectively. Also, in Fig. 3(a), we have two intervals of collapses. The behavior of the atomic population inversion in Fig. 3(c) and Fig. 3(d) is similar to that in Fig. 2(c) and Fig. 2(d) ($\Delta_1/\lambda = \Delta_3/\lambda = 15$). The behavior of the atomic population inversion in Fig. 3(e) and Fig. 3(f) completely changes compared with the previous cases. The mean value of oscillations is higher than the previous cases. This means that the energy increases in the atomic system. Also, the first time interval of collapse in Fig. 3(e) increases compared with Fig. 3(c), so we can consider the detuning parameter as a quantum control parameter. To discuss the influence of Kerr-medium on the atomic population inversion when $q \to 1$ in the absence of the time-dependent coupling parameter ($\mu/\lambda = 0$), and when $\Delta_1/\lambda = \Delta_2/\lambda = 0$, we have plotted Fig. 4. For a small value of Kerr-medium parameter ($\chi_1 = \chi_2 = 0.1$), the behavior of $W(\tau)$ in Fig. 4(a) is similar to that of $W(\tau)$ in Fig. 2(a) except for the mean value of oscillations is shifted upward and the amplitude of oscillations decreases. With the increase of the value of Kerr-medium, the behavior of $W(\tau)$ changes. The mean value of oscillations is shifted upward. For a great value of Kerr-medium we have a greatest positive mean value of oscillations and the maximum value of fluctuations approaches to one (see Fig. 4(f)).

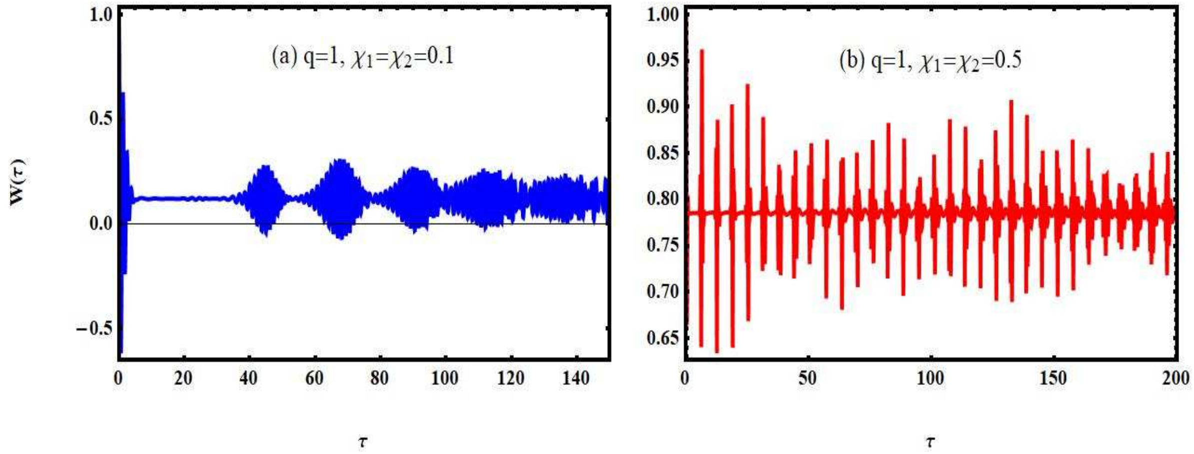

Fig. 4: Dynamics of the atomic population inversion $W(\tau)$ with the same conditions as stated in Fig. 2 but for $q = 1$, and for: (a) $\chi_1 = \chi_2 = 0.1$, (b) $\chi_1 = \chi_2 = 0.5$.

4 Purity

Several methods have been used to measure the degree of entanglement between the atom and the field. One of them is the purity that can be used as a tool that indicates the degree of the entanglement between the components of the system. So we devote this section to discuss the purity of the system under consideration. It can be determined from the relation [45,46]

$$P(t) = Tr\rho_F^2(t), \tag{24}$$

where $\hat{\rho}_F(t)$ is the field reduced density matrix. We note that $P(t) = 1$ corresponding to the pure state or completely disentangled atomic and field states. $P(t) = \frac{1}{d}$ corresponding to maximum entanglement degree, where d is the dimension of the density matrix. From Eq. (22), it is easy to show that

$$P(t) = 2(|\rho_{12}(t)|^2 + |\rho_{13}(t)|^2 + |\rho_{23}(t)|^2) + \rho_{11}^2(t) + \rho_{22}^2(t) + \rho_{33}^2(t). \tag{25}$$

Now, we address the evolution of the purity $P(\tau)$ versus the scaled time $\tau = \lambda t$ for the same parameters that we have used in Figs 2-4. An illustration of the time evolution of the purity for the classical $q \to 1$ limit (left plot), $q = 2$ (right plot) and for $\alpha_1 = \alpha_1 = \sqrt{10}$ is shown in Figs. 5-7. Fig. 5 shows that the collapse and revival time occurs in the same times in the corresponding Figs. 2 of atomic inversion. In Fig. 5(a) after a sufficient time, the purity becomes stable and less than 0.7. In Fig. 5(b), the mean value of oscillations is greater than 0.4 ($q = 2$). In Fig. 5 (c-f), the presence of the time-dependent coupling parameter pulled the purity up sharply. For the classical $q \to 1$ limit, we have only one peak for $\mu/\lambda = 5Pi$, which disappears as the parameter μ/λ increases in the considered time interval (see Fig. 5(c) and Fig. 5(e)). Also, the value of the purity increases as the parameter μ/λ increases.

