Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/100301

Some Characterizations and Applications of a Size-Biased Weighted Distribution useful in Lifetime Modelling

M. Shakil[1,*], *M. Ahsanullah*[2] *and B. M. G. Kibria*[3]

[1]Miami Dade College, Hialeah, Florida, USA
[2]Rider University, Lawrenceville, New Jersey, USA
[3]Florida International University, Miami, Florida, USA

Received: 2 Aug. 2020, Revised: 25 Aug. 2020, Accepted: 15 Sept. 2020.
Published online: 1 Nov. 2021.

Abstract: In this paper, we theoretically establish some new characterization results by left and right truncated moments, order statistics and upper record values, along with applications to some real life-time data, of a size-biased (weighted) distribution.
Keywords: Characterizations, Эрланга distribution, Order statistics, Record values, Size-Biased (weighted) distribution, truncated moments.

1 Introduction

The problems of characterizations of both discrete and continuous probability distributions have been addressed by many authors. According to Nagaraja [2], "A characterization is a certain distributional or statistical property of a statistic or statistics that uniquely determines the associated stochastic model". On the other hand, Koudou and Ley [3] points out that, "In probability and statistics, a characterization theorem occurs when a given distribution is the only one which satisfies a certain property. Besides their evident mathematical interest per se, characterization theorems also deepen our understanding of the distributions under investigation and sometimes open unexpected paths to innovations which might have been uncovered otherwise". Thus, it is very important to characterize a particular probability distribution subject to certain conditions before applying it to some real world data. For various methods of characterizations of probability distributions, we refer to Koudou and Ley [3], Nagaraja [2], Galambos and Kotz [4], Kotz and Shanbhag [5], Ahsanullah and Shakil [6,7], Ahsanullah et al. [8,9,10], and Ahsanullah [11].

After a thorough and careful analysis of various scientific research papers on the Эрланга distribution, we observed that, for a non-negative continuous random variable X, the Эрланга distribution (or, simply, Э distribution) first appeared in Chinese language in a paper by Lv et al. [1]. According to Lv et al. [1], Эрланга distribution has also been cited in some Russian literature. Now, the Эрланга distribution is well-known as the Ailamujia distribution in the English literature. Since then, many authors have investigated the Эрланга distribution, its properties and applications in modeling real lifetime data as an alternative to the exponential distribution. Recently, various weighted model versions of the Эрланга distribution have been developed and studied by many authors by taking different choices of the weight functions.

Besides the applications of the probability distributions to real lifetime data sets, the importance of the characterizations of probability distributions cannot be underemphasized. It appears from literature that, despite several studies on the weighted Эрланга distribution, its properties and applications in modeling real lifetime data in recent years, no attention has been paid to its characterizations. In this paper, we theoretically establish some new characterizations of a weighted version of the Эрланга distribution, namely, a size-biased (weighted) Эрланга distribution introduced by Rather et al. [12]. Our proposed characterizations of the Rather et al. [12]'s size-biased (weighted) Эрланга distribution are based on left and right truncated moments, order statistics and record values. Moreover, the goodness of fit test of the Rather et al. [12]'s size-biased (weighted) Эрланга distribution model will be further examined by considering another real-world data

*Corresponding author e-mail: mshakil@mdc.edu

example, namely, the data on a sample of 40 male cholesterol levels (as reported in Triola [13]) to justify and show its significant applications in more real life practical situations.

The organization of the paper is as follows: In Section 2, we provide the proposed size-biased (weighted) Эрланга distribution, and its several distributional properties, such as the moments, Shannon entropy, reliability analysis and computations of percentage points. In Section 3, the estimation of the parameters and the application of the size-biased (weighted) Эрланга distribution to some real life-time data are presented. The proposed characterization results of the size-biased (weighted) Эрланга distribution are presented in Section 4. Some concluding remarks are outlined in Section 5.

2 Size-Biased (Weighted) Эрланга Distribution

In this section, we will first define the Эрланга distribution introduced by Lv et al. [1]. Then, we shall discuss the size-biased (weighted) Эрланга distribution introduced by Rather et al. [12]. For the sake of completeness, we will provide some of its essential basic distributional properties which will be used in establishing our characterization results by left and right truncated moments, order statistics and record values. For more properties of the size biased Эрланга distribution and its applications to real lifetime data, the interested readers are referred to the paper of Rather et al. [12], and references therein.

Эрланга Distribution: For a non-negative continuous random variable, X, the probability density function (pdf) and the cumulative distribution function (cdf) of the Эрланга distribution are respectively given by

$$f_X(x) = \begin{cases} 4\alpha^2 x e^{-2\alpha x}, & (0 \le x < \infty, \alpha > 0) \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

and

$$F(x) = 1 - (1 + 2\alpha x) e^{-2\alpha x}, \quad (0 \le x < \infty, \alpha > 0). \tag{2}$$

As pointed out by Jayakumar and Elangovan [14], the Эрланга (Ailamujia) distribution is a versatile distribution to model the repair time and guarantee the distribution delay time. For detailed properties and applications of the Эрланга distribution, we refer to Lv et al. [1], Pan et al. [15], Long [16] and Li [17]. For the weighted Эрланга distribution and its applications, we refer to Jan et al. [18] and references therein.

Size-Biased (Weighted) Эрланга Distribution: For a non-negative random variable, X, with the probability density function (pdf), $f_X(x)$, the probability density function of the size biased (weighted) random variable, X_{sb}, is defined as

$$f_{sb}(x) = \frac{f_X(x)}{E(X)}, \tag{3}$$

where $E(X) < \infty$ denotes the expected value of X. Recently, using the pdf (1) of the Эрланга distribution and the Eq. (3), a new size-biased (weighted) Эрланга distribution has been developed by Rather et al. [12], with the probability density function (pdf) given by

$$f_{sb}(x) = \begin{cases} 4\alpha^3 x^2 e^{-2\alpha x}, & (0 \le x < \infty, \alpha > 0) \\ 0, & \text{otherwise} \end{cases}. \tag{4}$$

Clearly Eq. (4) defines a pdf since $\int_0^\infty \left(4\alpha^3 x^2 e^{-2\alpha x}\right) dx = 1$, in view of the integral $\int_0^\infty t^{\nu-1} e^{-\mu t}\, dt = \frac{1}{\mu^\nu}\Gamma(\nu), [\mu > 0, \nu > 0]$, see Gradshteyn and Ryzhik [19], Eq. 3.381.4, p. 317, where $\Gamma(\ )$ denotes the gamma function. The cumulative distribution function (cdf) corresponding to the pdf (4) is given by

$$F_{sb}(x) = \frac{1}{2}\gamma(3, 2\alpha x), \quad (0 \le x < \infty, \alpha > 0), \tag{5}$$

where $\gamma(\nu, \mu z) = (\mu^{\nu}) \int_0^z t^{\nu-1} e^{-\mu t}\, dt$ in the Eq. (5) denotes the incomplete gamma function, see Gradshteyn and Ryzhik [19], Eq. 3.381.1, p. 317. Since $\frac{d(\gamma(\nu, z))}{dz} = z^{\nu-1} e^{-z}$, see Gradshteyn and Ryzhik [19], Eq. 8.356.4, p. 942, it is easily verified, by direct differentiation, that $\frac{dF_{sb}(x)}{dx} = 4\alpha^3 x^2 e^{-2\alpha x}$, which is the pdf (4) under question.