To explore the effect of the detuning parameters Δ_1/λ, Δ_2/λ, in the absence of the time-dependent coupling parameter and Kerr-medium ($\mu/\lambda = \chi = 0$), we have plotted Fig. 6. In Fig 6(a), when $q \to 1$, $\Delta_1/\lambda = \Delta_2/\lambda = 10$, we have two sharp peaks and the value of the purity increases compared with Fig. 2(a). The mean value of oscillations in Fig. 6(b) is less than that in Fig. 6(a). When $q \to 1$, $\Delta_1/\lambda = \Delta_2/\lambda = 15$, the value of the purity is increased and we have only one sharp peak. When $q \to 1$, $\Delta_1/\lambda = \Delta_2/\lambda = 25$, we have only one sharp beak, but the collapse period is increased. To visualize the influence of Kerr-medium on the purity in the absence of both the time-dependent coupling parameter and detuning parameter, we have plotted Fig. 7. We notice that when $q \to 1$, $\chi_1/\lambda = \chi_2/\lambda = 0.1$, the nonlinear interaction of the Kerr-medium with the field modes decreases the value of purity and we have only four small peaks in the considered

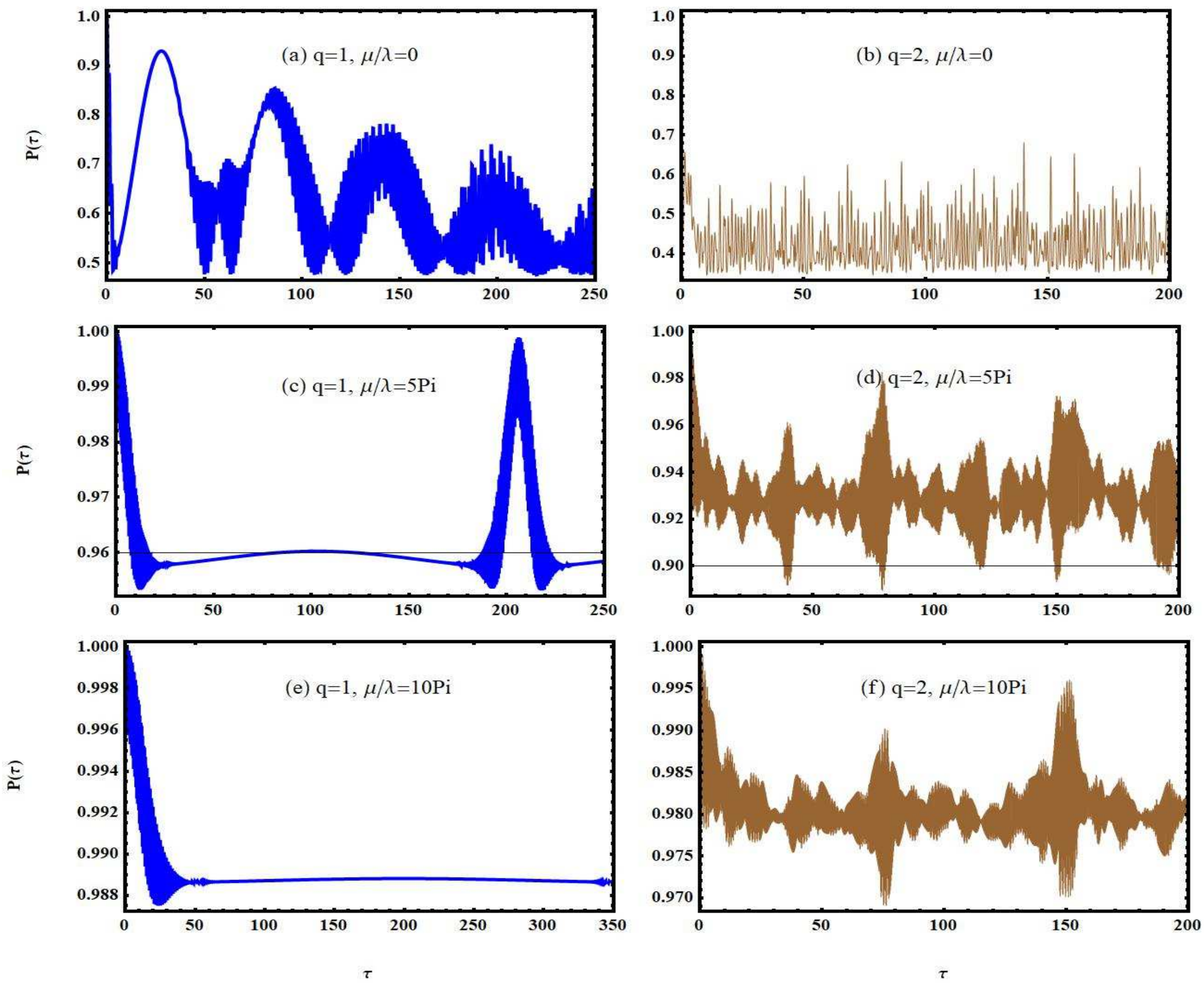

Fig. 5: Dynamics of the purity $P(\tau)$ with the same conditions as stated in Fig. 2.

time interval. With the increase of the nonlinear interaction of the Kerr-medium with field modes, the value of purity begins to increase. Taking $q \to 1,\ \chi_1/\lambda = \chi_2/\lambda = 0.5$, we see that purity increases and many oscillations appear.