2.1 Distributional Properties of the Size Biased (Weighted) Эрланга Distribution

In what follows, we will provide some important distributional properties of the proposed size biased (weighted) Эрланга distribution for the sake of completeness and to use these in our proposed characterizations.

2.1.1 Graphs of the pdf and cdf

The possible shapes of the pdf (4) and cdf (5) of the size-biased (weighted) Эрланга distribution are given for some selected values of the parameter, $\alpha > 0$, in Figures 1 – 2, respectively. The effects of the parameter, $\alpha > 0$, can be easily seen from these graphs. For example, it is clear from these plots that the size-biased (weighted) Эрланга distribution is positively right skewed with longer and heavier right tails for the selected values of the parameter, $\alpha > 0$.

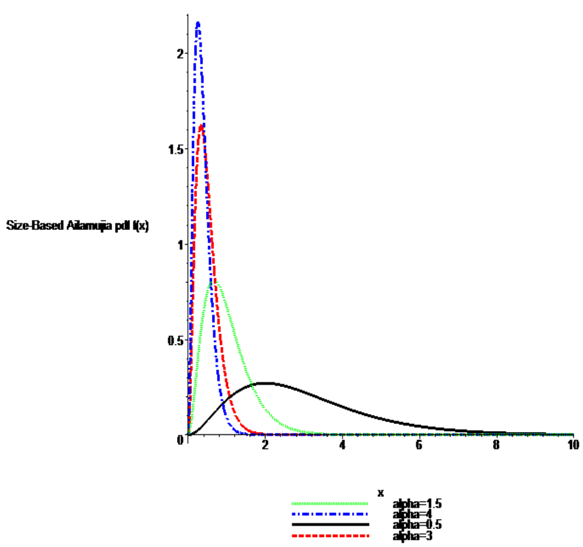

Fig. 1: Plots of the Size-Biased Эрланга Distribution pdf (4)

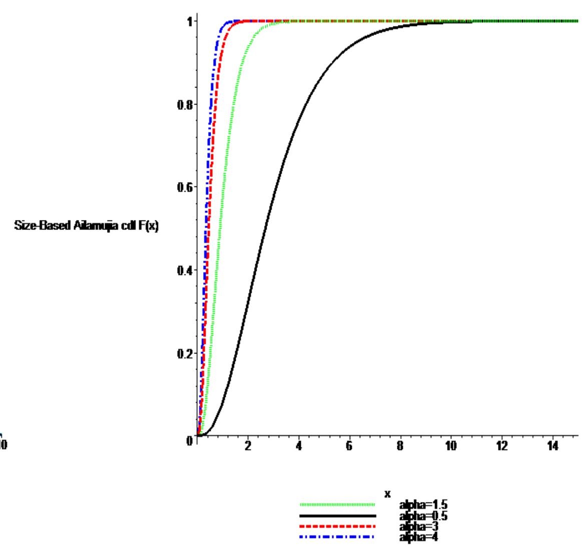

Fig. 2: Plots of the Size-Biased Эрланга Distribution cdf (5)

2.1.2 Moments: kth Moment of the Size-Biased (Weighted) Эрланга Distribution

For some integer $k > 0$, the kth moment of the above size-biased (weighted) Эрланга distribution is given by

$$E_{sb}(X^k) = \int_0^{\infty} x^k f_{sb}(x)\, dx = (4\alpha^3) \int_0^{\infty} x^{k+2} e^{-2\alpha x}\, dx \tag{6}$$

$$= \frac{\Gamma(k+3)}{2^{k+1}\alpha^k}, \ (\alpha > 0), \tag{7}$$

which easily follows by using the well-known integral $\int_0^\infty t^{\nu-1} e^{-\mu t}\, dt = \mu^{-\nu}\, \Gamma(\nu)$ in the Eq. (6), where $\Gamma(\nu)$ denotes the gamma function, see Gradshteyn and Ryzhik [19], Eq. 3.381.4, p. 317.

2.1.3 1st Moment of the Size Biased (Weighted -) Эрланга Distribution

Now, when $k=1$ in the Eq. (7), the 1st moment of the above size-biased (weighted) Эрланга distribution is given by

$$E(X)=\frac{3}{2\alpha},\ (\alpha>0). \tag{8}$$

It is obvious from Eq. (8) that $0<E(X)<\infty$, i.e. $E(X)$ exists and is finite for $\alpha>0$.

2.1.4 kth Incomplete Moment of the Size-Biased (Weighted) Эрланга Distribution

For some integer $k>0$, the kth incomplete moment of the above size-biased (weighted) Эрланга distribution is given by

$$I_x=\int_0^x u^k f_{sb}(u)\,du=\left(4\alpha^3\right)\int_0^x u^{k+2} e^{-2\alpha u}\,du\ . \tag{9}$$

Now, using $\int_0^z t^{\nu-1} e^{-\mu t}\, dt = \mu^{-\nu}\, \gamma(\nu, \mu z)$ in the Eq. (9), where $\gamma(.)$ denotes the incomplete gamma function, see Gradshteyn and Ryzhik [19], Eq. 3.381.1, p. 317, we have

$$I_x=\frac{\gamma(k+3, 2\alpha x)}{2^{k+1}\alpha^k},\ (\alpha>0), \tag{10}$$

$$=P_k(x),\ \text{say.} \tag{11}$$

2.1.5 1st Incomplete Moment of the Size-Biased (Weighted) Эрланга Distribution

When $k=1$ in the Eq. (10), the 1st incomplete moment of the above size-biased (weighted) Эрланга distribution is given by

$$P_1(x)=\frac{\gamma(4, 2\alpha x)}{4\alpha},\ (\alpha>0). \tag{12}$$

2.1.6 Shannon Entropy of the Size-Biased (Weighted) Эрланга Distribution

The Shannon entropy measure of a random variable X is a measure of variation of uncertainty and has been used in many fields such as physics, engineering and economics. According to Shannon [20], the entropy measure of a continuous real random variable X is defined as

$$H_X\left[f_X(X)\right] = E[-\ln(f_X(X)] = -\int_{-\infty}^{\infty} f_X(x)\ln\left[f_X(x)\right]dx.$$

Thus, in view of the integral $\int_0^{\infty} t^{\nu-1}e^{-\mu t}\ln(t)\,dt = \frac{1}{\mu^{\nu}}\Gamma(\nu)\{\psi(\nu)-\ln(\mu)\}, [\mu>0, \nu>0]$, see Gradshteyn and Ryzhik [19], Eq. 4.352.1, p. 576, the Shannon entropy of the size-biased (weighted) Эрланга distribution, with pdf (4), is easily given by

$$\begin{aligned} H_X\left[f_{sb}(X)\right] &= -\int_0^{\infty}\left[4\alpha^3 x^2 e^{-2\alpha x}\right]\left[\ln\left(4\alpha^3 x^2 e^{-2\alpha x}\right)\right]dx \\ &= (2\alpha)E(X) - \left(128\alpha^6\right)\left[\psi(3)-\ln(2\alpha)\right] - \ln\left(4\alpha^3\right), \end{aligned}$$

from which, by using the Eq. (8) for the 1st moment, $E(X) = \frac{3}{2\alpha}$, and simplifying, we have

$$\begin{aligned} H_X\left[f_{sb}(X)\right] &= 3 - \left(128\alpha^6\right)\left[\psi(3)-\ln(2\alpha)\right] - \ln\left(4\alpha^3\right) \\ &= \ln\left(\frac{e^3}{4\alpha^3}\right) - \left(128\alpha^6\right)\left[\psi(3)-\ln(2\alpha)\right], \end{aligned} \tag{13}$$

where $\psi(\ )$ denotes the digamma(psi) function.

2.1.7 Reliability Analysis

The reliability analysis is also important in modelling many phenomena in several fields of applied sciences. The hazard rate (or the failure rate) is defined for a non-repairable population as the instantaneous rate of failure for the survivors to time t during the next instant of time.

Survival Function, Hazard Rate Function, and Reversed Failure Rate Function: Using the pdf (4) and the cdf (5), the corresponding survival (or reliability) function, $S(x)$, hazard rate function (or failure rate function), $h(x)$, and reversed failure rate function, $\eta(x)$, of the size-biased (weighted) Эрланга distribution are, respectively, given by

$$\begin{aligned} S_{sb}(x) &= R_{sb}(x) = 1 - F_{sb}(x) \\ &= 1 - \frac{1}{2}\gamma(3, 2\alpha x), \quad (0<x<\infty, \alpha>0), \end{aligned} \tag{14}$$

$$h_{sb}(x) = \frac{f_{sb}(x)}{1-F_{sb}(x)} = \frac{4\alpha^3 x^2 e^{-2\alpha x}}{1-\frac{1}{2}\gamma(3, 2\alpha x)}, (0<x<\infty, \alpha>0), \tag{15}$$

and

$$\eta_{sb}(x) = \frac{f_{sb}(x)}{F_{sb}(x)}$$

$$= \frac{8\alpha^3 x^2 e^{-2\alpha x}}{\gamma(3, 2\alpha x)}, (0 < x < \infty, \alpha > 0).$$

(16)

The possible shapes of the hazard rate function (or the failure rate function) (15), $h_{sb}(x)$, of the size-biased (weighted) Эрланга distribution are given for some selected values of the parameter, $\alpha > 0$, in Figure 3 below. The effects of the parameter, $\alpha > 0$, can be easily seen from these graphs. For example, the increasing and upside-down bathtub shape behaviors of the hazard rate function (or the failure rate function), $h_{sb}(x)$, of the size-biased (weighted) Эрланга distribution are evident from Figure (3).

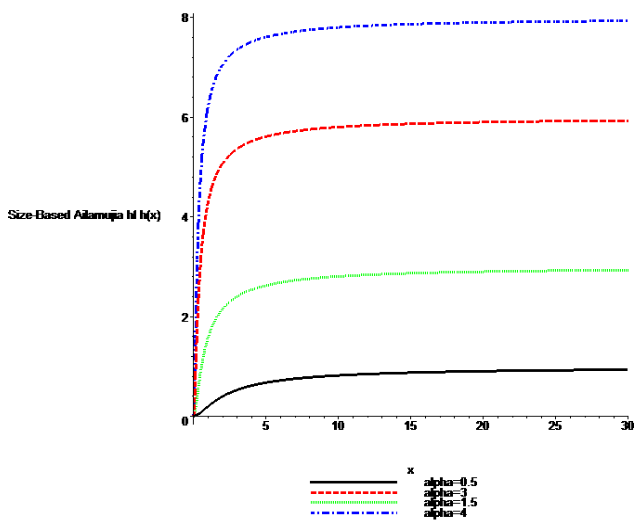

Fig. 3: Plots of the Size-Biased Эрланга Distribution hf (15).

Furthermore, differentiating the Eq. (15) with respect to x, we have

$$h_{sb}^{/}(x) = \frac{f_{sb}^{/}(x)}{f_{sb}(x)} h_{sb}(x) + \left[h_{sb}(x)\right]^2 \tag{17}$$

for $x > 0$, where $f_{sb}(x)$ and $h_{sb}(x)$ are given by Eqs. (4) and (15) respectively, and $f_{sb}^{/}(x)$ is obtained by differentiating the Eq. (4) with respect to x, i.e.

$$f_{sb}^{/}(x) = 8\alpha^3 x e^{(-2\alpha x)} [1 - \alpha x]. \tag{18}$$

To discuss the behavior of the failure rate function, $h_{sb}(x)$, let $h_{sb}^{/}(x) = 0$. We observe that the nonlinear equation $h_{sb}^{/}(x) = 0$ does not have a closed form solution, but could be solved numerically using some mathematical software such as Maple, or Mathematica, or R. It is obvious from Eqs. (17) and (18) that $h_{sb}^{/}(x)$ is positive provided $x < \frac{1}{\alpha}, \alpha > 0$. This shows that the size-biased (weighted) Эрланга distribution model has the increasing failure rate (IFR) property when $x < \frac{1}{\alpha}, \alpha > 0$. In addition, it is also sometimes useful to find the average failure rate function

(AFR), over any interval, say, $(0, t)$, see, for example, Barlow and Proschan [21]. Thus, using Eq. (14), the average failure rate function (AFR) of the size-biased (weighted) Эрланга distribution model, over the interval $(0, t)$, is given by

$$AFR = \frac{-\ln\left(R_{sb}(t)\right)}{t} = \frac{-\ln\left(1 - F_{sb}(t)\right)}{t} = -\frac{1}{t}\ln\left(1 - \frac{1}{2}\gamma(3, 2\alpha t)\right),$$

which in view of the expansion of logarithmic function as a power series, is seen to be positive irrespective of the values of the parameter, $\alpha > 0$. Hence the size-biased (weighted) Эрланга distribution model is increasing failure rate on average (IFRA). Also, recall that a life distribution $F(.)$ is NBU (New Better than Used) if $R(x+y) \le R(x)R(y), \forall\ x, y \ge 0$, and NWU (new worse than used) if the reversed inequality holds, see, for example, Barlow and Proschan [21]. We note that, for the size-biased (weighted) Эрланга distribution model, since

$$R(x+y) = 1 - \frac{1}{2}\gamma\left(3, 2\alpha(x+y)\right),$$

and

$$R(x).R(y)$$
$$= \left\{1 - \frac{1}{2}\gamma(3, 2\alpha x)\right\} \times \left\{1 - \frac{1}{2}\gamma(3, 2\alpha y)\right\},$$

it is easy to see that $R(x+y) \le R(x).R(y)$, which implies that the distribution of the size-biased (weighted) Эрланга distribution model has the property of New Better than Used (NBU).