5 Mandel Q-Parameter

A good measure of statistical and nonclassical properties of the field is the Mandel parameter. Which is a useful criterion to check the quantum statistics of any quantum state. It is defined as [47]

$$Q_i = \frac{\langle \hat{n}_i^2 \rangle - \langle \hat{n}_i \rangle^2 - \langle \hat{n}_i \rangle}{\langle \hat{n}_i \rangle},\ i = 1,2. \tag{26}$$

This quantity gets zero, positive and negative values when the state is respectively standard coherent state, classical state (super-Poissonian) and nonclassical state (sub-Poissonian). To proceed further, paying attention to the second mode of the

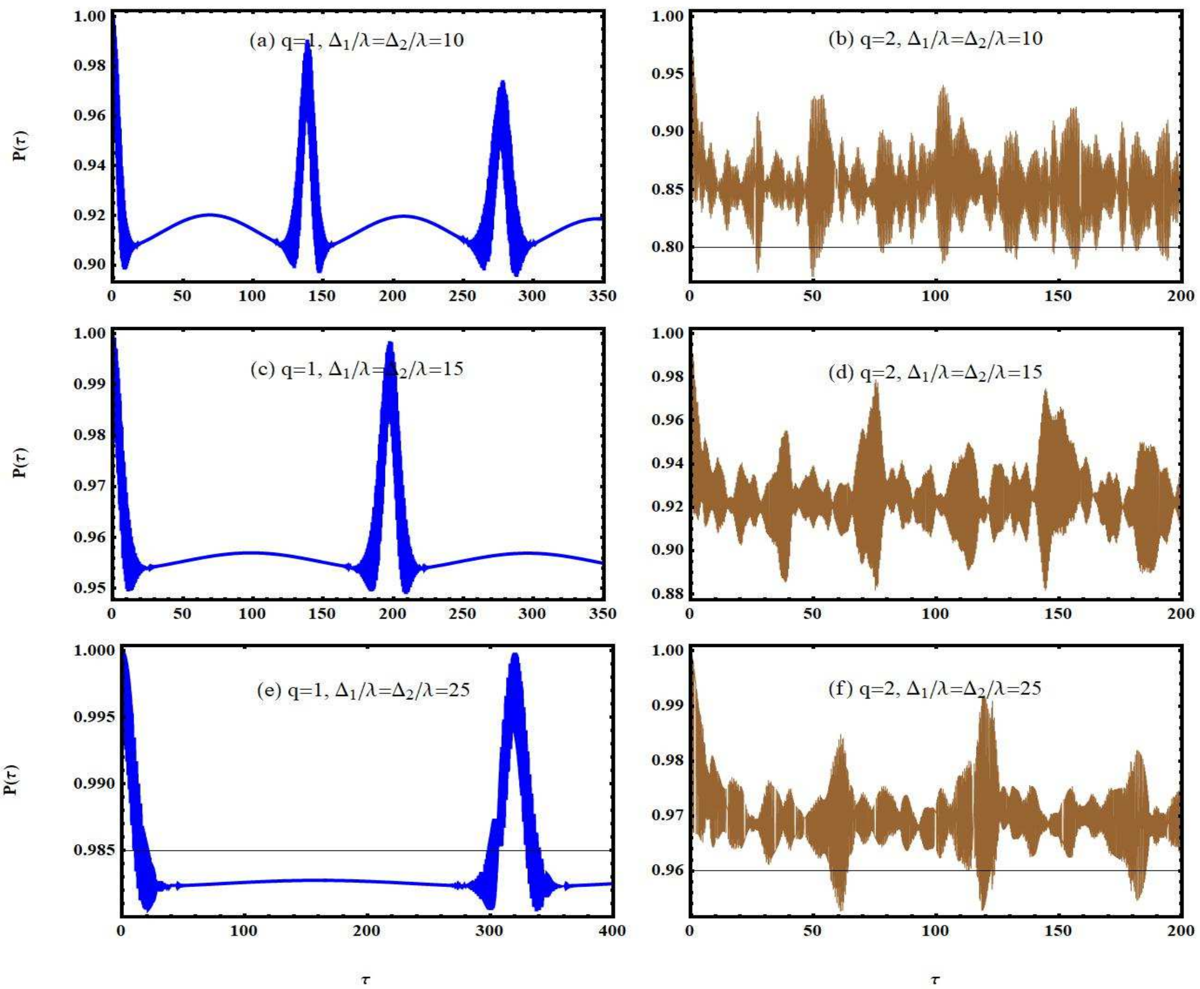

Fig. 6: Dynamics of the purity $P(\tau)$ with the same conditions as stated in Fig. 3.