2.1.8 Percentiles

The percentile points of a given distribution are also important to be known before any statistical applications of it. As such, the statisticians would be interested in knowing the median (50%), 25%, or 75% quartiles, or, in the computations of the 90%, 95%, or 99% confidence levels for other applications in order to assess the statistical significance of an observation whose distribution is known. Thus, in view of these facts, we have computed the percentage points of the size-biased (weighted) Эрланга distribution model with the pdf (4). The $100\,p\,th$ percentile or the quantile of order p, for any $0 < p < 1$, of the size-biased (weighted) Эрланга distribution model with the pdf (4) is defined as a number x_p such that the area under $f_{sb}(x)$ to the left of x_p is p. In other words, x_p is any solution of the equation $F_{sb}(x_p) = \int_1^{x_p} f_{sb}(u)\,du = p$, where $F_{sb}(x_p)$ denotes the cdf (5). Thus, using the Maple 11 program, we have numerically solved the equation $F_{sb}(x_p) = \int_1^{x_p} f_{sb}(u)\,du = p$, and computed the percentage points x_p associated with the cdf, $F_{sb}(x_p)$, of X for different sets of values of the parameter, $\alpha > 0$, which are provided in the Table 1 below.

Table 1: Percentage Points of the Size-Biased (Weighted) Эрланга Distribution Model (α).

α	**75 %**	**80 %**	**85 %**	**90 %**	**95 %**	**99 %**
0.5	3.92040	4.27903	4.72305	5.32232	6.29580	8.40594
1	1.96020	2.13951	2.36153	2.66116	3.14790	4.20297
2	0.98010	1.06976	1.18076	1.33058	1.57395	2.10149
3	0.65340	0.71317	0.78718	0.88705	1.04930	1.40099
4	0.49005	0.53488	0.59038	0.66529	0.78697	1.05074
5	0.39204	0.42790	0.47231	0.53222	0.62958	0.84059

3 Estimation of Parameter, Simulation and Applications

3.1 Estimation and Simulation: Remark 1

For detailed studies on the estimation of the parameter $\{\alpha\}$ of the said size-biased (weighted) Эрланга distribution model (by the methods of the moments and maximum likelihood), see Rather et al. [12]. If $\{X_i\}_{i=1}^{n}$ bean iid sample from a the said size-biased (weighted) Эрланга distribution with the parameter $\{\alpha\}$. According to Rather et al. [12], both the moment and maximum likelihood estimation of the parameter $\{\alpha\}$ coincide and is given by $\hat{\alpha} = \frac{3}{2\bar{X}}$, where

$$\bar{X} = \left(\frac{\sum_{i=1}^{n} X_i}{n} \right).$$

Remark 2: For details on the simulation study of ML estimators, the interested readers are also referred to Rather et al. [12].

3.2 Applications

Rather et al. [12] applied their newly proposed size-biased (weighted) Эрланга distribution model to two different real life data sets, namely, "the data on the 23 ball bearings in the life tests studied by Lawless [22]" and "the data on the lifetimes (in days) of 40 patients suffering from blood cancer (leukemia)", to show that it is a better fit than its sub models. In this section, the goodness of fit test of the said size-biased (weighted) Эрланга distribution model will be further examined by considering another real-world data example, namely, the data on a sample of 40 male cholesterol levels as reported in Triola [13], provided in the following Table 2.

Table 2: Male Cholesterol Levels Data**.**

522, 127, 740, 49, 230, 316, 590, 466, 121, 578, 78, 265, 250, 265, 273, 272, 972, 75, 138, 139, 638, 613, 762, 303, 690, 31, 189, 957, 339, 416, 120, 702, 1252, 288, 176, 277, 649, 113, 656, 172

The mean, median and skewness of this data are 395.225, 282.5 and 0.967 respectively. We can see that the data is positively skewed. Maple 11 has been used for computing the data moment, estimating the parameter (by employing the method of moments), and chi-square test for goodness-of-fit. The data moments are computed as $\hat{\mu} = 395.225$. The estimation of the parameters and chi-square goodness-of-fit test are provided in Tables 3 and 4 respectively.

Table 3: Parameter Estimates for the Male Cholesterol Levels DataAssuming Different Models.

Model			
Size-Biased (Weighted) Эрланга Distribution Model (α **)**	**Эрланга (or Ailamujia) Distribution Model (** α **)**	**Rayleigh Distribution Model (** b **)**	**Exponential Distribution Model (** a **)**
$\hat{\alpha} = 0.0037953$	$\hat{\alpha} = 0.0025302$	$\hat{b} = 315.34393$	$\hat{a} = 395.225$

Table 4: Comparison Criteria (Chi-Square Test for Goodness-of-Fit) (at the Level of Significance = 0.5).

	Model			
	Size-Biased (Weighted) Эрланга Distribution Model (α)	**Rayleigh Distribution Model (b)**	**Exponential Distribution Model (a)**	**Эрланга (or Ailamujia) Distribution Model (α)**
Test Statistic	1.33398	2.63421	3.94577	6.10828
Critical Value	5.99146	5.99146	5.99146	5.99146
P-Value	0.51325	0.26791	0.13906	0.04716
RANK	**1**	**2**	**3**	**4**

From the chi-square goodness-of-fit test, we observed that the size-biased (weighted) Эрланга distribution model, along with the Rayleigh, exponential and Эрланга distributions fit the male cholesterol levels data reasonably well. However, the size-biased (weighted) Эрланга distribution model produces the highest p-value and the smallest test statistic, so fitted better than the Rayleigh, exponential and Эрланга distributions. Also, for the parameters estimated in Table 3, the size-biased (weighted) Эрланга distribution model, along with the Rayleigh, exponential and Эрланга distributions respectively have been superimposed on the histogram of the male cholesterol levels data as in Figure 4. It shows that the size-biased (weighted) Эрланга distribution model fits the male cholesterol levels data reasonably well.