field necessitates the following expectation values:

$$\langle \hat{b}_q^{\dagger}\hat{b}_q\rangle = \sum_{n_1,n_2}^{\infty} ([n_2]_q|G_1(n_1-1,n_2,t)|^2 + [n_2]_q|G_2(n_1-1,n_2,t)|^2 + [n_2+1]_q|G_3(n_1-1,n_2+1,t)|^2), \quad (27)$$

$$\langle (\hat{b}_q^{\dagger}\hat{b}_q)^2\rangle = \sum_{n_1,n_2}^{\infty} ([n_2]_q^2|G_1(n_1-1,n_2,t)|^2 + [n_2]_q^2|G_2(n_1-1,n_2,t)|^2 + [n_2+1]_q^2|G_3(n_1-1,n_2+1,t)|^2). \quad (28)$$

To examine the behavior of Mandel Q-parameter for the second field mode, we have plotted Figs. 8-10 using the same initial data as in the previous sections except for $\theta = 0,\ \phi = Pi/4$. In Fig. 8(a), when $q \to 1$ and all the other parameters are zero, the behavior of Mandel Q-parameter shows fluctuations between classical and nonclassical behavior and the mean value of oscillations is zero. However, when $q \to 1$ and taking in our consideration the excistance of the time-dependent coupling parameter, the number of oscillations decreases and the time interval of collapse increases as the parameter μ/λ increases. Also, the mean value of oscillations takes the nonclassical state (sub-Poissonian) as

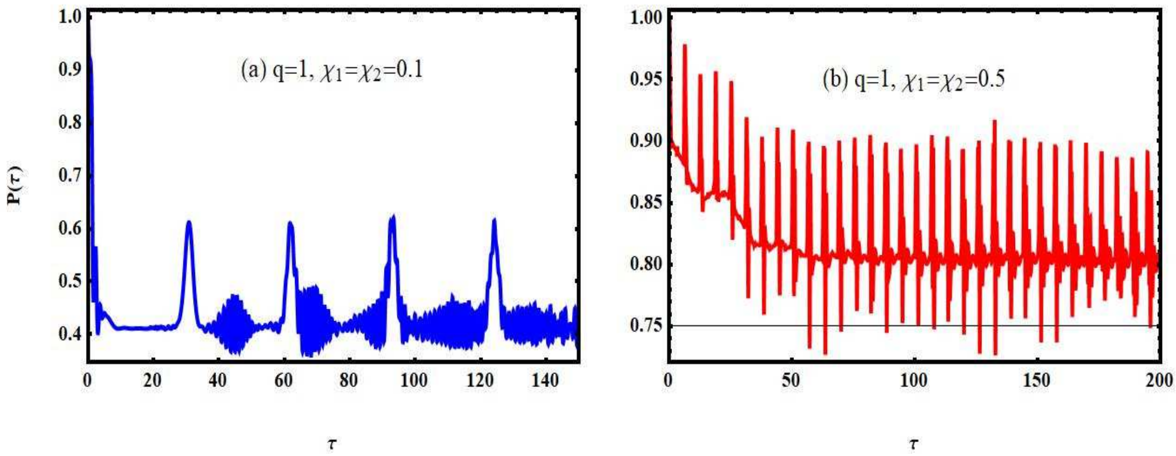

Fig. 7: Dynamics of the purity $P(\tau)$ with the same conditions as stated in Fig. 4.

the parameter μ/λ increases (see Fig. 8(c) and Fig. 8(e)). In Fig. 8(b, d, f), the behavior of Mandel parameter shows nonclassical behavior (sub-Poissonian).

To discuss the effect of the detuning parameters $\Delta_1/\lambda,\ \Delta_2/\lambda$, for $q \to 1$ and $q = 2$ in the absence of both of the time-dependent coupling parameter and Kerr-medium ($\mu/\lambda = \chi = 0$), we have plotted Fig. 9. When $q \to 1,\ \Delta_1/\lambda = \Delta_2/\lambda = 5$, the behavior of Mandel parameter shows fluctuations between between classical and nonclassical behavior and the mean value of oscillations is shifted towards the nonclassical area compared with that in Fig. 8(a) (see Fig. 9(a)). However, when $q \to 1,\ \Delta_1/\lambda = \Delta_2/\lambda = 10,\ 17$, the numbers of oscillations decrease compared with that in Fig. 9(a). Also, the mean value of oscillations is shifted towards the nonclassical state greater than that in Fig. 9(a) (see Fig. 9(c) and Fig. 9(e)). For $q = 2,\ \Delta_1/\lambda = \Delta_2/\lambda = 5,\ 10$, and 17, the behavior of Mandel parameter shows nonclassical behavior (sub-Poissonian) (see Figs. 9(b, d, f)). The influence of Kerr-like medium on the Mandel Q-parameter in the absence of both the time-dependent coupling parameter and detuning parameter has plotted Fig. 10. We observe that when $q \to 1$, $\chi_1/\lambda = \chi_2/\lambda = 0.1,\ 0.7$, the effect of Kerr-medium on the Mandel parameter behavior is in contrast with the effect of both of the time-dependent parameter and detuning parameters. By the increase of the value of Kerr-medium, Mandel parameter shows a classical behavior and the number of oscillations increases with the decrease in its amplitudes as the time develops (see Fig. 10(a) and Fig. 10(b)).

6 Conclusion

We have considered a three-level Λ-type atom interacting with a two-mode deformed field. The Kerr-medium and the detuning parameter were considered. Also, the coupling parameter was modulated to be time-dependent. Under an approximation similar to that of the Rotating-Wave Approximation (RWA), the exact expression of atom-field wave function was obtained. After obtaining the exact analytical form of the state vector of the whole system; the influence of the time-dependent coupling parameter, detuning parameter and Kerr nonlinearity on the atomic inversion, purity and Mandel Q-parameter for undeformed/deformed systems have been studied. The investigations have shown that the atomic inversion has the quantum collapse-revival behavior. The purity of a three-level atomic system has been introduced and its time evolution has been studied. This allowed to explore the degree of entanglement of the available systems for $q \to 1$ and $q = 2$. For the undeformed case ($q \to 1$) and the deformed case ($q = 2$), the increase of the time-dependent coupling parameter created a decrease in the degree of entanglement between the atom and the field. Also, the Kerr-medium resulted in a new behavior of Mandel Q-parameter for the considered system. Finally, we can

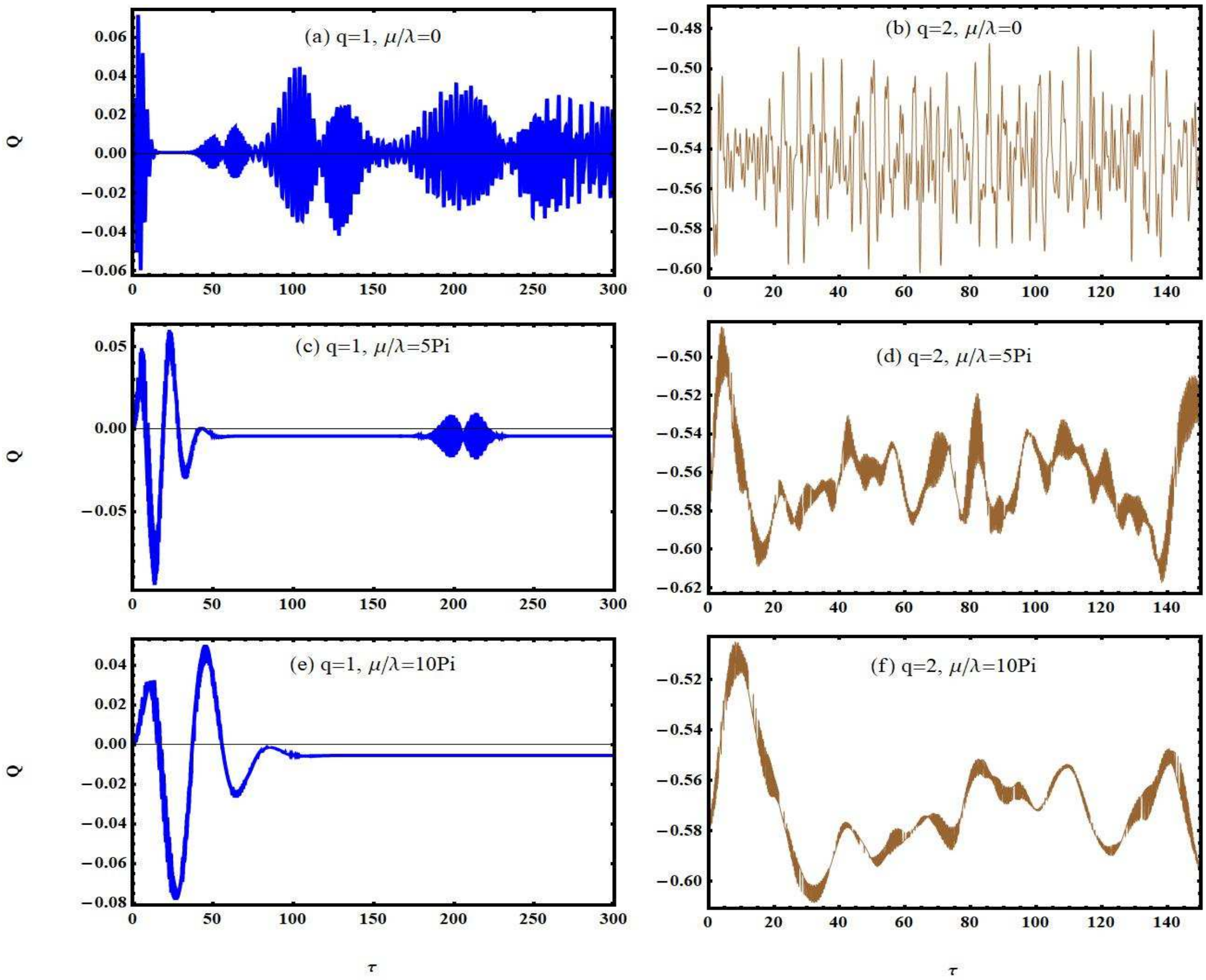

Fig. 8: Dynamics of the Mandel Q-parameter with the same conditions as stated in Fig. 2 but for $\theta = 0,\ \phi = Pi/4$.

deduce that the presence of the time-dependent coupling parameter, detuning parameter, Kerr nonlinearity led to a noticeable effect in the quantum entanglement and the quantum statistics of the considered systems. It is interesting to mention that one can study this system when both the field and the atom are initially prepared in other states.