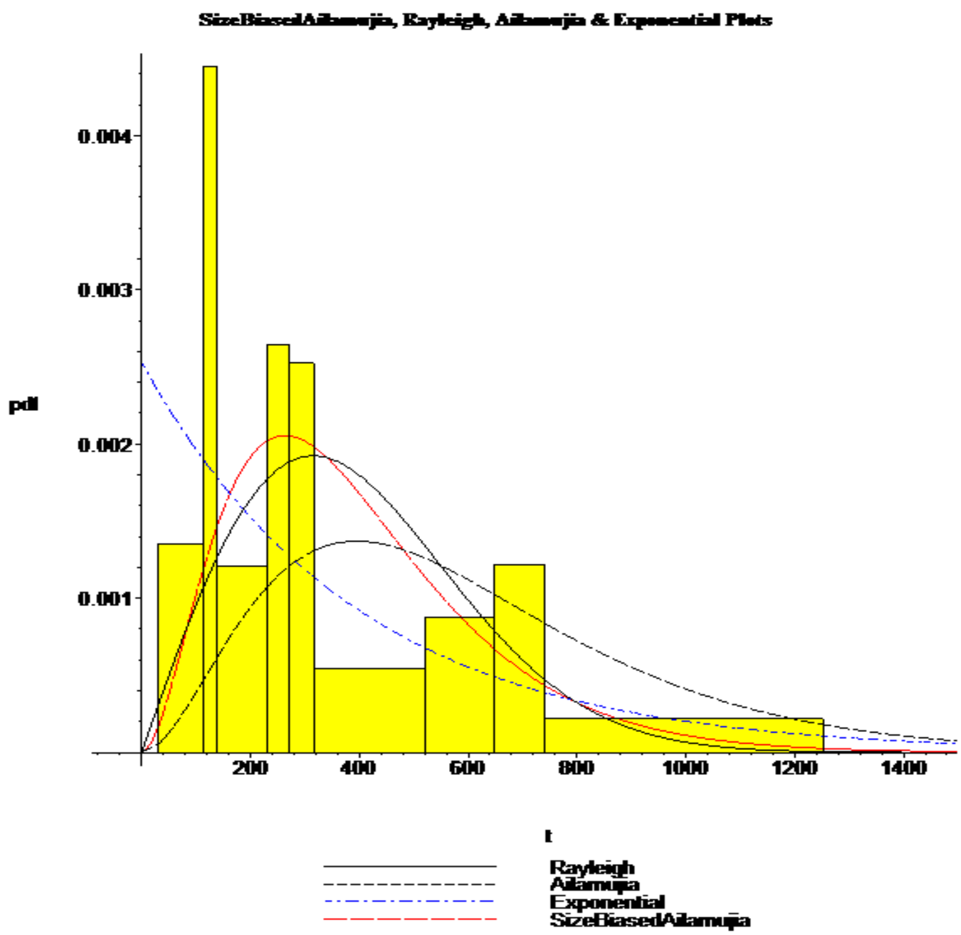

Fig. 4: Fitting of the pdfs of the Size-Biased (Weighted) Эрланга, Rayleigh, Exponential and Эрланга Distributions to the Male Cholesterol Levels Data.

4 Characterizations Results

In what follows, in this section, we will provide our proposed characterizations of the said size biased (weighted) Эрланга distribution by left and right truncated moments, order statistics and record values.

4.1 Characterization by Truncated Moment

In this subsection, we provide two new characterization results of the above-mentioned size biased Эрланга distribution by truncated moments. The first characterization result (Theorem 4.1) is based on a relation between left truncated moment and failure rate function. The second characterization result (Theorem 4.2) is based on a relation between right truncated moment and reversed failure rate function. For this, we will need the following assumption and lemmas.

Assumption 4.1 Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\omega = \inf\{x \mid F(x) > 0\}$, and $\delta = \sup\{x \mid F(x) < 1\}$. We also assume that $f(x)$ is a differentiable for all x, and $E(X)$ exists.

Lemma 4.1 If the random variable X satisfies the Assumption 4.1 with $\omega = 0$ and $\delta = \infty$, and if $E(X|X \le x) = g(x)\tau(x)$, where $\tau(x) = \dfrac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_0^x \dfrac{u - g'(u)}{g(u)}du$ is finite for $x > 0$, then $f(x) = c\, e^{\int_0^x \frac{u - g'(u)}{g(u)}du}$, where c is a constant determined by the condition $\int_0^\infty f(x)dx = 1$.

Proof For proof, see Shakil, et al. [23].

Lemma 4.2 If the random variable X satisfies the Assumption 4.1 with $\omega = 0$ and $\delta = \infty$, and if $E(X \mid X \ge x) = \tilde{g}(x) r(x)$, where $r(x) = \dfrac{f(x)}{1 - F(x)}$ and $\tilde{g}(x)$ is a continuous differentiable function of x with the condition that $\int_x^\infty \dfrac{u + \left[\tilde{g}(u)\right]'}{\tilde{g}(u)}du$ is finite for $x > 0$, then $f(x) = c\, e^{-\int_x^\infty \frac{u + \left[\tilde{g}(u)\right]'}{\tilde{g}(u)}du}$, where c is a constant determined by the condition $\int_0^\infty f(x)dx = 1$.

Proof See Shakil et al. [23].

Theorem 4.1 If the random variable X satisfies the Assumption 4.1 with $\omega = 0$ and $\delta = \infty$, then $E(X|X \le x) = g(x)\dfrac{f_{sb}(x)}{F_{sb}(x)}$, where

$$g(x) = \frac{P_1(x)}{4\alpha^3 x^2 e^{-2\alpha x}}, \tag{19}$$

where $P_1(x)$ is given by Eq. (12), if and only if X has the pdf

$$f_{sb}(x)=\begin{cases}4\alpha^3x^2e^{-2\alpha x}, & (0\le x<\infty, \alpha>0)\\ 0, & \text{otherwise.}\end{cases}$$

Proof Suppose that $E\left(X\middle|X\le x\right)=g\left(x\right)\dfrac{f_{sb}\left(x\right)}{F_{sb}\left(x\right)}$. Then, since $E\left(X\middle|X\le x\right)=\dfrac{\int_0^x u\,f_{sb}\left(u\right)du}{F_{sb}\left(x\right)}$, we have $g\left(x\right)=\dfrac{\int_0^x u\,f_{sb}\left(u\right)du}{f_{sb}\left(x\right)}$. Now, if the random variable X satisfies the Assumption 4.1 and has the distribution with the pdf (4), then applying Eq. (12), we have

$$g\left(x\right)=\frac{\int_0^x u\,f_{sb}\left(u\right)du}{f_{sb}\left(x\right)}=\frac{\dfrac{\gamma\left(4,2\alpha x\right)}{4\alpha}}{4\alpha^3x^2e^{-2\alpha x}}=\frac{P_1\left(x\right)}{4\alpha^3x^2e^{-2\alpha x}}.$$

Consequently, the proof of "if" part of the Theorem 4.1 follows from Lemma 4.1.

Conversely, suppose that

$$g\left(x\right)=\frac{P_1\left(x\right)}{4\alpha^3x^2e^{-2\alpha x}},$$

where $P_1\left(x\right)$ is given by Eq. (12). Now, from Lemma 4.1, we have

$$g\left(x\right)=\frac{\int_0^x u\,f_{sb}\left(u\right)du}{f_{sb}\left(x\right)},$$

or

$$\int_0^x u\,f_{sb}\left(u\right)du=f_{sb}(x)g(x).$$

Differentiating the above equation with respect to respect to x, we obtain

$$x\,f_{sb}(x)=f_{sb}^{\prime}\left(x\right)g(x)+f_{sb}\left(x\right)g^{\prime}\left(x\right),$$

from which, using the definition of the pdf (4) and noting that

$$f_{sb}^{\prime}\left(x\right)=8\alpha^3x\,e^{(-2\alpha x)}\left[1-\alpha x\right],$$

we easily obtain

$$g^{\prime}(x)=x-g\left(x\right)\frac{2x\left(1-\alpha x\right)e^{-2\alpha x}}{x^2e^{-2\alpha x}},$$

or, $$\frac{x-g^{/}(x)}{g(x)}=\frac{2x\,(1-\alpha x)e^{-2\alpha x}}{x^{2}\,e^{-2\alpha x}}. \tag{20}$$

Since, by Lemma 4.1, we have

$$\frac{x-g^{/}(x)}{g(x)}=\frac{f_{sb}{}^{/}(x)}{f_{sb}(x)}, \text{ see Shakil et al. [23],} \tag{21}$$

so from Eqs. (20) and (21), it follows that

$$\frac{f_{sb}{}^{/}(x)}{f_{sb}(x)}=\frac{2x\,(1-\alpha x)e^{-2\alpha x}}{x^{2}\,e^{-2\alpha x}},$$

or

$$d\left(\ln\left(f_{sb}(x)\right)\right)=d\left(\ln\left(x^{2}\,e^{-2\alpha x}\right)\right). \tag{22}$$

Now, integrating Eq. (22) with respect to x and simplifying, we easily have

$$\ln\left(f_{sb}(x)\right)=\ln\left(c\left(x^{2}\,e^{-2\alpha x}\right)\right),$$

or

$$f_{sb}(x)=c\left(x^{2}\,e^{-2\alpha x}\right), \tag{23}$$

where c is the normalizing constant to be determined. Thus, on integrating the above Eq.(23) with respect to x from $x=0$ to $x=\infty$, using the condition $\int_0^\infty f_{sb}(x)\,dx=1$ and noting that $\int_0^\infty t^{\nu-1}e^{-\mu t}\,dt=\mu^{-\nu}\,\Gamma(\nu)$, where $\Gamma(\nu)$ denotes the gamma function, see Gradshteyn and Ryzhik [19], Eq. 3.381.4, p. 317, we obtain $c=4\alpha^{3}$. Thus $f_{sb}(x)=4\alpha^{3}\,x^{2}\,e^{-2\alpha x}$, $(0\le x<\infty,\alpha>0)$, which is the required pdf (4) of the random variable X. This completes the proof of Theorem 4.1.

Theorem 4.2 If the random variable X satisfies the Assumption 4.1 with $\omega=0$ and $\delta=\infty$, then $E\left(X\,|\,X\ge x\right)=\tilde{g}(x)\frac{f_{sb}(x)}{1-F_{sb}(x)}$, where

$$\tilde{g}(x)=\frac{\left(E(X)-g(x)\,f_{sb}(x)\right)}{4\alpha^{3}\,x^{2}\,e^{-2\alpha x}},$$

where $g(x)$ is given by Eq. (19) and $E(X)$ is given by Eq. (8), if and only if X has the pdf

$$f_{sb}(x)=\begin{cases}4\alpha^{3}\,x^{2}\,e^{-2\alpha x}, & (0\le x<\infty,\alpha>0)\\ 0, & \text{otherwise.}\end{cases}$$

Proof Suppose that $E\left(X\middle|X\geq x\right)=\tilde{g}\left(x\right)\dfrac{f_{sb}\left(x\right)}{1-F_{sb}\left(x\right)}$. Then, since $E\left(X\middle|X\geq x\right)=\dfrac{\int_x^\infty u\,f_{sb}\left(u\right)du}{1-F_{sb}\left(x\right)}$, we have $\tilde{g}\left(x\right)=\dfrac{\int_x^\infty u\,f_{sb}\left(u\right)du}{f_{sb}\left(x\right)}$. Now, if the random variable X satisfies the Assumptions 4.1 and has the distribution with the pdf (4), then we have

$$\tilde{g}\left(x\right)=\frac{\int_x^\infty u\,f_{sb}\left(u\right)du}{f_{sb}\left(x\right)}=\frac{\int_0^\infty u\,f_{sb}\left(u\right)du-\int_0^x u\,f_{sb}\left(u\right)du}{f_{sb}\left(x\right)}$$

$$=\frac{\left(E\left(X\right)-g\left(x\right)f\left(x\right)\right)}{4\alpha^3\,x^2\,e^{-2\alpha x}}.$$

Consequently, the proof of “if” part of the Theorem 4.2 follows from Lemma 4.2.

Conversely, suppose that $\tilde{g}\left(x\right)=\dfrac{\left(E\left(X\right)-g\left(x\right)f_{sb}\left(x\right)\right)}{4\alpha^3\,x^2\,e^{-2\alpha x}}$. Now, from Lemma 4.2, we have

$$\tilde{g}\left(x\right)=\frac{\int_x^\infty u\,f_{sb}\left(u\right)du}{f_{sb}\left(x\right)},$$

or

$$\int_x^\infty u\,f_{sb}\left(u\right)du=f_{sb}(x).\tilde{g}\left(x\right).$$

Differentiating the above equation with respect to respect to x, we obtain

$$-x\,f_{sb}(x)=f_{sb}^{/}\left(x\right).\tilde{g}\left(x\right)+f_{sb}\left(x\right).\left(\tilde{g}\left(x\right)\right)^{/},$$

from which, using the definition of the pdf (4) and noting that

$$f_{sb}^{/}\left(x\right)=8\alpha^3\,x\,e^{(-2\alpha x)}\left[1-\alpha\,x\right],$$

we easily obtain

$$\left(\tilde{g}\left(x\right)\right)^{/}=-x-\tilde{g}\left(x\right)\frac{2x\left(1-\alpha\,x\right)e^{-2\alpha x}}{x^2\,e^{-2\alpha x}},$$

or,
$$\frac{x+\left(\tilde{g}\left(x\right)\right)^{/}}{\tilde{g}\left(x\right)}=-\frac{2x\left(1-\alpha\,x\right)e^{-2\alpha x}}{x^2\,e^{-2\alpha x}}. \tag{24}$$

Since, by Lemma 4.2, we have

$$\frac{f_{sb}{}^{/}(x)}{f_{sb}(x)} = -\frac{x+\left[\tilde{g}(x)\right]^{/}}{\tilde{g}(x)}, \quad \text{see Shakil et al. [23],} \tag{25}$$

so from Eqs. (24) and (25), it follows that

$$\frac{f_{sb}{}^{/}(x)}{f_{sb}(x)} = \frac{2x\,(1-\alpha x)e^{-2\alpha x}}{x^2\,e^{-2\alpha x}}$$

or

$$d\left(\ln\left(f_{sb}(x)\right)\right) = d\left(\ln\left(x^2\,e^{-2\alpha x}\right)\right). \tag{26}$$

Now, integrating Eq. (26) with respect to x and simplifying, we easily have

$$\ln\left(f_{sb}(x)\right) = \ln\left(c\left(x^2\,e^{-2\alpha x}\right)\right),$$

or

$$f_{sb}(x) = c\left(x^2\,e^{-2\alpha x}\right), \tag{27}$$

where c is the normalizing constant to be determined. Thus, on integrating the above Eq. (27) with respect to x from $x=0$ to $x=\infty$, using the condition $\int_0^\infty f_{sb}(x)\,dx = 1$ and noting that $\int_0^\infty t^{\nu-1}e^{-\mu t}\,dt = \mu^{-\nu}\,\Gamma(\nu)$, where $\Gamma(\nu)$ denotes the gamma function, see Gradshteyn and Ryzhik [19], Eq. 3.381.4, p. 317, we obtain $c = 4\alpha^3$, and so $f_{sb}(x) = 4\alpha^3\,x^2\,e^{-2\alpha x}$, $(0 \le x < \infty, \alpha > 0)$, which is the required pdf (4) of the random variable X. This completes the proof of Theorem 4.2.