Conflict of Interest

The authors declare that they have no conflict of interest.

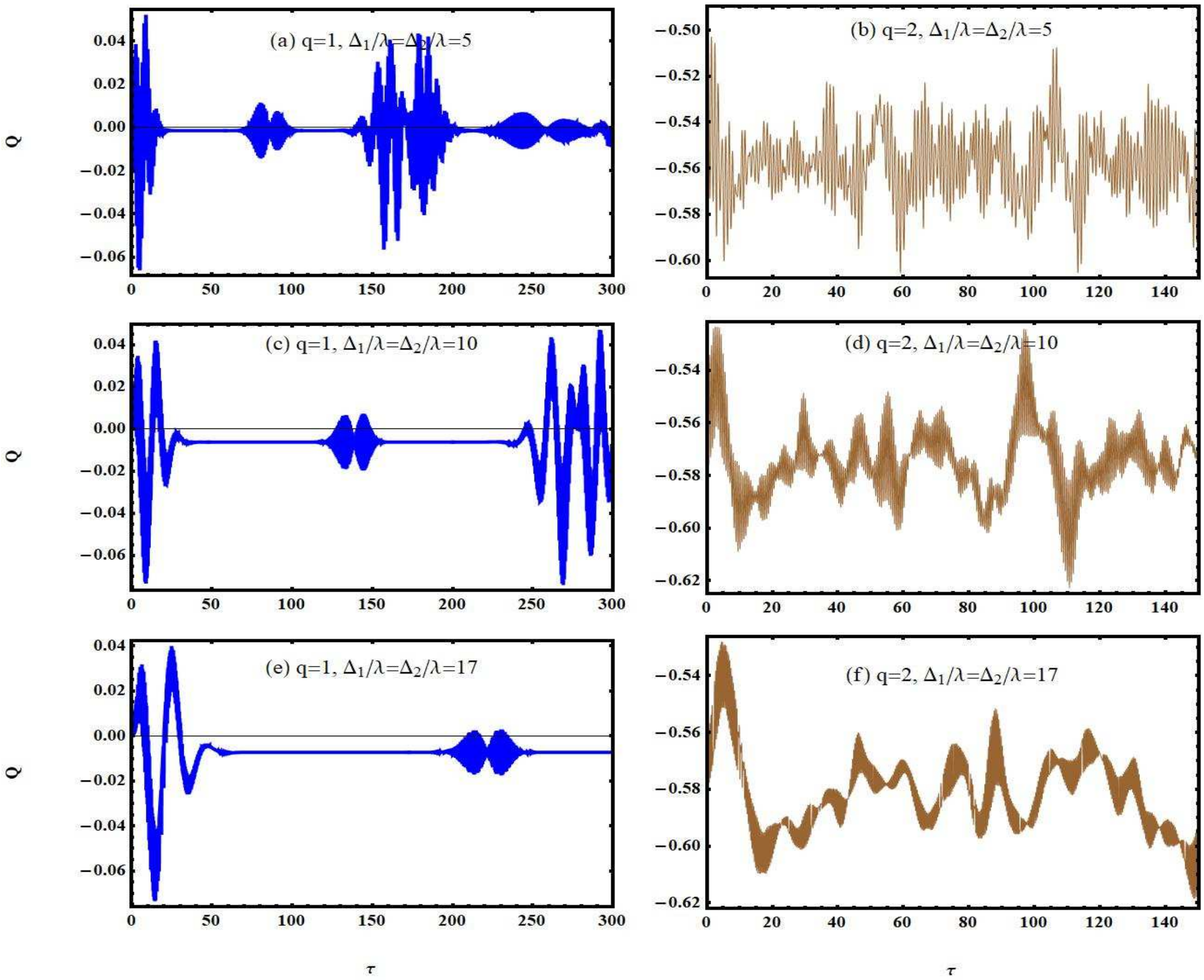

Fig. 9: Dynamics of the Mandel Q-parameter with the same conditions as stated in Fig. 3 but for $\theta = 0$, $\phi = Pi/4$, and for: (a,b) $\Delta_1/\lambda = \Delta_2/\lambda = 5$, (c,d) $\Delta_1/\lambda = \Delta_2/\lambda = 10$, (e,f) $\Delta_1/\lambda = \Delta_2/\lambda = 17$..

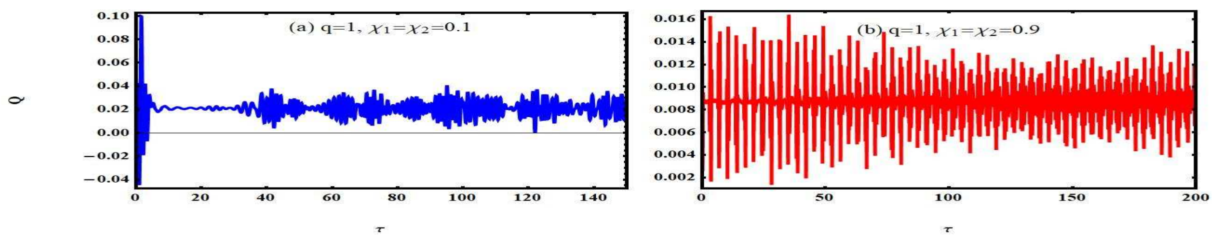

Fig. 10: Dynamics of the Mandel Q-parameter with the same conditions as stated in Fig. 4 but for $\theta = 0$, $\phi = Pi/4$, and for: (a) $\chi_1 = \chi_2 = 0.1$, (b) $\chi_1 = \chi_2 = 0.9$.