4.2 Characterizations by Order Statistics

If $X_1, X_2, \ldots, X_n$ are the n independent copies of the random variable X with absolutely continuous distribution function $F(x)$ and pdf $f(x)$, and if $X_{1,n} \le X_{2,n} \le \ldots \le X_{n,n}$ are the corresponding order statistics, then it is known from Ahsanullah et al. [24], chapter 5, or Arnold et al. [25], chapter 2, that $X_{j,n} \mid X_{k,n} = x$, for $1 \le k < j \le n$, is distributed as the $(j-k)th$ order statistics from $(n-k)$ independent observations from the random variable V having the pdf $f_V(v \mid x)$ where $f_V(v \mid x) = \frac{f(v)}{1-F(x)}$, $0 \le v < x$, and $X_{i.,n} \mid X_{k,n} = x, 1 \le i < k \le n$, is distributed as ith order statistics from k independent observations from the random variable W having the pdf $f_W(w \mid x)$ where $f_W(w \mid x) = \frac{f(w)}{F(x)}$, $w < x$.

Let $S_{k-1} = \frac{1}{k-1}\left(X_{1,n} + X_{2,n} + \ldots + X_{k-1,n}\right)$, and $T_{k,n} = \frac{1}{n-k}\left(X_{k+1,n} + X_{k+2,n} + \ldots + X_{n.n}\right)$.

Theorem 4.3 Suppose the random variable X satisfies the Assumption 4.1 with $\omega = 0$ and $\delta = \infty$, then $E(S_{k-1} \mid X_{k,n} = x) = g(x)\,\tau(x)$, where $\tau(x) = \frac{f_{sb}(x)}{F_{sb}(x)}$ and

$$g(x)=\frac{P_1(x)}{4\alpha^3\,x^2\,e^{-2\alpha x}},$$

where $P_1(x)$ is given by the Eq. (12), if and only if X has the pdf

$$f_{sb}(x)=\begin{cases}4\alpha^3\,x^2\,e^{-2\alpha x}, & (0\le x<\infty,\alpha>0)\\ 0, & \text{otherwise.}\end{cases}$$

Proof It is known that $E(S_{k-1}\mid X_{k,n}=x)=E(X\mid X\le x)$; see Ahsanullah et al. [24], and David and Nagaraja [26]. Hence, by Theorem 4.1, the result follows.

Theorem 4.4 Suppose the random variable X satisfies the Assumption 4.1 with $\omega=0$ and $\delta=\infty$, then $E(T_{k,n}\mid X_{k,n}=x)=\tilde{g}(x)\dfrac{f_{sb}(x)}{1-F_{sb}(x)}$, where $\tilde{g}(x)=\dfrac{\left(E(X)-g(x)f_{sb}(x)\right)}{4\alpha^3\,x^2\,e^{-2\alpha x}}$,

$g(x)$ is given by Eq. (19) and $E(X)$ is given by the Eq. (8), if and only if X has the pdf

$$f_{sb}(x)=\begin{cases}4\alpha^3\,x^2\,e^{-2\alpha x}, & (0\le x<\infty,\alpha>0)\\ 0, & \text{otherwise.}\end{cases}$$

Proof Since $E(T_{k,n}\mid X_{k,n}=x)=E(X\mid X\ge x)$, see Ahsanullah et al. [24], and David and Nagaraja [26], the result follows from Theorem 4.2.

4.3 Characterization by Upper Record Values

For details on record values, see Ahsanullah [27]. Let $X_1,X_2,\ldots$ be a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and pdf $f(x)$. If $Y_n=\max(X_1,X_2,\ldots,\ X_n)$ for $n\ge 1$ and $Y_j>Y_{j-1}$, $j>1$, then X_j is called an upper record value of $\{X_n,\ n\ge 1\}$. The indices at which the upper records occur are given by the record times $\{U(n)>\min(j\mid j>U(n+1),X_j>X_{U(n-1)},n>1)\}$ and $U(1)=1$. Let the nth upper record value be denoted by $X(n)=X_{U(n)}$.

Theorem 4.5 Suppose the random variable X satisfies the Assumption 4.1 with $\omega=0$ and $\delta=\infty$, then $E(X(n+1)\mid X(n)=x)=\tilde{g}(x)\dfrac{f_{sb}(x)}{1-F_{sb}(x)}$, where

$\tilde{g}(x)=\dfrac{\left(E(X)-g(x)f_{sb}(x)\right)}{4\alpha^3\,x^2\,e^{-2\alpha x}}$, $g(x)$ is given by the Eq. (19) and $E(X)$ is given by Eq. (8), if and only if X has the pdf

$$f_{sb}(x)=\begin{cases}4\alpha^3\, x^2\, e^{-2\alpha x}, & (0\le x<\infty, \alpha>0)\\ 0, & \text{otherwise.}\end{cases}$$

Proof It is known from Ahsanullah et al. [24], and Nevzorov [28] that $E(X(n+1)\,|\,X(n)=x)=E(X\,|\,X\ge x)$. Then, the result follows from Theorem 4.2.

4.4 Some Remarks on Characterization by Truncated Moment

Since a characterization of a particular probability distribution states that it is the only distribution that satisfies some specified conditions, our characterization results may serve as a basis for parameter estimation, see Glänzel et al. [29] and Glänzel [30, 31]. Moreover, Glänzel [31] points out that the characterizations by truncated moments may also be useful in developing some goodness-of-fit tests of distributions by using data whether they satisfy certain properties given in the characterizations of distributions. These conditions are used by various authors to test goodness of fit, efficiency of a particular test of hypothesis and the power of a particular estimating, etc. For example, Volkova and Nikitin [32] used a well-known characterization result of Ahsanullah [33] to test exponentiality of a distribution. For more on the goodness-of-fit and symmetry tests based on the characterization properties of distributions, the interested readers are referred to recent nice papers of Nikitin [34], Miloševic [35] and Akbari [36], and references therein

5 Concluding Remarks

In this paper, we have considered a size-biased (weighted) version of Lv et al. [1]'s Эрланга distribution. Some characteristics of the size-biased (wighted) Эрланга distribution are obtained. The plots for the cdf, pdf and hazard function, and table for percentile points for selected values of the parameter have been provided. It is noticeable that the size-biased (wighted) Эрланга distribution is skewed to the right and bears most of the properties of skewed distributions. The statistical applications of the results to a problem of the male cholesterol levels data from medical science have been provided. It was found that the size-biased (wighted) Эрланга distribution model fits better than the Rayleigh, exponential and Эрланга distributions.