References

[1] E. T. Jaynes, F. W. Cummings, *Comparison of quantum and semiclassical radiation theories with application to the beam maser*, Proceeding of IEEE **51**, 89-109 (1963).

[2] H. I. Yoo, J. H. Eberly, *Dynamical Theory of an atom with two or three levels interacting with quantized cavity fields*, Physics Report: Review Section of Physics Letters **118**, 239-337 (1985).

[3] I. Jex, *Emission spectra of a two-level atom under the presence of another two-level atom*, Journal of Modern Optics **39**, 835-844 (1990).

[4] M. Abdel-Aty, *Influence of a Kerr-like medium on the evolution of field entropy and entanglement in a three-level atom*, J. Phys. B: Atomic, Molecular and Optical Physics **33**, 2665-2676 (2000).

[5] M. Abdel-Aty, *General formalism of interaction of a two-level atom with cavity field in arbitrary forms of nonlinearities*, Physica A **313**, 471-487 (2002).

[6] M. Abdel-Aty, G. M. Abd Al-Kader, A.-S. F. Obada, *Entropy and entanglement of an effective two-level atom interacting with two quantized field modes in squeezed displaced fock states*, Chaos, Solitons and Fractals **12**, 2455-2470 (2001).

[7] M. Abdel-Aty, *Engineering entanglement of a general three-level system interacting with a correlated two-mode nonlinear coherent state*, The European Physical Journal D **23**, 155-165 (2003).

[8] M. Abdel-Aty, *An investigation of entanglement and quasiprobability distribution in a generalized Jaynes-Cummings model*, Journal of Mathematical Physics **44**, 1457-1471 (2003).

[9] M. Abdel-Aty, *Quantum information entropy and multi-qubit entanglement*, Progress in Quantum Electronics **31**, 1-49 (2007).

[10] T. M. El-Shahat, S. Abdel-Khalek, M. Abdel-Aty, A.-S. F. Obada, *Aspects on entropy squeezing of a two-level atom in a squeezed vacuum*, Chaos, Solitons and Fractals **18**, 289-298 (2003).

[11] M. Abdel-Aty, M. S. Abdalla, A.-S. F. Obada, *Entropy and phase properties of isotropic coupled oscillators interacting with a single atom: one- and two-photon processes*, Journal of Optics B: Quantum and Semiclassical Optics **4**, S133-S141 (2002).

[12] M. Sebawe Abdalla, S. S. Hassan, M. Abdel-Aty, *Entropic uncertainty in the Jaynes-Cummings model in presence of a second harmonic generation*, Optics Communications **244**, 431-443 (2005).

[13] M. Abdel-Aty, M. S. Abdalla, A.-S. F. Obada, *Uncertainty relation and information entropy of a time-dependent bimodal two-level system*, Journal of Physics B: Atomic, Molecular and Optical Physics **35**, 4773-4786 (2002).

[14] A.-S. F. Obada, D. A. M. Abo-Kahla, Metwally, M. Abdel-Aty, *The quantum computational speed of a single Cooper-pair box*, Physica E: Low-Dimensional Systems and Nanostructures **43**, 1792-1797 (2011).

[15] S. Korashy, A. S. Abdel-Rady, Abdel-Nasser A. Osman, *Influence of Stark shift and Kerr-medium on the interaction of a two-level atom with two quantized field modes: a time-dependent system*, Quantum Information Review **5**, 9-14 (2017).

[16] M. Abdel-Aty, *Quantum phase entropy and entanglement of a multiphoton three-level atom near the edge of a photonic band gap*, Laser Physics **16**, 1381 (2006).

[17] A. Joshi, M. Xiao, *On the dynamical evolution of a three-level atom with atomic motion in a lossless cavity*, Optics Communications **232**, 273-287 (2004).

[18] M. Abdel-Aty, *Information entropy of a time-dependent three-level trapped ion interacting with a laser field*, Journal of Physics A: Mathematical and General **38**, 8589-8602 (2005).

[19] C. Huang , L. Tang, F. Kong, J. Fang, M. Zhou, *Entropy evolution of field interacting with V-type three-level atom via intensity-dependent coupling*, Physica A **368**, 25-30 (2006).

[20] M. Abdel-Aty, *Quantum field entropy and entanglement of a three-level atom two-mode system with an arbitrary nonlinear medium*, Journal of Modern Optics **50**, 161-177 (2003).

[21] S. Abdel-Khalek, Y. S. El-Saman, M. Abdel-Aty, *Geometric phase of a moving three-level atom*, Optics Communications **283**, 1826-1831 (2010).

[22] M. J. Faghihi and M. K. Tavassoly, *Quantum entanglement and position-momentum entropic squeezing of a moving Lambda-type three-level atom interacting with a single-mode quantized field with intensity-dependent coupling*, Journal of Physics B: Atomic, Molecular and Optical Physics. **46**, 145506 (2013).

[23] M. P. Sharma, J. A. Roversi, *Dynamics of population of a four-level atom due to one- and three-photon processes*, Physical Review A **29**, 3264-3270 (1984).

[24] M. Abdel-Aty, *Quantum information entropy and multi-qubit entanglement*, Progress in Quantum Electronics **31**, 1-49 (2007).

[25] M. Abdel-Aty, *Quantum mutual entropy of a single four-level atom strongly coupled to a cavity field and driven by a laser field*, Optics Communications **275**, 129-134 (2007).

[26] L. E. Thabet, T. M. El-Shahat, M. Abdel-Aty, B. S. H. Rababh, *Dynamics of entanglement and non-classicality features of a single-mode nonlinear Jaynes-Cummings model*, Chaos, Solitons and Fractals **126**, 106-115 (2019).

[27] X. X. Li, X. M. Hu, X. X. Cheng, H. N. Cui, *Two-collective mode entanglement in a four-level atomic ensemble*, International Journal of Theoretical Physics **56**, 2774-2782 (2017).

[28] N. H. Abdel-Wahab, *The interaction between a four-level N-type atom and two-mode cavity field in the presence of a Kerr medium*, Journal of Physics B: Atomic, Molecular and Optical Physics **40**, 4223-4233 (2007).

[29] H. R. Baghshahi, M. K. Tavassoly, A. Behjat, *Entanglement of a damped non-degenerate $\diamondsuit$-type atom interacting nonlinearly with a single-mode cavity*, European Physical Journal Plus **131**, 80 (2016).

[30] M. Akremi, S. T. Korashy, T. M. El-Shahat, R. Nekhili, Inamuddin, M. R. Gorji, I. Khan, *New features of non-linear time-dependent two-level atoms*, Journal of the Taiwan Institute of Chemical Engineers **105**, 171-181 (2019).

[31] S. T. Korashy, T. M. El-Shahat, N. Habiballah, H. El-Sheikh, M. Abdel-Aty, *Dynamics of nonlinear time-dependent two two-level atoms in a two-mode cavity*, International Journal of Quantum Information **18**, 2050003 (2020).

[32] M. A. Anton, F. Carreno, O. G. Calderón, S. Melle, I. Gonzalo, *Optical switching by controlling the double-dark resonancesin a N-tripod five-level atom*, Optics Communications **281**, 6040-6048 (2008).

[33] H. R. Hamedi, A. H. Gharamaleki, M. Sahrai, *Colossal Kerr nonlinearity based on electromagnetically induced transparency*, Applied Optics **55**, 5892-5899 (2016).

[34] N. H. Abdel-Wahab, *A study of the interaction between a five-level atom and a single-mode cavity field: Fan-type*, Modern Physics Letters B **25**, 1971-1982 (2011).

[35] N. H. Abdel-Wahab, *A moving four-level N-type atom interacting with cavity fields*, Journal of Physics B: Atomic, Molecular and Optical Physics **41**, 105502 (2008).

[36] S. Abdel-Khalek, *dynamics of entanglement of a moving five-level atom interacting with cavity fields*, Journal of Russian Laser Research **32**, 86-93 (2011).

[37] S. Abdel-Khalek, A. A. Mousa, T. A. Nofal, *Dynamics properties of Wehrl information entropy and Wehrl phase distribution for a moving four-level atom*, Journal of Russian Laser Research **33**, 547-558 (2012).

[38] N. Zidan, S. Abdel-Khalek, M. Abdel-Aty, *Geometric phase and disentanglement of a moving four-level atom in the presence of nonlinear medium*, International Journal of Quantum Information **10**, 1250007 (2012).

[39] T. M. El-Shahat, L. E. Thabet, *On a moving four-level λ-type atom interacting with a single-mode quantized field in intensity-dependent coupling*, Applied Mathematics and Information Sciences **12**, 179-191 (2018).

[40] S. J. Anwar, M. Ramzan, M. Usman, M. K. Khan, *Entanglement dynamics of three and four level atomic system under Stark effect and Kerr-like medium*, Quantum Reports **1**, 23-36 (2019).

[41] C. Tsallis, *Possible generalization of Boltzmann-Gibbs statistics*, J. Stat .Phys. 52, 479-487(1988).

[42] K. Berrada, S. Abdel-Khalek, *Entanglement of atom-field interaction for nonlinear optical fields*, Physica E **44**, 628-634 (2011).

[43] L. N. Childs, *A Concrete Introduction to Higher Algebra*, 3rd edition (Springer, Berlin, 2008).

[44] Q. C. Zhou and S. N. Zhu, *Dynamics of a single-mode field interacting with Λ-type three-level atom*, Optics Communication **248**, 437-448 (2005).

[45] A.-S. F. Obada and A. A. Eied. *Entanglement in a system of an Ξ-type three level atom interacting with a non-correlated two-mode cavity field in the presence of nonlinearities*, Optics Communication **282**, 2184-2191 (2009).

[46] M. S. Abdalla, E. M. Khalil, and A.-S. F. Obada, *Statistical properties of a two-photon cavity mode in the presence of degenerate parametric amplifier*, Annals Physics **322**, 2554-2568 (2007).

[47] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics*, (Cambridge University Press, 1995).