Furthermore, based on the characteristics of the size-biased (wighted) Эрланга distribution, we have established two new characterization results of the size-biased (weighted) Эрланга distribution by truncated moments. The first characterization result is based on a relation between left truncated moment and failure rate function. The second characterization result is based on a relation between right truncated moment and reversed failure rate function. In addition, we have characterized the size-biased (weighted) Эрланга distribution by order statistics and record values.

Acknowledgements: The authors are grateful to the editor and editorial review for comments and suggestions that improved the paper.

Conflict of Interest: The authors declare that they have no conflict of interest.

References

[1] Lv H. Q., Gao L. H., and Chen C. L. (2002). Эрланга distribution and its application in supportability data analysis, Journal of Academy of Armored Force Engineering., **16(3),** 48 – 52(2002).

[2] Nagaraja, H. (2006). Characterizations of Probability Distributions, In Springer Handbook of Engineering Statistics (79 - 95), Springer, London, UK., 79-95.

[3] Koudou, A. E., and Ley, C. (2014). Characterizations of GIG laws: A survey, Probability surveys., **11**, 161 – 176(2014)

[4] Galambos, J., and Kotz, S. (1978). Characterizations of probability distributions. A unified approach with an emphasis on exponential and related models, Lecture Notes in Mathematics, 675, Springer, Berlin, Germany.

[5] Kotz, S., and Shanbhag, D. N. (1980). Some new approaches to probability distributions, Advances in Applied Probability., **12**, 903 – 921(1980).

[6] Ahsanullah, M., and Shakil, M. (2012). A note on the characterizations of Pareto distribution by upper record values, Communications of the Korean Mathematical Society., **27(4)**, 835 – 842(2012).

[7] Ahsanullah, M., and Shakil, M. (2015). Characterizations of continuous probability distributions occurring in physics and allied

sciences by truncated moment, International Journal of Advanced Statistics and Probability., **3(1)**, 100 – 114(2015).

[8] Ahsanullah, M., Kibria, B. M. G., and Shakil, M. (2014). Normal and Student´s t Distributions and Their Applications, Atlantis Press, Paris, France.

[9] Ahsanullah, M., Shakil, M., and Kibria, B. G. (2015). Characterizations of folded student's t distribution, Journal of Statistical Distributions and Applications., **2(1),** 15(2015).

[10] Ahsanullah, M., Shakil, M., and Kibria, B. M. (2016). Characterizations of Continuous Distributions by Truncated Moment, Journal of Modern Applied Statistical Methods., **15(1)**,316 -331(2016).

[11] Ahsanullah, M. (2017). Characterizations of Univariate Continuous Distributions, Atlantis Press, Paris, France.

[12] Rather, A. A., Subramanian, C., Shafi, S., Malik, K. A., Ahmad, P. J., Para, B. A., and Jan, T. R. (2018). A new Size Biased Distribution with applications in Engineering and Medical Science, International Journal of Scientific Research in Mathematical and Statistical Sciences., **5(4),** 75 – 85(2018).

[13] Triola, M. F. (2009). Elementary Statistics Using Excel, 4th Edition. Pearson Education, Inc., Boston, MA, USA.

[14] Jayakumar, B., and Elangovan, R. (2019). A New Generalization of Ailamujia Distribution with Applications in Bladder Cancer Data, International Journal of Scientific Research in Mathematical and Statistical Sciences., **6(1),** 61 – 68(2019).

[15] Pan, G. T., Wang, B. H., Chen, C. L., Huang, Y. B., and Dang, M. T. (2009). The research of interval estimation and hypothetical test of small sample of Эрланга distribution, Application of Statistics and Management., **28(3),** 468 – 472(2009).

[16] Long, B. (2015). Bayesian estimation of parameter on Эрланга distribution under different prior distribution, Mathematics in Practice & Theory., **4**, 186 – 192(2015).

[17] Li, L. P. (2016). Minimax estimation of the parameter of ЭРланга distribution under different loss functions, Science Journal of Applied Mathematics and Statistics., **4(5),** 229 - 235.

[18] Jan, U., Fatima, K., and Ahmad, S. P. (2017). On Weighted Ailamujia Distribution and Its Applications to Lifetime Data, J. Stat. Appl. Pro., **6(3)**, 619 - 633.

[19] Gradshteyn, I. S., and Ryzhik, I. M. (2000). Table of integrals, series, and products, Academic Press, Inc., San Diego, California, USA.

[20] Shannon, C. E. (1948). A mathematical theory of communication. Bell System Tech. J., **27(379** – **423)**, 623 – 656(1984).

[21] Barlow, R. E., and Proschan, F. (1965). Mathematical Theory of Reliability. John Wiley, New York, USA.

[22] Lawless, J. F. (2003). Statistical models and methods for lifetime data. John Wiley and Sons, New York, USA.

[23] Shakil, M., Ahsanullah, M., and Kibria, B. G. (2018). On the Characterizations of Chen's Two-Parameter Exponential Power Life-Testing Distribution, Journal of Statistical Theory and Applications., **17(3)**,393 – 407(2018).

[24] Ahsanullah, M., Nevzorov,V. B., and Shakil, M. (2013). An Introduction to Order Statistics, Atlantis Press, Paris, France.

[25] Arnold, B.C., Balakrishnan, and Nagaraja, H. N. (2005). First Course in Order Statistics, Wiley, New York, USA.

[26] David, H. A., and Nagaraja, H. N. (2003). Order Statistics, Third Edition, Wiley, New York, USA.

[27] Ahsanullah, M. (1995). Record Statistics, Nova Science Publishers, New York, USA.

[28] Nevzorov, V. B. (2001). Records: Mathematical Theory, Translation of Mathematical Monograph, American Mathematical Society, Rhode Island, USA.

[29] Glänzel, W., Teles, A., and Schubert, A. (1984). Characterization by truncated moments and its application to Pearson-type distributions. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete., **66(2),** 173-183(1984).

[30] Glänzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families, In Mathematical Statistics and Probability Theory (pp. 75-84), Springer, Dordrecht, Germany.

[31] Glanzel, W. (1990). Some consequences of a characterization theorem based on truncated moments. Statistics., **21(4)**, 613-618(1990).

[32] Volkova, K. Y., and Nikitin, Y. Y. (2015). Exponentiality tests based on Ahsanullah's characterization and their efficiency, Journal of Mathematical Sciences., **204(1)**, 42-54(2015).

[33] Ahsanullah, M. (1978). A characterization of the exponential distribution by spacings. Journal of Applied Probability., **15(3)**, 650-653(1978).

[34] Nikitin, Y. Y. (2017). Tests based on characterizations, and their efficiencies: A survey, Acta et Commentationes Universitatis Tartuensis de Mathematica., **21(1)**, 3-24(2017).

[35] Miloševic, B. (2017). Some recent characterization based goodness of fit tests. In the Proceedings of the 20th European Young Statisticians Meeting (pp. 67-73) (Uppsala, Sweden, September, 2017).

[36] Akbari, M. (2020). Characterization and Goodness-of-Fit Test of Pareto and Some Related Distributions Based on Near-Order Statistics, Journal of Probability and Statistics, Volume 2020, Article ID 4262574, 1–9(2020).